Von Neumann-Morgenstern-Solutions for Semi Orthogonal Games

Joachim Rosenmüller

Institute of Mathematical Economics
IMW
University of Bielefeld
D–33615 Bielefeld
Germany
e-mail: jr@wiwi.uni-bielefeld.de

1 Notations and Definitions ****************

1.1 Games and Solutions

A game :

$$(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$$

- $\boldsymbol{I}$ "the players" : $\{1, \ldots, n\}$ or an interval of $\mathbb{R}$,
- $\underline{\underline{\mathbf{F}}}$ "the coalitions" ($\mathcal{P}(\boldsymbol{I})$ or Borelian subsets of $\boldsymbol{I}$).
- $\boldsymbol{v}$ "the coalitional function"; $\boldsymbol{v} := \underline{\underline{\mathbf{F}}} \to \mathbb{R}_+$
  $(\boldsymbol{v}(\emptyset) = 0, \quad \boldsymbol{v}(\boldsymbol{I}) = 1)$
  (a.c. w.r.t. $\boldsymbol{\lambda}$ – hence nonatomic).

Players cooperate in $S \in \underline{\underline{\mathbf{F}}}$, $\to$ monetary value $\boldsymbol{v}(S)$.

Imputations :

$$\mathcal{I} := \{\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \text{ is a probability on } \underline{\underline{\mathbf{F}}}\}.$$

("Total worth" $\boldsymbol{v}(\boldsymbol{I})$ is distributed).

Definition 1.1. *A Solution Concept* *is a mapping from a class* $\boldsymbol{V}$ *of coalitional functions resulting in a set of imputations, i.e.*

$$\mathcal{S} : \boldsymbol{V} \to \mathcal{P}(\mathcal{I})$$

Example 1.2.

- The *Core*
  $$\mathcal{C}(\boldsymbol{v}) = \{\boldsymbol{\xi} \in \mathcal{I} \,|\, \boldsymbol{\xi} \geq \boldsymbol{v}\ .\}$$
- The *Shapley value:*
  a *linear* mapping
  $$\boldsymbol{\Phi} := \boldsymbol{V} \to \mathcal{I}$$
  axiomatically defined.
  Shapley****[1954], Aumann–Shapley****[1966],
  Kannai****[1966],

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The ***vNM–Stable Set*** ("Von Neumann–Morgenstern Solution"):
(VON NEUMANN–MORGENSTERN**** [1944], LUCAS****[1968])

$\boldsymbol{\xi}$ ***dominates*** $\boldsymbol{\eta}$ w.r.t $S \in \underline{\underline{\mathbf{F}}}$
if

$$\boldsymbol{\lambda}(S) > 0 \quad \text{and} \quad \boldsymbol{\xi}(S) \leq \boldsymbol{v}(S) \tag{1.1}$$

and

$$\boldsymbol{\xi}(T) > \boldsymbol{\eta}(T) \quad (T \in \underline{\underline{\mathbf{F}}},\ T \subseteq S, \boldsymbol{\lambda}(T) > 0). \tag{1.2}$$

Every subcoalition of S (almost every player in S) strictly improves its payoff at $\boldsymbol{\xi}$ versus $\boldsymbol{\eta}$.

Write $\boldsymbol{\xi}\,\text{dom}_S\,\boldsymbol{\eta}$.

Definition 1.3. ***vNM–Stable Set:***

A set $\mathcal{S}$ of imputations such that:

- *(*"Internal stability":*)*
  no $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{S}$ with $\boldsymbol{\xi}\,\text{dom}_{\bullet}\,\boldsymbol{\eta}$.
- *(*"external stability":*)*
  for $\boldsymbol{\eta} \notin \mathcal{S}$ there exists $\boldsymbol{\xi} \in \mathcal{S}$ such that $\boldsymbol{\xi}\,\text{dom}_{\bullet}\,\boldsymbol{\eta}$.

Example 1.4. [the 3–person majority game]

$\boldsymbol{I} = \{1,2,3\}$

$$\boldsymbol{v}(S) = \begin{cases} 1 & (|S| \geq 2) \\ 0 & \text{otherwise} \end{cases}$$

Imputations:

$$\mathfrak{J}(\boldsymbol{v}) = \{\boldsymbol{x} \in \mathbb{R}^3_+ \,\Big|\, \sum_{i \in \boldsymbol{I}} = 1\} \quad ,$$

(unit simplex in $\mathbb{R}^3$).

