Regular Tournaments and their spectra

progress report

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Rehovot, December 2012

Abstract

Tournament graphs are directed graphs with an (asymmetric) adjacency matrix D which summarizes the result of regular round-robin tournaments between N players: Every player plays against all the others, if i wins against j then $D_{i,j} = 1$ and $D_{j,i} = 0$. Clearly $D_{i,i} = 0$. If N is odd then a tournament can be regular each player wins exactly half the times.

To construct regular tournaments we introduced a random walk in the space of the tournament adjacency matrices which will be argued to be ergodic.

The spectrum of D for regular tournaments consists of one point on the real axis, the rest are in the complex plane, all with real part $= 1/2$. The spectral statistics on the ”critical lime” will be studied using a trace formula for the spectral density. For large N it will be shown that the mean spectral density approaches the semi-circle law. Moreover, numerical simulations supported by theoretical arguments derived by using the trace formula, show that the spectral statistics is consistent with the predictions of the Gaussian Unitary Ensemble of random matrices.

Round Robin Tournaments:

A round-robin tournament (or all-play-all tournament) is a competition "in which each contestant meets all other contestants in turn".

The term *round-robin* is derived from the term *ruban*, meaning "ribbon". Over a long period of time, the term was corrupted and idiomized to *robin*.

Irregular Tournament

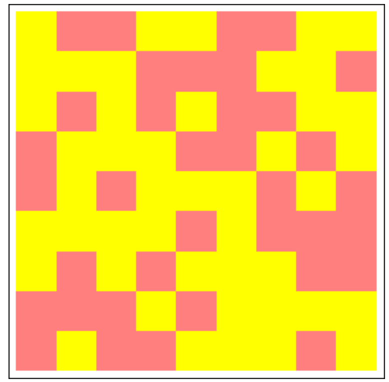
A kind of a Polo game, Afghanistan (Photo graph Courtesy N. A.)

Regular Tournament:

Round Robin tournament where each player wins (and loses) exactly $\frac{1}{2}$ of his/her games.

-> N : Number of players in a regular tournament is odd

$$D = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} =$$



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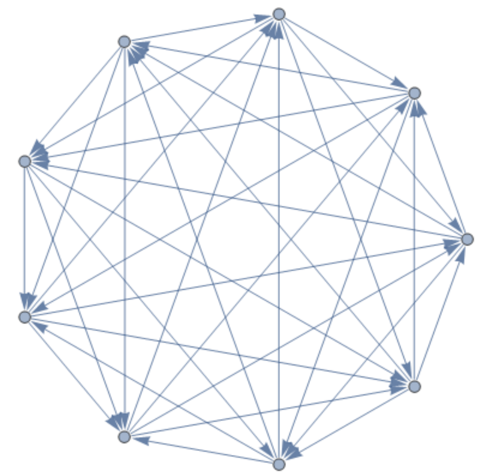


In a recent lecture by Bojan Mohar I learnt:

1. The spectrum of D consists of the trivial value ½(N-1) and N-1 complex values on the line $-\frac{1}{2} + i\,y$
2. The spectral density (properly scaled) approaches the semi-circle distribution for large N.

Being intrigued and with no clue of Bojan's (unpublished) results I decided to look at it myself – the more I studied the more intrigued I am. I'll discuss here my present state of ignorance and a few results I obtained along this way.

The set of regular tournaments of (odd) N players.

1. Equivalent classes – same tournament with a different enumeration of the players

2. The number of non-equivalent tournaments cannot be greater than $\frac{2^{N(N-1)/2}}{N!}$
this estimates misses the requirement that the tournament is regular.

3. B. McKay proved *As $n \to \infty$ with n odd,*

$$RT(n) = \frac{2^{(n^2-1)/2} e^{-1/2}}{\pi^{(n-1)/2} n^{n/2-1}} \left(1 + O(n^{-1/2+\epsilon})\right)$$

Random walk in the space of regular tournaments

Random walk :

Choose entries (i,j) , (i,k) , (k,j)

at random however check that:

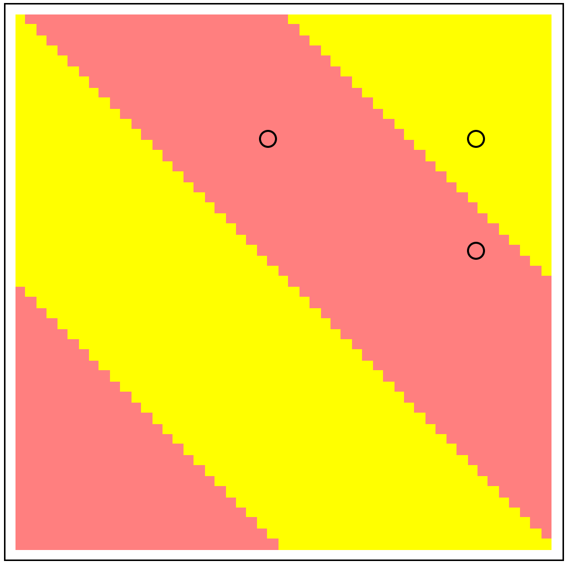
If $D_{i,j} = 0 \quad \to \quad D_{i,k} = D_{k,j} = 1$

If $D_{i,j} = 1 \quad \to \quad D_{i,k} = D_{k,j} = 0$

Then swap $1 \longleftrightarrow 0$ on all positions

(i,j) (j,i) (i,k) (k,i) (k,j) (j,k)

(hitting probability $=1/4$)

$D_0 =$ 

Generating random regular tournaments

Random walk :

Choose entries (i,j) , (i,k) , (k,j)

at random however check that:

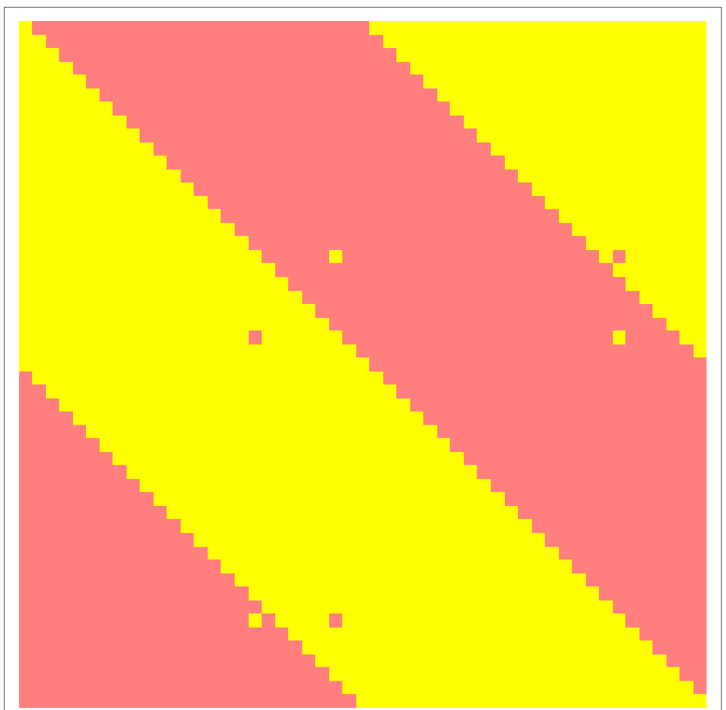
If $D_{i,j} = 0 \quad \rightarrow \quad D_{i,k} = D_{k,j} = 1$

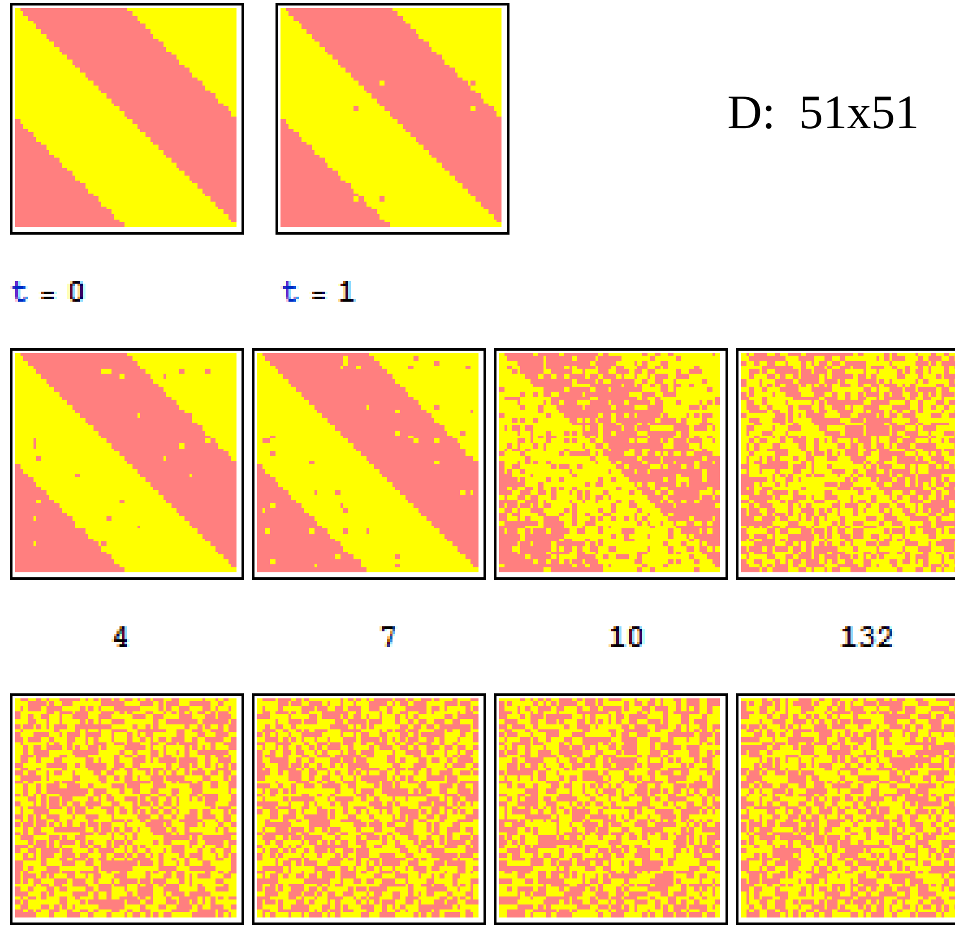
If $D_{i,j} = 1 \quad \rightarrow \quad D_{i,k} = D_{k,j} = 0$

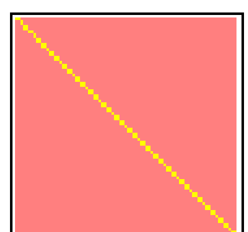
Then swap $1 \longleftrightarrow 0$ on all positions

(i,j) (j,i) (i,k) (k,i) $k,j)$ (j,k)

(hitting probability $=1/4$)