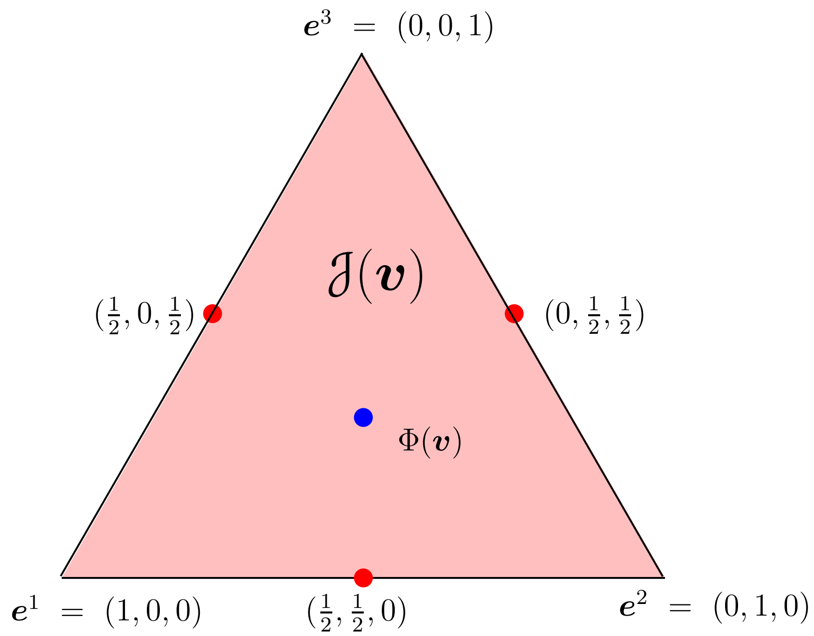

Figure 1.1: Imputations and Solutions for the Majority Game

We have

The Shapley Value: $\Phi(\boldsymbol{v}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$;

The Core: $\mathcal{C}(\boldsymbol{v}) = \emptyset$;

A vNM–Stable Set: $\mathcal{S} = \left\{(\frac{1}{2}, \frac{1}{2}, 0)(\frac{1}{2}, 0, \frac{1}{2})(0, \frac{1}{2}, \frac{1}{2})\right\}$;

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1.2 Linear Production Games

$\boldsymbol{v} = \boldsymbol{v}^{\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}}$ given by

$$\boldsymbol{v}(S) := \max\left\{\boldsymbol{c}\boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \mathbb{R}^l_+ \,,\ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}(S)\right\}$$

with (all nonnegative):

$\boldsymbol{c} \in \mathbb{R}^l_+$,
$\boldsymbol{A}$ an $m \times l$ matrix ,
$\boldsymbol{b} : \underline{\underline{\mathbf{F}}} \to \mathbb{R}^m_+$ a (nonatomic) vectorvalued measure .

The Core is nonempty !

$\bar{\boldsymbol{y}} \in \mathbb{R}^m_+$: an optimal solution for the dual problem of the grand coalition (“shadow prices” for production factors),

$$\bar{\boldsymbol{y}}\boldsymbol{b}(\boldsymbol{I}) = \min\left\{\boldsymbol{y}\boldsymbol{b}(\boldsymbol{I}) \,\middle|\, \boldsymbol{y} \in \mathbb{R}^m_+ \,,\ \boldsymbol{y}\boldsymbol{A} \geq \boldsymbol{c}\right\}$$

then

$$\bar{\boldsymbol{y}}\boldsymbol{b}(\bullet) \in \mathcal{C}(\boldsymbol{v}^{\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}})$$

Remark 1.5.

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  Core and the Shapley value "converge" towards equilibria.

  (i.e. $\boldsymbol{yb}(\bullet)$ the shadow price evaluated worth of the factors = equilibrium solution)

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  Both favor the short side of the market

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  Equivalence Theorems

  .) The vNM-Stable Set does not satisfy equivalence theorems - respects the cartel power of the long side.

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Remark 1.6. Any LP.–game $\boldsymbol{v}$ is a "**glove game**", i.e.,

$$\boldsymbol{v}(S) := \min\{\boldsymbol{\lambda}^{\rho}(S) \mid \rho = 1, \ldots, r\} \quad (S \in \underline{\underline{\mathbf{F}}}). \tag{1.3}$$

($\boldsymbol{\lambda}^{\rho}$ are assumed copies of Lebesgue measure on some interval).

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2 Linear Production Games: Continuum of Players

2.1 The Orthogonal Case

For $\boldsymbol{v}(\bullet) := \min\{\boldsymbol{\lambda}^\rho(\bullet) \mid \rho = 1, \ldots, r\}$ the core is well known(BILLERA–RAANAN *** [1981]):

Theorem 2.1. $\mathcal{C}\{\boldsymbol{v}\} = \boldsymbol{ConvH}\{\boldsymbol{\lambda}^\rho \mid \boldsymbol{\lambda}^\rho(\boldsymbol{I}) = 1 = \boldsymbol{v}(\boldsymbol{I})\}$

In the **orthogonal** and **exact** case, the Core is vNM–Stable. (EINY, HOLZMAN, MONDERER, SHITOVITZ ****[1996]).