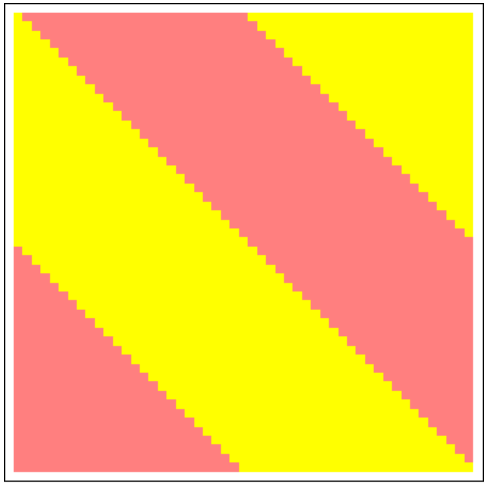


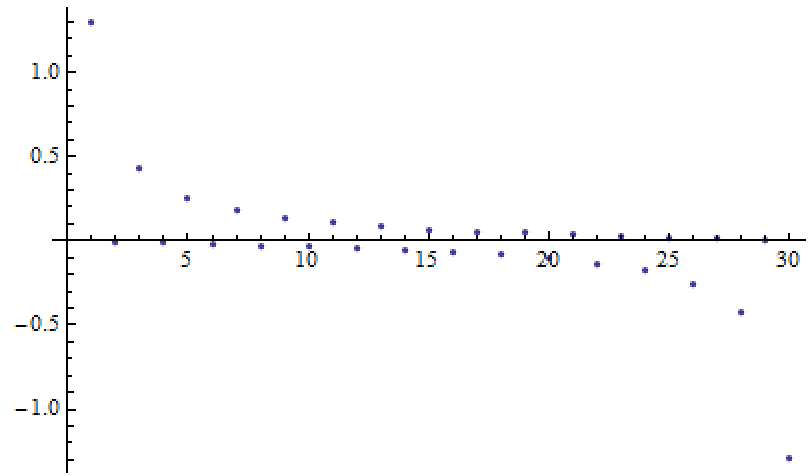




Check: $D + D^T$

Ergodicity of the scrambling mapping

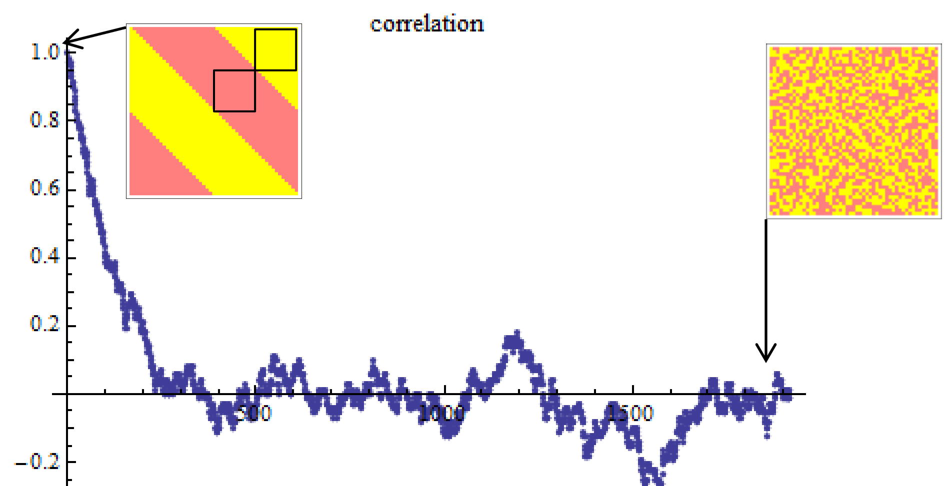
$D_0 =$ 



The scrambling transformation is a rank two (effective one) perturbation, of random sign.
The starting spectrum is simple, and therefore the successive matrices generated by the transformation are interlacing backwards and forwards depending on the sign. ?
Hence successive tournaments are not equivalent under permutations.

Correlation =

$$-\frac{\sum_{i,j=1}^{\lfloor N/4\rfloor} \tilde{D}_{q+i,q+j}\tilde{D}_{q+\lfloor N/4\rfloor+i,q+\lfloor N/4\rfloor+j}}{(N/4)^2}$$



The spectral properties of Tournament graphs

$E = |1\rangle\langle 1| - I \quad ; \quad \langle 1| = (1, 1, \cdots, 1)$

$D = \frac{1}{2i} A^{(M)} + \frac{1}{2} E = \frac{1}{2i} A^{(M)} + \frac{1}{2}(|1\rangle\langle 1| - I)$

where $A^{(M)} = 2i(D - \frac{1}{2}E)$ is a $\boxed{\text{Hermitian}}$ matrix with entries:

$A^{(M)}_{r,r} = 0 \quad ; \quad A^{(M)}_{r,s} = e^{i\phi_{r,s}} \quad \phi_{r,s} = -\phi_{s,r} = \frac{\pi}{2}$.

$\boxed{A^{(M)}\text{: "Magnetic" Laplacian on a complete graph on } N \text{ vertices } d = (N-1) .}$

From now on, discuss the spectrum of $A^{(M)}$: $\sigma(A^{(M)})$.

$\forall$ row (column) $\sharp(+i) = \sharp(-i) = \frac{N-1}{2} \rightarrow 0 \in \sigma(A^{(M)})$, with eigenvector $|1\rangle$.

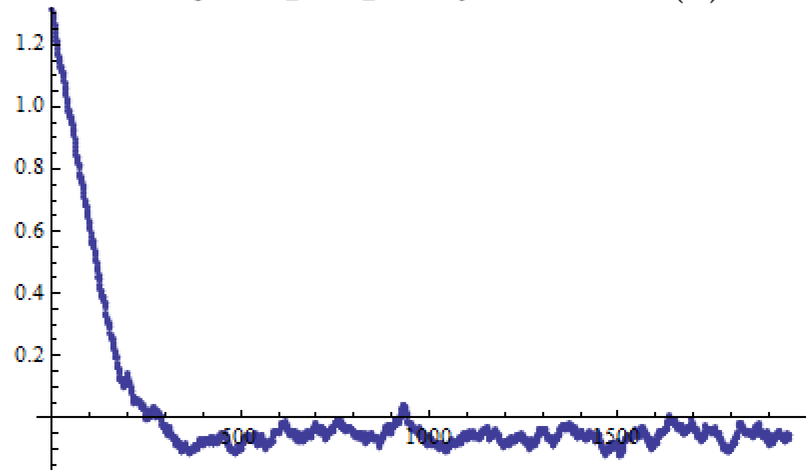
$\sigma(A^{(M)})/0$ is symmetric about 0 since $\{A^{(M)}\}^{\star} = -A^{(M)}$.

The spectrum of D consists of $\{\frac{1}{2}(N-1), -\frac{1}{2} + \frac{1}{2i}\mu_n\}$, $\mu_n \in \sigma(A^{(M)})/0$.

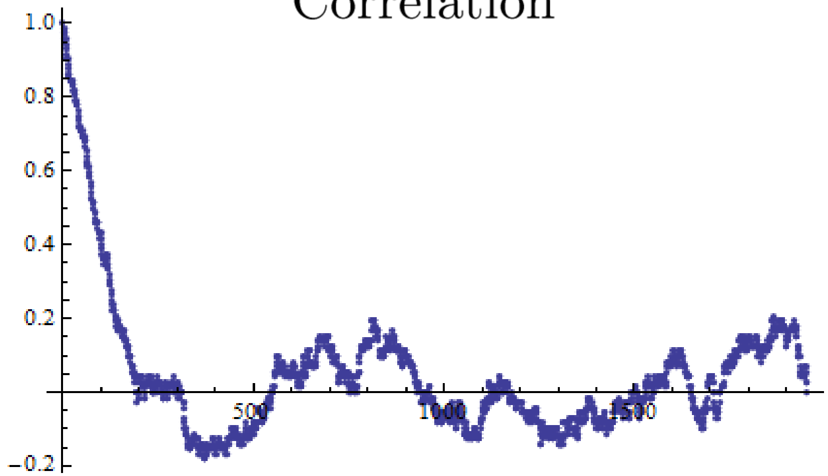
"Ramanujan" : $|\mu_n| \leq 2\sqrt{N-2}$. ?

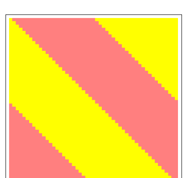
Scaled spectral parameters $\epsilon_n = \frac{\mu_n}{2\sqrt{N-2}}$.

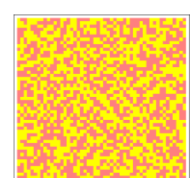
Ramanujan property : $Max(\epsilon) - 1$



Correlation







A trace formula for the "Magnetic Laplacian" spectrum

Reminder

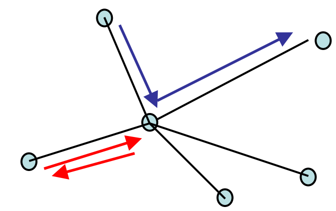
Directed - edge connectivity *vs.* vertex connectivity

$A_{i,j}$: Adjacency Matrix, $\dim A = V$; $B_{e',e} = \delta_{o(e'),\tau(e)}$, $\dim B = 2E$.

$J_{e',e} = \delta_{e',\hat{e}}$; $Y = B - J$: The Hashimoto connectivity matrix.

$$\sharp\{t\text{-periodic walks }\} = \mathrm{tr}B^t = \mathrm{tr}A^t.$$

$$\sharp\{t\text{-periodic non-backscattering walks }\} = \mathrm{tr}Y^t$$



Bass' (Bartholdi's) identity for d-regular graphs:

For any η arbitrary complex number.

$$\det(\eta I^{(2E)} - Y) = (\eta^2 - 1)^{E-V} \det(I^{(V)}(\eta^2 + (d-1)) - \eta A) \ .$$

$I^{(2E)}$ and $I^{(V)}$ unit matrix in $2E$ and V dimensions.

In the Magnetic case

$$B^{(M)}_{e',e} = \delta_{o(e'),\tau(e)} e^{\frac{i}{2}(\phi_e + \phi_{e'})}, \quad \text{and} \quad Y^{(M)}_{e',e} = B^{(M)} - J = Y_{e',e} e^{\frac{i}{2}(\phi_e + \phi_{e'})}$$

For regular tournaments the connectivities A and Y correspond to a complete graph

Bass' (Bartholdi's) identity for the "Magnetic Laplacian":

For any η arbitrary complex number.

$$\det(\eta I - Y^{(M)}) = (\eta^2-1)^{\frac{N(N-3)}{2}} \det(I^{(N)}(\eta^2+(N-2)) - \eta A^{(M)}) \ .$$

$$B^{(M)}_{e',e} = i\ \delta_{o(e'),\tau(e)}\mathrm{Sign}(e), \quad \text{and} \quad Y^{(M)}_{e',e} = B^{(M)} - J = i\ Y_{e',e}\boxed{\mathrm{Sign(e)}}$$

Denote the spectrum of $A^{(M)}$ by $\{\mu_k\}_{k=1}^{N}$

Denote the spectrum of $Y^{(M)}$ by $\{\eta_r\}_{r=1}^{N(N-1)/2}$

The η spectrum consists of

$$\sqrt{N-2}\ e^{\pm i \arccos \frac{\mu_k}{2\sqrt{(N-2)}}} \quad \text{for all } k$$

and the trivial ± 1 with multiplicity $N(N-3)/2$.

Thus:

$$\begin{aligned} y_t^{(M)} &= \frac{1}{N}\frac{\mathrm{tr}(Y^{(M)})^t}{(N-2)^{t/2}} \\ &= \frac{2}{N}\sum_{k=1}^{N} \cos t \arccos \frac{\mu_k}{2\sqrt{(N-2)}} + \frac{N(N-3)}{2N}\frac{1+(-1)^t}{(N-2)^{t/2}} \\ &= \frac{2}{N}\sum_{k=1}^{N} T_t(\frac{\mu_k}{2\sqrt{(N-2)}}) + \frac{N(N-3)}{2N}\frac{1+(-1)^t}{(N-2)^{t/2}} \end{aligned}$$

Making use of the Bartholdi identity, and $\epsilon = \frac{\mu}{\sqrt{2(N-2)}}$

the trace formula for the "Magnetic Laplacian" spectrum follows:

$$\rho(\epsilon) = \frac{1}{N} \sum_{\epsilon_j \in \mathcal{R}_{\mathcal{M}}} \delta(\epsilon - \epsilon_j) =$$

$$= \frac{2}{\pi}\sqrt{1-\epsilon^2}(1+\mathcal{O}(\frac{1}{N})) + \frac{1}{\pi}\frac{1}{\sqrt{1-\epsilon^2}} \sum_{t=3}^{\infty} y_t^{(M)}\ T_t(\epsilon)\ .$$

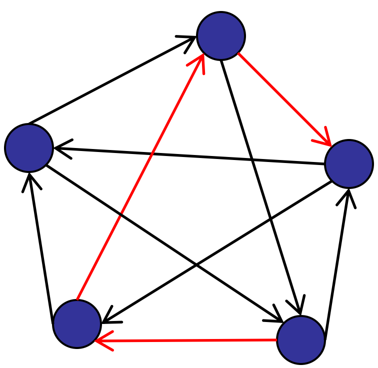
Where : $T_t(x) = \cos(t \arccos x) \quad x \in [-1,1] \ ; \ y_t^{(M)} = \frac{1}{N} \frac{\mathrm{tr}(Y^{(M)})^t}{(N-2)^{\frac{t}{2}}}$

$\mathrm{tr}(Y^{(M)})^t = \sum_{w \in \mathcal{W}_t} (-1)^{k_t(w)} \ ; \ \mathrm{tr} Y^t = \sum_{w \in \mathcal{W}_t} \approx (N-2)^t$

$\mathcal{W}_t$ - The set of t-periodic non back-tracking walks on the magnetic graph