Theorem 2.2. *Let*

$$\boldsymbol{v}(\bullet) = \min\{\boldsymbol{\lambda}^\rho(\bullet) \mid \rho = 1, \ldots, r\}$$

and suppose that the $\boldsymbol{\lambda}^\rho$ *are* ***orthogonal probabilities****. Then the core is (the unique) vNM–Stable Set.*

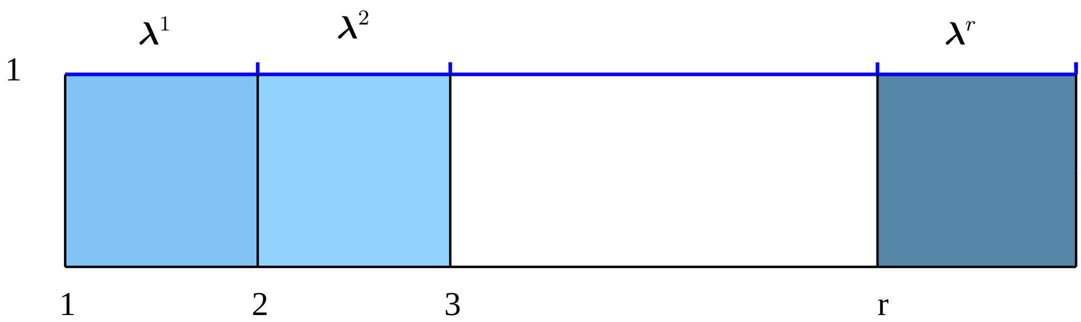

Figure 2.1: The exact and orthogonal case

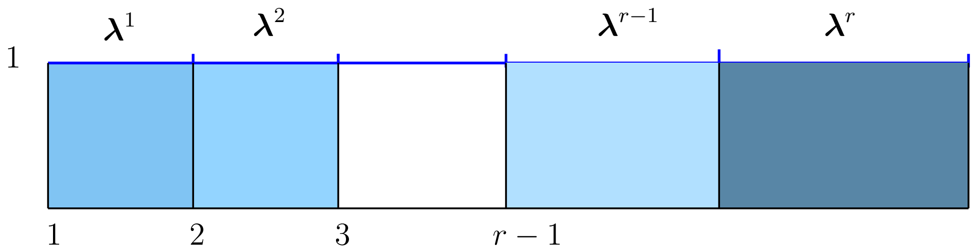

Figure 2.2: The not exact and orthogonal case

If the game is not exact, then the Core is not vNM–stable.

Characterization of convex vNM–Stable Sets (Rosenmüller–Shitovitz *** [2000,2010]):

Theorem 2.3 (***Characterization***)**.**
Let μ^ρ be probabilities satisfying

1. $\mu^\rho \ll \lambda^\rho \quad (\rho = 1, \ldots, r)$,
2. $\dot{\mu}^\rho \leq 1$ *a.e.* $\quad (\rho = 1, \ldots, r)$,

Then

$$\mathcal{S} \;:=\; \boldsymbol{ConvH}\{\mu^\rho \mid \rho = 1, \ldots, r\}$$

is a vNM–Stable set.
All convex vNM–Stable sets *are generated this way.*

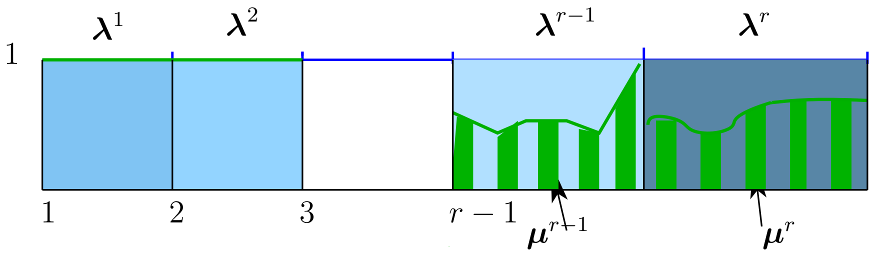

Figure 2.3: The orthogonal case – a vNM–Stable Set

.) The vNM-Stable Set does not favor the short side of the market unconditionally - respects the cartel power of the long side.

2.2 ε–Relevant Coalitions

For orthogonal $\boldsymbol{\lambda}^\rho$ and $S = \bigcup_{\rho=1}^r S^\rho$ (disjoint !).

$$\boldsymbol{v}(S) \;=\; \min_{\rho=1}^{r} \boldsymbol{\lambda}^\rho(S) \;=\; \min_{\rho=1}^{r} \boldsymbol{\lambda}^\rho(S^\rho)$$

Choose $T^\rho \subseteq S^\rho$ with $\boldsymbol{\lambda}(T^\rho) = \boldsymbol{v}(S)$.