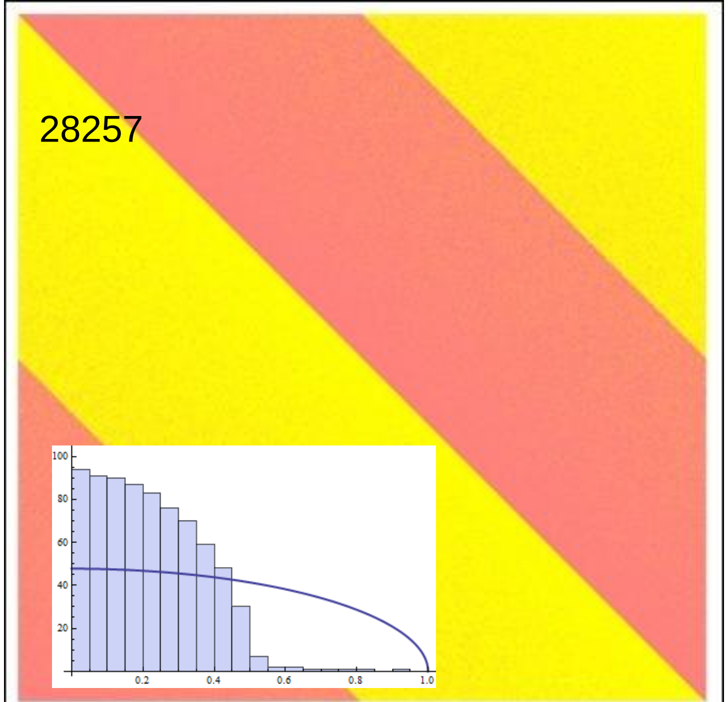
$k_t(w)$ - the number of negative bonds on the t - periodic walk.

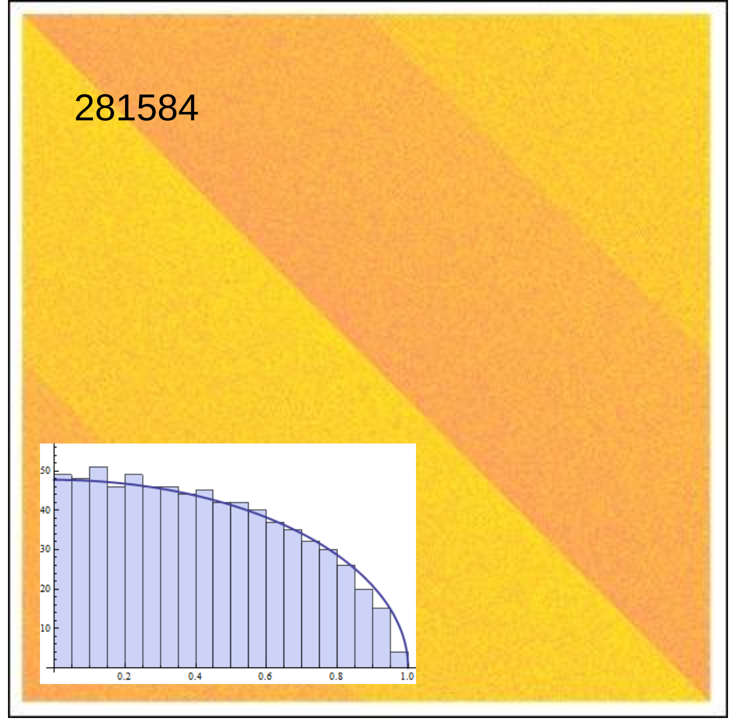
$$\boxed{y_t^{(M)} = \frac{1}{N} \frac{\sum_{w \in \mathcal{W}_t} (-1)^{k_t(w)}}{\sqrt{|\mathcal{W}_t|}}}$$



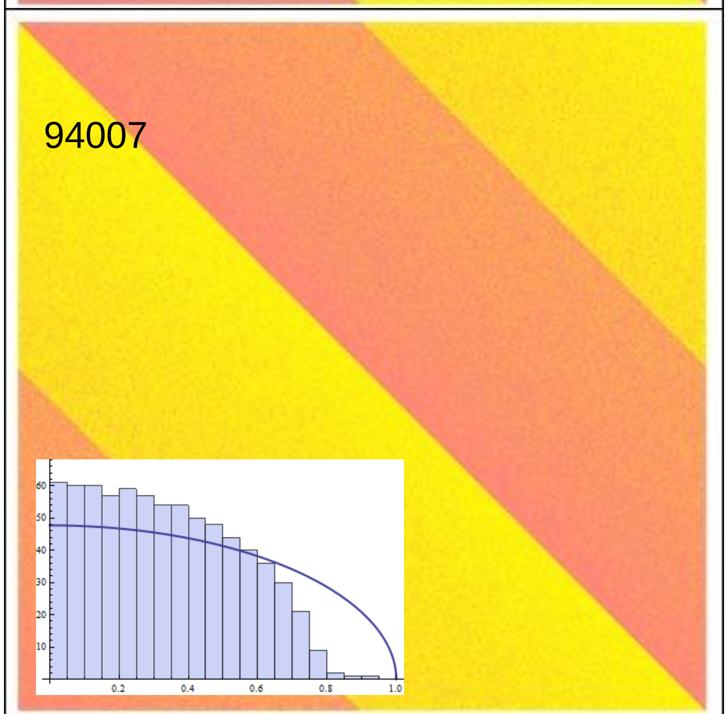
Note: For **odd** t, $y_t^{(M)} = 0$,

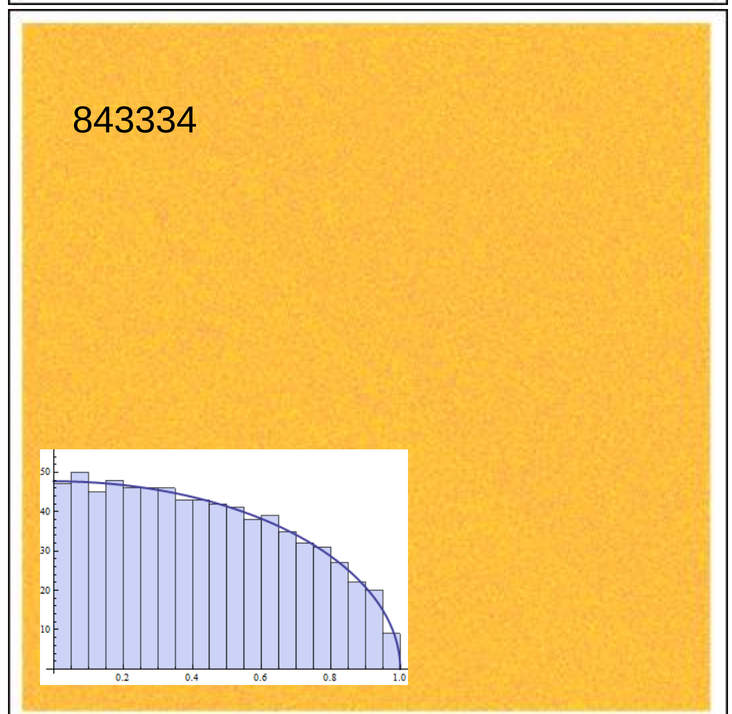
consistent with the reflection symmetry of the spectrum.