Then $T \;=\; \cup_{\rho=1}^r T^\rho$ yields

$$\boldsymbol{v}(T) = \boldsymbol{v}(S) \;.$$

Now: if $\boldsymbol{\vartheta} \operatorname{dom}_S \boldsymbol{\eta}$, then $\boldsymbol{\vartheta} \operatorname{dom}_T \boldsymbol{\eta}$.

Moreover:

Theorem 2.4 (***The Inheritance Theorem***). *Let $\boldsymbol{\vartheta}, \boldsymbol{\eta}$ be imputations and let $\boldsymbol{\vartheta} \operatorname{dom}_S \boldsymbol{\eta}$. Then, for all sufficiently small $\varepsilon > 0$ there is a coalition $T \subseteq S$ satisfying*

$$(\boldsymbol{\lambda}^1(T), \ldots, \boldsymbol{\lambda}^r(T)) \;=\; \varepsilon(1, \ldots, 1) \quad \textit{and} \quad \boldsymbol{\vartheta} \operatorname{dom}_T \boldsymbol{\eta} \;.$$

*I.e., with respect to domination, it is sufficient and necessary to consider "ε–**relevant coalitions**" only.*

3 LP Games – Continuum of Players The Semi Orthogonal Game

3.1 The non–cornered commodity

Now $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r$: orthogonal probabilities (Lebesgue measure)
also : $\boldsymbol{\lambda}^0$ with piecewise constant density, $\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1$.

I.e.

$$\dot{\boldsymbol{\lambda}}^0 = \sum_{\tau \in \mathbf{T}} h_\tau \, \mathbb{1}_{\boldsymbol{D}^\tau} \; ; \tag{3.1}$$

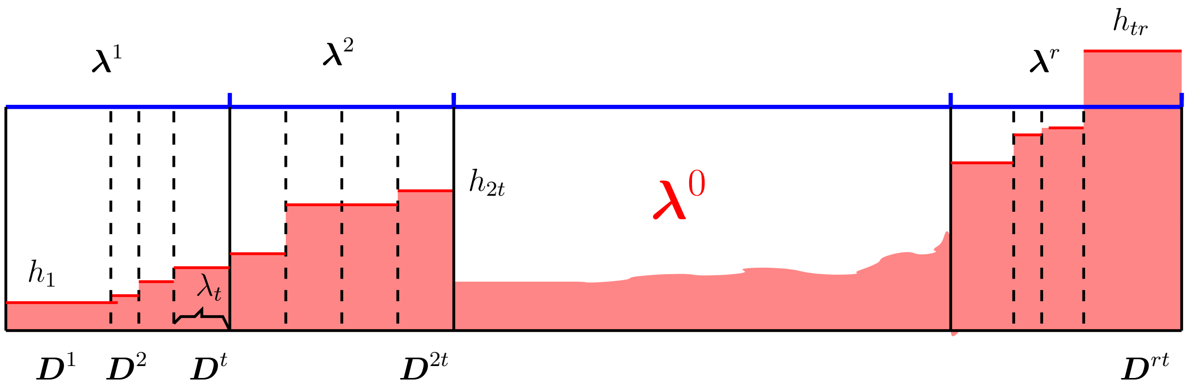

Figure 3.1: The density of $\boldsymbol{\lambda}^0$

Again consider

$$\boldsymbol{v}(\bullet) = \min \{\boldsymbol{\lambda}^\rho(\bullet) \mid \rho = 0, 1, \ldots, r\}$$

Then

$$\mathcal{C}(\boldsymbol{v}) = \mathbf{ConvH} \{\boldsymbol{\lambda}^\rho \mid \rho = 1, \ldots, r \;\} \; .$$

For $r = t = 2$:

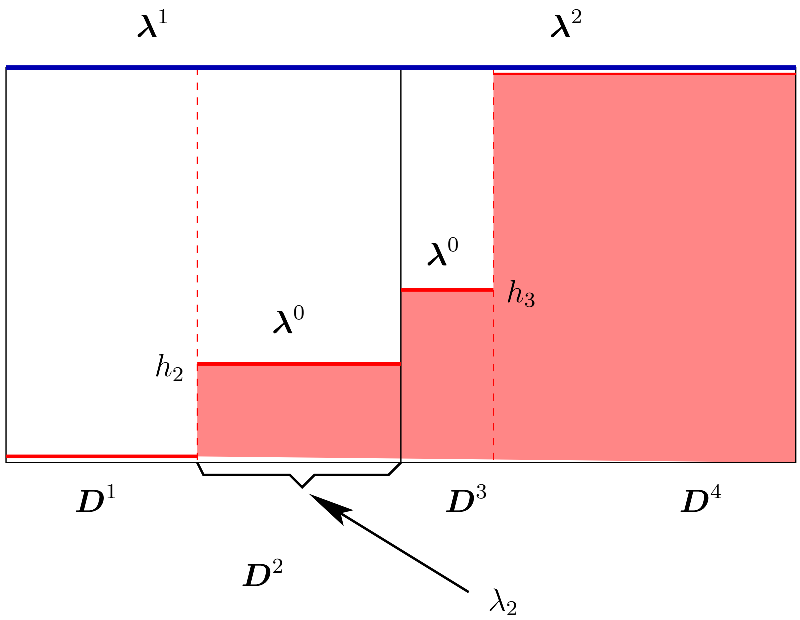

Figure 3.2: The density of $\boldsymbol{\lambda}^0$ with 4 steps

3.2 ε–Relevant Coalitions

“Discrete analogues” : $\quad (\lambda_\tau \;:=\; \boldsymbol{\lambda}(\boldsymbol{D}^\tau)\;)$

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Pre–imputations:

$$\boldsymbol{x} = (x_\tau)_{\tau\in\mathbf{T}} \in \mathbb{R}^{rt}_{+} \;\Big|\; \sum_{\tau\in\mathbf{T}} \lambda_\tau x_\tau = 1$$

Then

$$\dot{\boldsymbol{\vartheta}}^{\boldsymbol{x}} \;:=\; \sum_{\tau\in\mathbf{T}} x_\tau \mathbb{1}_{\boldsymbol{D}^\tau}\;, \tag{3.2}$$

constitutes $\boldsymbol{\vartheta}^{\boldsymbol{x}} \in \mathfrak{I}(\boldsymbol{v})$

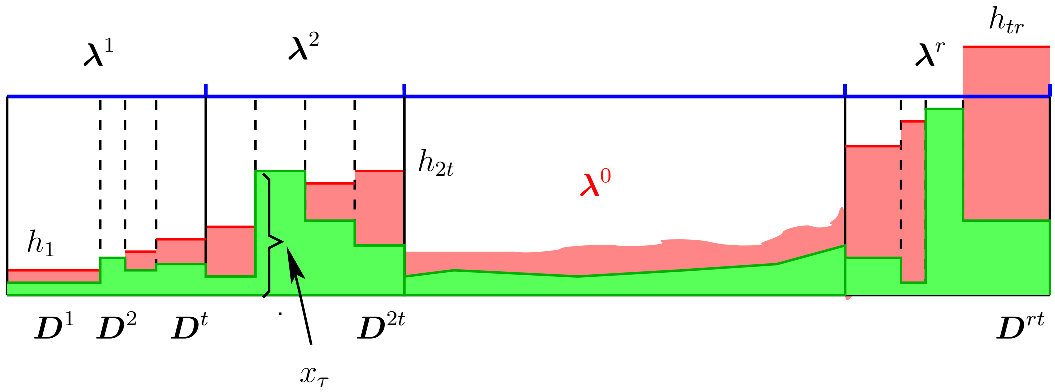

Figure 3.3: A preimputation

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Pre–coalitions:.

$$\boldsymbol{a} \;=\; (a_\tau)_{\tau\in\mathbf{T}}$$

For some $\varepsilon > 0$ choose a coalition $T^{\varepsilon\boldsymbol{a}}$

$$\boldsymbol{\lambda}(T^{\varepsilon\boldsymbol{a}} \cap \boldsymbol{D}^\tau) = \varepsilon a_\tau \quad (\tau \in \mathbf{T}) \ .$$

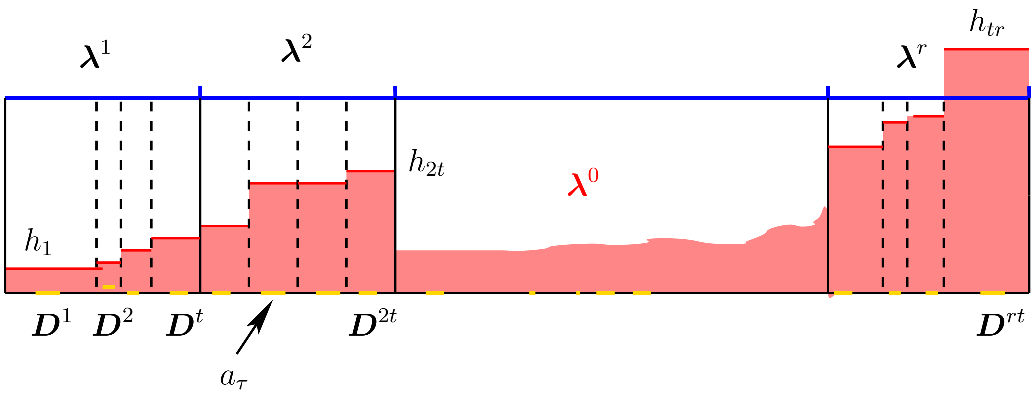

Figure 3.4: A pre–coalition

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pre–measures , i.e., functionals on pre–coalitions:

$$\boldsymbol{c}^\rho \ : \ \mathbb{R}^{rt} \to \mathbb{R} \ , \quad \boldsymbol{c}^\rho(\boldsymbol{a}) \ := \ \sum_{\tau \in \mathbf{T}^\rho} a_\tau \quad .$$

corresponds to $\boldsymbol{\lambda}^\rho$ as

$$\boldsymbol{\lambda}^\rho(T^{\varepsilon \boldsymbol{a}}) \ = \ \varepsilon \boldsymbol{c}^\rho(\boldsymbol{a})$$

also

$$\boldsymbol{c}^0 \ : \ \mathbb{R}^{rt} \to \mathbb{R} \ , \quad \boldsymbol{c}^\rho(\boldsymbol{a}) \ := \ \sum_{\tau \in \mathbf{T}} h_\tau a_\tau$$

corresponds to $\boldsymbol{\lambda}^0$, as

$$\boldsymbol{\lambda}^0(T^{\varepsilon \boldsymbol{a}}) \ = \ \varepsilon \boldsymbol{c}^0(\boldsymbol{a}) \ .$$

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pre–game

$$v(\boldsymbol{a}) \ := \ \min\{\boldsymbol{c}^\rho(\boldsymbol{a}) \mid \quad (\rho = 0, 1, \ldots, r)\ \}$$

(positively homogenous: $v(t\boldsymbol{a}) = tv(\boldsymbol{a}) \ \ t > 0$).

Definition 3.1. *The extremal points of the convex set*

$$\boldsymbol{A} \ : \ \left\{\boldsymbol{a} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{c}^\rho(\boldsymbol{a}) \geq 1 \ \ (\rho = 0, 1, \ldots, r)\right\} \tag{3.3}$$

are called the ***relevant vectors****.*

Theorem 3.2 (***The Inheritance Theorem***).

. *Let* $\boldsymbol{\vartheta}\,\mathrm{dom}_S\,\boldsymbol{\eta}$.

Then there is $\delta > 0$ *such that for all* $0 < \varepsilon < \delta$ *there is a relevant vector* $\boldsymbol{a}^e \in \boldsymbol{A}^e$ *and a coalition* $T \subseteq S$ *satisfying*

$$\vec{\boldsymbol{\lambda}}(T) = \varepsilon \boldsymbol{a}^e \qquad \text{and} \qquad \boldsymbol{\vartheta}\,\mathrm{dom}_T\,\boldsymbol{\eta} \ .$$

It is sufficient and necessary to consider ε*–relevant coalitions only.*

3.3 vNM–Stable Sets for the Semi–Orthogonal Game

With some conditions to the h_τ we can construct a "candidate".