nmax = 1501
# of attempted
hexagons 1125750



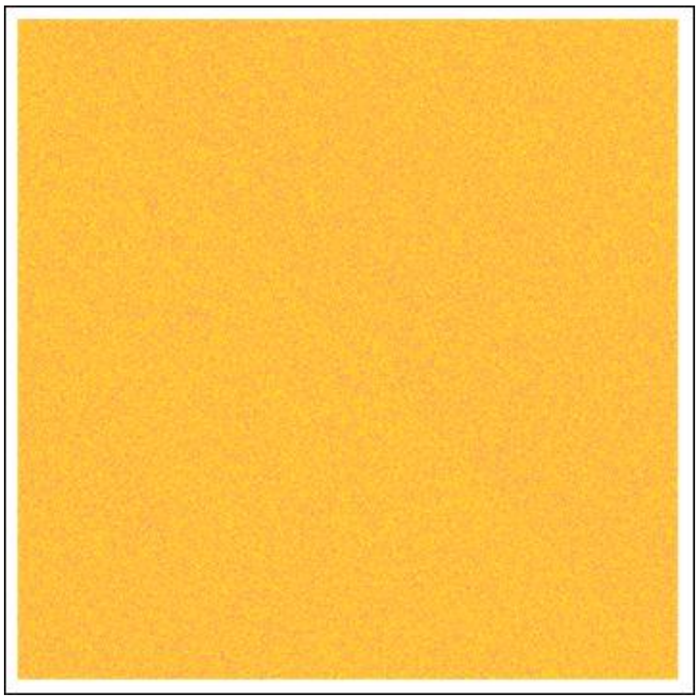


Spectral statistics

Unfolding the spectrum (positive half only)

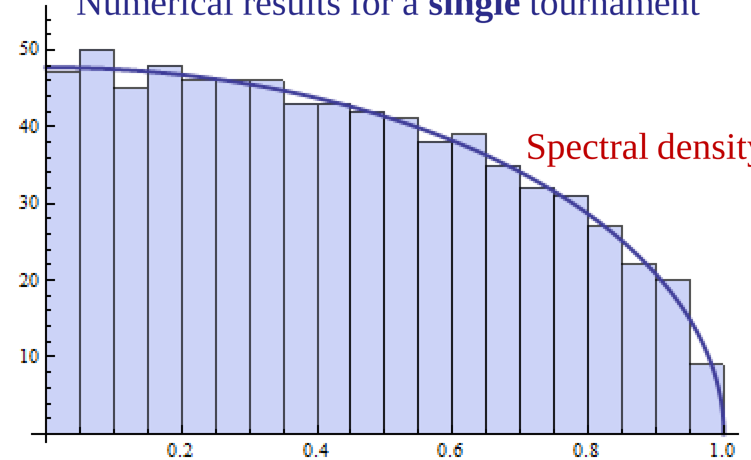
$$\mathcal{N}(\epsilon) = \frac{N}{\pi}(\arcsin\epsilon + \epsilon\sqrt{1-\epsilon^2})$$

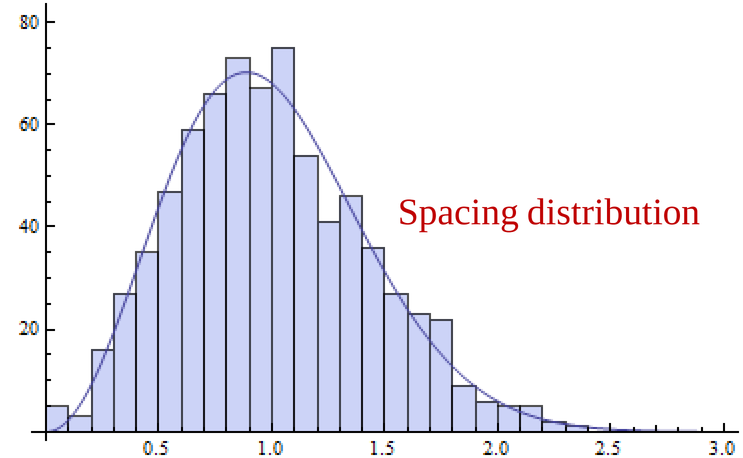
$$\theta_j = 2\pi\frac{\mathcal{N}(\epsilon_j)}{N}$$ Uniformly distributed on the unit circle

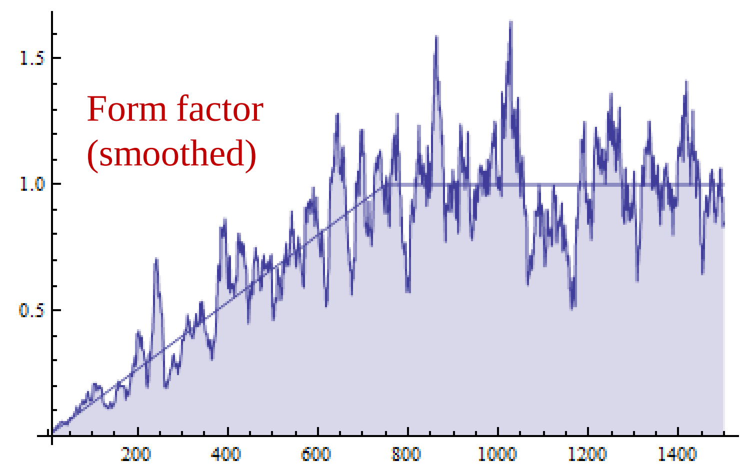


nmax = 1500
Spectral points in the support 749
Scrambling 843334 iterations

Numerical results for a **single** tournament







Random Matrix Theory in a nut shell.

I. The Gaussian Orthogonal Ensemble (GOE)

The set of $N \times N$ **symmetric real** normally distributed random matrices H

$$P_{GOE}(H)\mathrm{d}H = C_N \exp(-\mathrm{tr}H^2) \prod_{i \geq j} \mathrm{d}H_{i,j}$$

II. The Gaussian Unitary Ensemble (GUE)

The set of $N \times N$ **Hermitian complex** normally distributed random matrices H

$$P_{GUE}(H)\mathrm{d}H = C_N \exp(-\mathrm{tr}HH^\dagger) \prod_{i \geq j} \mathrm{d}H_{i,j}$$

Eigenvalues distribution:

The spectral density: $\rho(\lambda) = \frac{1}{N} \sum_{j=1}^{N} \delta(\lambda - \lambda_j)$.