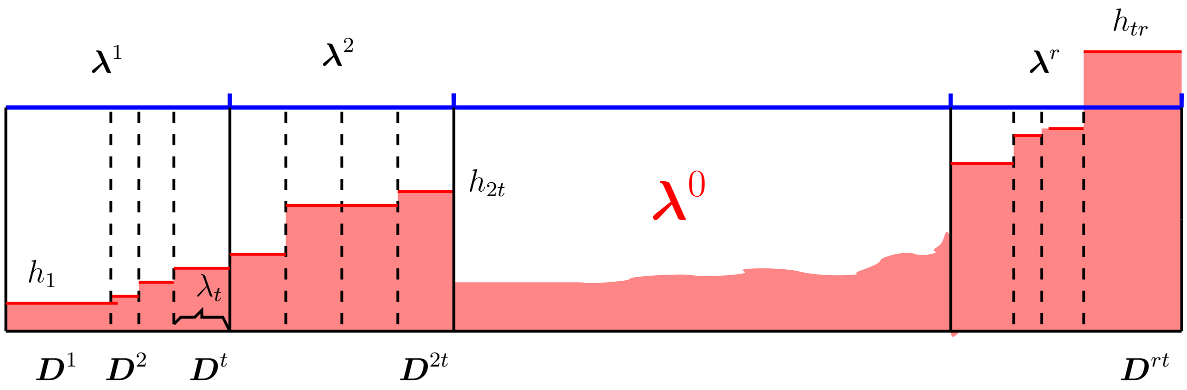

Figure 3.5: The density of $\boldsymbol{\lambda}^0$

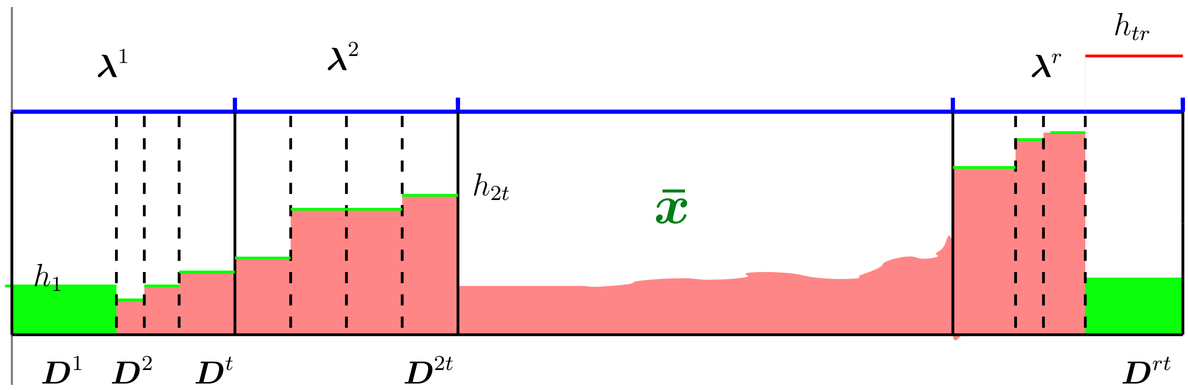

Figure 3.6: The canditate $\bar{\boldsymbol{x}}$

Definition 3.3. *Define $\bar{\boldsymbol{x}}$:*
First

$$\overline{x}_\tau = h_\tau \quad (\tau \in \mathbf{T} \setminus \{\overline{\tau}_1, \overline{\tau}_r\}) \quad . \tag{3.4}$$

Then $\overline{x}_{\overline{\tau}_1}$ and $\overline{x}_{\overline{\tau}_r}$ by two equations

$$\overline{x}_{\overline{\tau}_1} + h_{\widehat{\tau}_2} \ldots + h_{\widehat{\tau}_{r-1}} + \overline{x}_{\overline{\tau}_r} = 1 \tag{3.5}$$

and

$$\sum_{\tau \in \mathbf{T}} \lambda_{\overline{\tau}_\tau} \overline{x}_{\overline{\tau}_\tau} = 1 \ . \tag{3.6}$$

That is, $\bar{\boldsymbol{x}}$ reflects $\boldsymbol{\lambda}^0$ up to some normalising to a preimputation AND some equation for a relevant vector.

Theorem 3.4. *(With some conditions to $h_\bullet$ and $\lambda_\bullet$)*

$$\boldsymbol{ConvH}\{\boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^\rho\}$$

constitutes a vNM Stable set.

.) Again: vNM-Stable Set does not favor the short side of the market unconditionally - respects the cartel power of the long side.

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The 2×2–example (scarce central measure):

$$h_2 + h_3 \leq 1 \qquad \textbf{and} \qquad \lambda_1 + \lambda_3 \leq 1 \ .$$

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Then,

$$\bar{\boldsymbol{x}} \ := \ (\frac{\lambda_3 + \lambda^0_{23})}{\lambda_4 - \lambda_1}, h_2, h_3, \frac{(\lambda_2 - \lambda^0_{23})}{\lambda_4 - \lambda_1}).$$

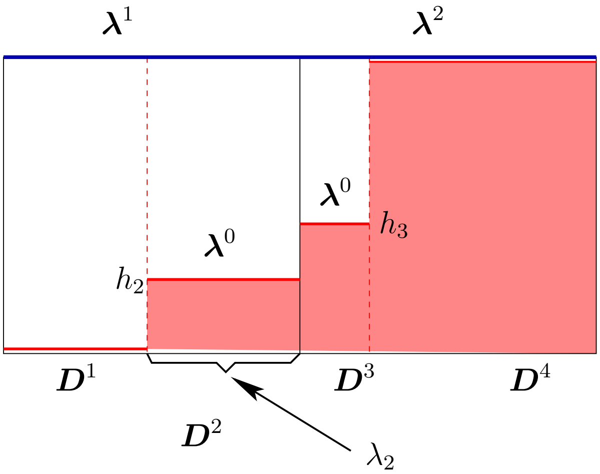

Figure 3.7: $\boldsymbol{\lambda}^0$ for the 2×2 case

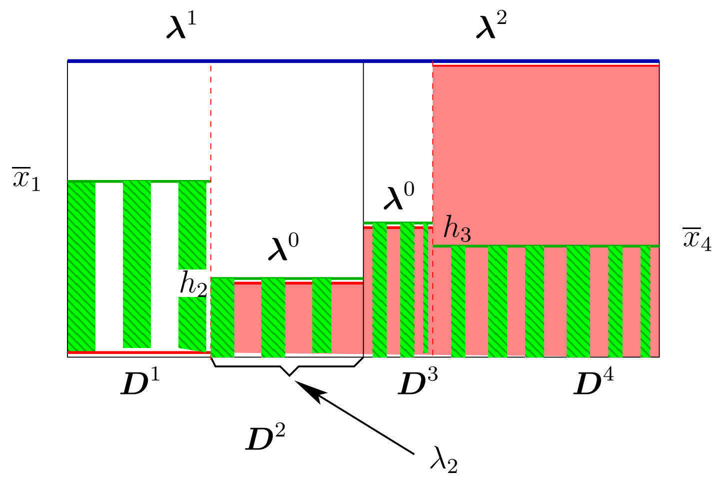

Figure 3.8: $\bar{\boldsymbol{x}}$ (or $\boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}$) for the 2×2 case

Theorem 3.5 (***Existence and Uniqueness***).

$$\mathcal{H} := \boldsymbol{ConvH}\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}\}$$

is a vNM–Stable Set. For the 2×2–case this is the **unique convex** *vNM–Stable Set.*

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The 2×2–example: (abundance of central commodity)

$$h_2 + h_3 \geq 1 \tag{3.7}$$

and

$$\lambda_1 + \lambda_3 \leq 1 \ . \tag{3.8}$$

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Theorem 3.6. $\mathcal{C}(\boldsymbol{v})$ *is the unique vNM–Stable Set.*

(For $h_1 = 0, h_2 = h_3 = \frac{1}{2}, h_4 = 1, \ \lambda_1 = ..., \lambda_2 = ...$:
Einy, Holzman, Monderer, Shitovitz ****[1996]).

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The 2×2– example (rich central commodity):

$$h_2 + h_3 \geq 1 \qquad \textbf{and} \qquad \lambda_1 + \lambda_3 \geq 1 \ .$$

** In this case

$$\widehat{\boldsymbol{x}} := \left(0, (1-h_3)\frac{\lambda_3}{\lambda_2}, h_3, 1)\right) \quad = \quad \left(h_2, (1-h_3)\frac{\lambda_3}{\lambda_2}, h_3, 1)\right)$$

$(h_1 + h_3 < 1, h_1 + h_4 = 1)$.

Theorem 3.7 (***Existence Theorem***).

$$\mathcal{H} \ := \ \boldsymbol{ConvH}\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}}\}$$

is a vNM–Stable Set. (NOT unique !)

Definition 3.8.

ϑ vNM–extremal:

$$\dot{\boldsymbol{\vartheta}} \;=\; \widehat{\boldsymbol{x}} \text{ on } \boldsymbol{D}^1 \cup \boldsymbol{D}^3 \cup \boldsymbol{D}^4$$

and

$$(1-h_3) \leq \boldsymbol{\vartheta} \;\leq\; h_2 \text{ on } \boldsymbol{D}^2$$

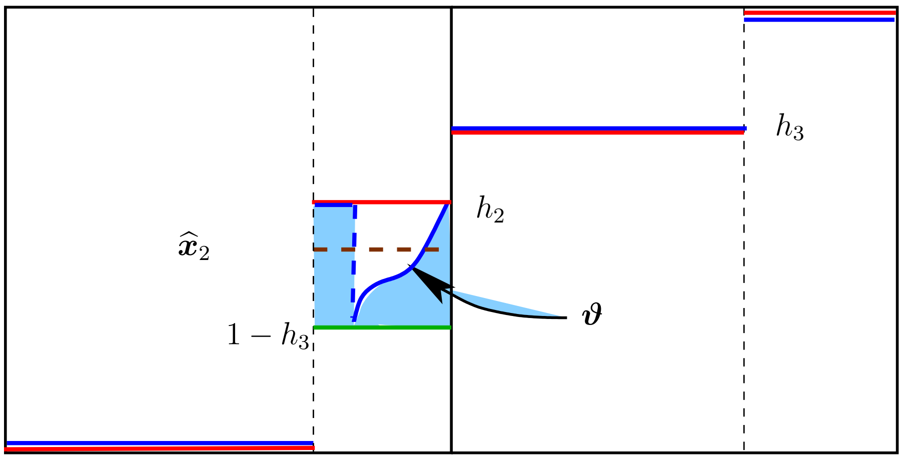

Figure 3.9: A vNM–extremal imputation

Theorem 3.9 (***Existence and Characterization***). *Let $\bar{\boldsymbol{\vartheta}}$ be a vNM–extremal imputation and let*

$$\mathcal{G} \;:=\; \boldsymbol{ConvH}\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \bar{\boldsymbol{\vartheta}}\}\ .$$

then $\mathcal{G}$ is a vNM–Stable Set.

Every convex *vNM–Stable Set is generated this way by a suitable vNM–extremal imputation $\bar{\boldsymbol{\vartheta}}$.*

Complete characterization !!