The Wigner Semi-circle law: For $N \to \infty \quad \langle \rho(\lambda) \rangle_{GOE,GUE} \to \frac{1}{2\pi}\sqrt{4N - \lambda^2}$

Example: Nearest neigbour spectral distribution

$$s_n = \frac{\lambda_n - \lambda_{n-1}}{\text{mean spacing}} \quad ; \quad P(s) = \frac{1}{\Delta K} \sum_{k=K}^{K+\Delta K} \delta(s - s_k) \quad ; \quad N > \Delta K \gg 1$$

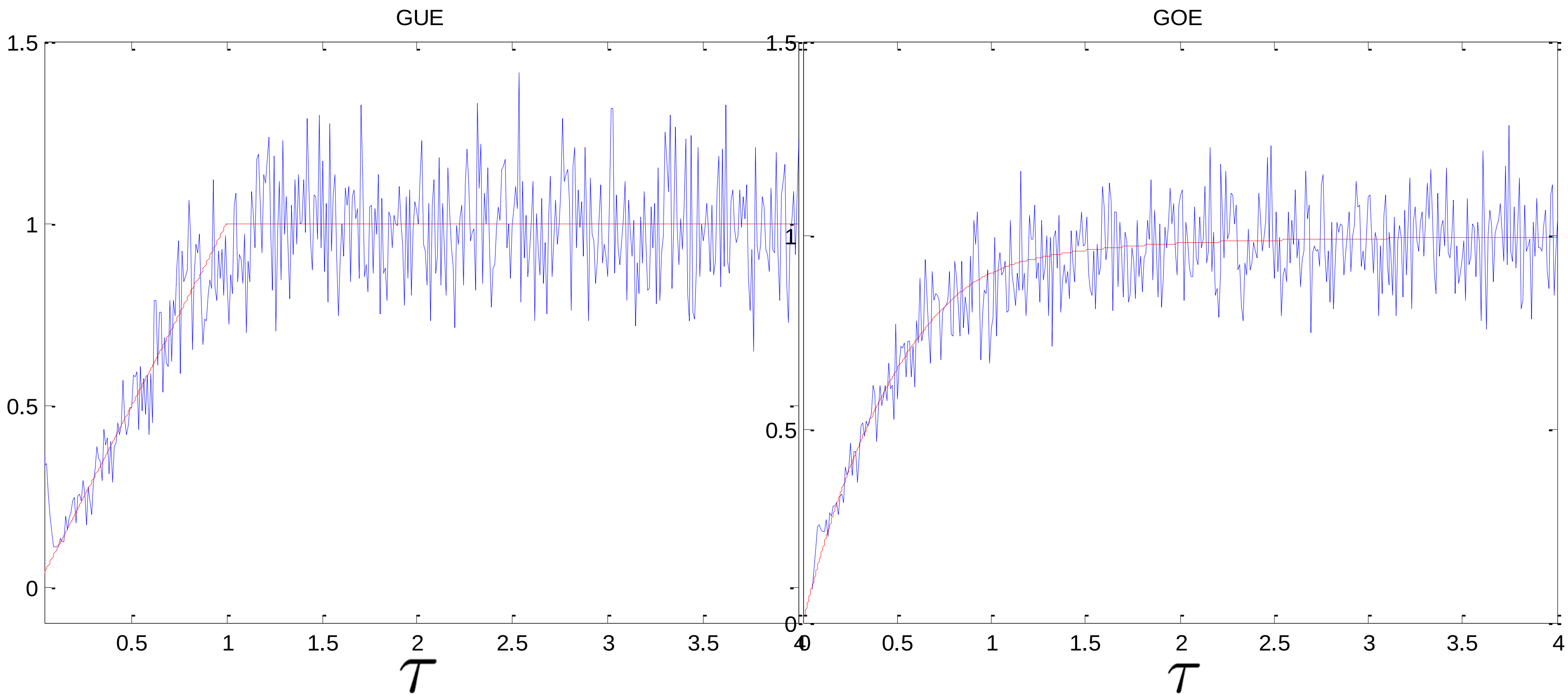
$$P_{GOE}(s) = \frac{\pi s}{2} \exp(-\frac{\pi s^2}{4}) \quad ; \quad P_{GUE}(s) = \frac{32 s^2}{\pi^2} \exp(-\frac{4s^2}{\pi})$$

Spectral 2-points correlations:

The Circular ensembles (COE,CUE):

$$\theta_j = 2\pi \frac{\mathcal{N}_{MK}(\lambda_j)}{V-1}$$ (mapping the spectrum on the unit circle)

$$K(\tau) = \frac{1}{V-1} \left\langle \sum_{i,j=1}^{V-1} \cos(\theta_i - \theta_j)t \right\rangle , \quad \text{and} \quad t = (V-1)\tau$$ (scaling property)



Two versions of the spectral form-factor.

$$\phi_j = \arccos \epsilon_j \quad ; \quad \langle \rho(\phi) \rangle = \tfrac{2}{\pi} \sin^2 \phi$$

$$\widetilde{K}(t;N) = \tfrac{1}{N} \left\langle \left| \sum_{j=1}^{N} e^{it\phi_j} \right|^2 \right\rangle$$

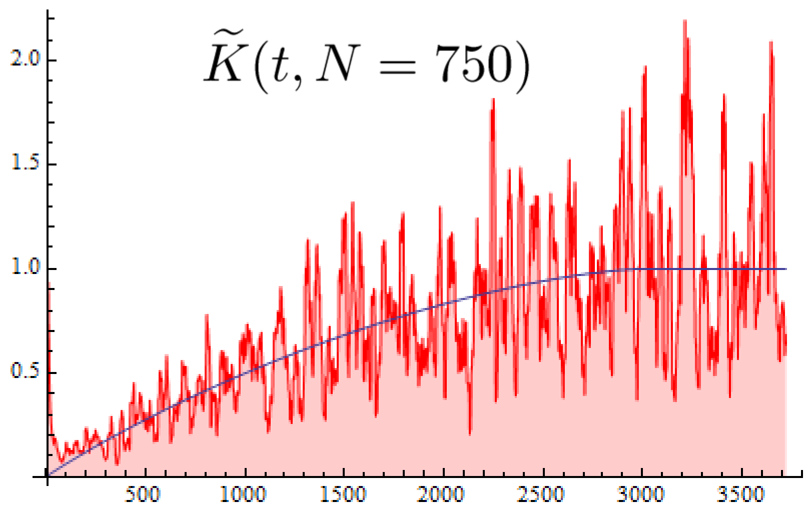
$$\theta_j = 2\pi \frac{\mathcal{N}(\epsilon_j)}{N} \quad ; \quad \langle \rho(\theta) \rangle = \tfrac{1}{2\pi}$$

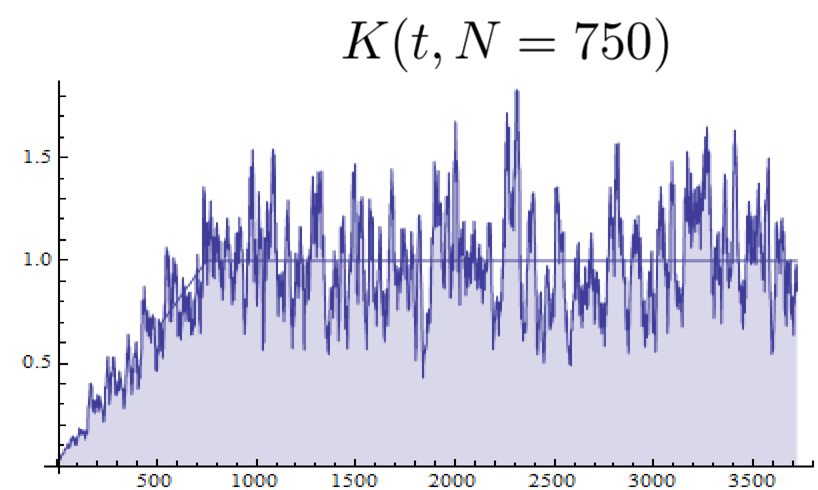
$$K(t;N) = \tfrac{1}{N} \left\langle \left| \sum_{j=1}^{N} e^{it\theta_j} \right|^2 \right\rangle$$

For large N

$$\widetilde{K}(t;N) = 2 \int_0^{\frac{\pi}{2}} \langle \rho(\phi) \rangle K\left(\frac{\tau}{2\pi \langle \rho(\phi) \rangle} \right) \mathrm{d}\phi \quad ; \quad \tau = \frac{t}{N} .$$

$$\begin{aligned} \widetilde{K}_{GUE}(\tau) &= \frac{\tau}{2} - \frac{1}{2\pi}\sqrt{\tau(4-\tau)} + \frac{1}{\pi}(2-\tau) \arccos \frac{\sqrt{\tau}}{2} \quad \text{for} \quad \tau < 4 \\ &\approx \frac{\tau}{2} - \frac{\tau^{3/2}}{3\pi} - \frac{\tau^{5/2}}{120\pi} + \mathcal{O}(\tau^{7/2}) . \\ &= 1 \quad \text{for} \quad \tau > 4 \end{aligned}$$





The fluctuating part of the spectral density :

$$\tilde{\rho}(\epsilon) = \rho(\epsilon) - \frac{2}{\pi}\sqrt{1-\epsilon^2} = \frac{1}{\pi}\frac{1}{\sqrt{1-\epsilon^2}}\sum_{t=3}^{\infty} T_t(\epsilon) y_t^{(M)}$$

Using the Orthogonality of the Chebyshev Polynomials:

$$y_t^{(M)} = 2\int_{-1}^{1} \mathrm{d}\epsilon \; \tilde{\rho}(\epsilon) \; T_t(\epsilon)$$

$$\langle (y_t^{(M)})^2 \rangle = 4\int_{-1}^{1}\int_{-1}^{1} T_t(\epsilon)T_t(\epsilon') \;\; \langle \tilde{\rho}(\epsilon)\tilde{\rho}(\epsilon')\rangle \;\; d\epsilon d\epsilon'$$

Map the spectrum to the unit circle: $\phi = \arccos\epsilon$, $\phi \in [0,\pi]$

♡ $$\langle (y_t^{(M)})^2 \rangle = 4\int_{0}^{\pi}\int_{0}^{\pi} \cos t\phi \cos t\psi \; \boxed{\langle \tilde{\rho}(\phi)\tilde{\rho}(\psi)\rangle} \; d\phi d\psi$$

However : $\widetilde{K}_(t;N) \equiv \frac{2}{N}\left\langle \left(\sum_{k=1}^{N} \cos(t\phi_k)\right)^2 \right\rangle$

Therefore

$$\widetilde{K}(t;N) \;\; = \;\; \frac{N}{2}\langle (y_t^{(M)})^2 \rangle = \frac{N}{2}\left\langle \left(\frac{1}{N}\frac{\sum_{w\in\mathcal{W}_t} (-1)^{k_t(p)}}{\sqrt{|\mathcal{W}_t|}} \right)^2 \right\rangle \;\; =$$

$$\widetilde{K}(t;N) = \frac{N}{2}\langle (y_t^{(M)})^2\rangle = \frac{N}{2}\left\langle \left(\frac{1}{N} \frac{\sum_{w\in\mathcal{W}_t}(-1)^{k_t(p)}}{\sqrt{|\mathcal{W}_t|}} \right)^2 \right\rangle$$

$$\approx \frac{N}{2}\frac{1}{N^2}\frac{\left\langle \left(t\sum_{p\in\mathcal{P}_t}(-1)^{k_t(p)} \right)^2 \right\rangle}{t|\mathcal{P}_t|} \approx \frac{\tau}{2}$$

$\mathcal{P}_t$: The set of non back tracking t-periodic orbits.

The remaining terms in the Taylor expansion of the form-factor remain to be computed. This is a non-trivial problem in combinatorics.

"Diagonal Approximation":
Assuming signs on periodic orbits are uncorrelated, and remembering that walks are of length t.

Assuming that the spectral fluctuations are of the GUE type,

$$\frac{\left\langle \left(\sum_{p\in\mathcal{P}_t}(-1)^{k_t(p)} \right)^2 \right\rangle}{|\mathcal{P}_t|} = \frac{2}{\tau}\widetilde{K}(\tau) \to \frac{2}{\tau} \quad \text{for} \quad N,t\to\infty,\ \frac{t}{N} = \tau > 4\ .$$

Open questions and future work

Prove uniform coverage of tournament space by the random walk.

“Ramanujan” property for the Magnetic Laplacian spectrum

GUE spectral statistics vis-à-vis periodic orbits statistics, beyond the diagonal approximation

Nodal counts

Irregular tournaments and almost regular tournaments (even N)

Thank you for your attention