Geometric optimization of eigenvalues of the Laplace operator

Alexei V. Penskoi

Moscow State University

Independent University of Moscow

Workshop “Applications of analysis:
game theory, spectral theory and beyond”
in honor of Yakar Kannai’s 70th birthday
Weizmann Institute of Science — December 25–27, 2012

Outline

Laplace operator

- Laplace operator in $\mathbb{R}^n$

$$\Delta f = -\frac{\partial^2 f}{\partial x_1^2} - \cdots - \frac{\partial^2 f}{\partial x_n^2}$$

Spectral problems for the Laplace operator

- Let Ω be a domain in $\mathbb{R}^n$
- Dirichlet spectral problem

$$\begin{cases} \Delta f = \lambda f \\ f|_{\partial\Omega} = 0 \end{cases}$$

- The spectrum consists only of eigenvalues

$$0 \leqslant \lambda_1(\Omega, D) \leqslant \lambda_2(\Omega, D) \leqslant \dots$$

Spectral problems for the Laplace operator

- Neumann spectral problem

$$\begin{cases} \Delta f = \lambda f \\ \frac{\partial f}{\partial \vec{n}}|_{\partial\Omega} = 0 \end{cases}$$

- For nice domains Ω the spectrum consists only of eigenvalues

$$0 = \lambda_1(\Omega, N) < \lambda_2(\Omega, N) \leqslant \lambda_3(\Omega, N) \leqslant \dots$$

Standing waves

- Wave equation

$$\frac{\partial^2}{\partial t^2} v(x,t) + \Delta v(x,t) = 0, x = (x^1, \ldots, x^n) \in \Omega \subset \mathbb{R}^n$$

- Separation of variables

$$v(x,t) = u(x)T(t) \quad \Longrightarrow \quad u(x)T''(t) + T(t)\Delta u(x) = 0$$

$$-\frac{T''(t)}{T(t)} = \frac{\Delta u(x)}{u(x)}$$

- Spectral problem for Δ

$$\Delta u(x) = \lambda u(x)$$

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Boundary conditions

- Fixed boundary $\Rightarrow$ Dirichlet spectral problem

$$\begin{cases} \Delta u = \lambda u \\ u|_{\partial\Omega} = 0 \end{cases}$$

- Free boundary $\Rightarrow$ Neumann spectral problem

$$\begin{cases} \Delta u = \lambda u \\ \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = 0 \end{cases}$$

Example: dim $=1$ (string)

- $\Omega = [0, l] \subset \mathbb{R}$
- Fixed endpoints $\Rightarrow$ Dirichlet spectral problem

$$\begin{cases} -u'' = \lambda u \\ u(0) = u(l) = 0 \end{cases}$$

- Free endpoints $\Rightarrow$ Neumann spectral problem

$$\begin{cases} -u'' = \lambda u \\ u'(0) = u'(l) = 0 \end{cases}$$

Example: dim $= 1$ (string)

- $-u'' = \lambda u \Longrightarrow u(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$
- Dirichlet spectral problem

$$u(0) = 0 \Longleftrightarrow B = 0$$

$$u(l) = 0 \Longleftrightarrow \sin(\sqrt{\lambda}l) = 0 \Longleftrightarrow \sqrt{\lambda}l = \pi n, \quad n \in \mathbb{Z}$$

- Eigenvalues $\lambda_n = \left(\frac{\pi n}{l}\right)^2$, where $n = 1, 2, 3, \ldots$
- Eigenfunctions $u_n = \sin(\frac{\pi n}{l}x)$

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- Neumann spectral problem

$$u(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

$$u'(x) = A\sqrt{\lambda}\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

$$u'(0) = 0 \iff A = 0$$

$$u'(l) = 0 \iff \sin(\sqrt{\lambda}l) = 0 \iff \sqrt{\lambda}l = \pi n, \quad n \in \mathbb{Z}$$

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- Eigenfunctions $u_n = \cos(\frac{\pi n}{l}x)$

Example: dim $= 2$, rectangular membrane

- Rectangular membrane $[0, a] \times [0, b] \subset \mathbb{R}^2$

$$-\frac{\partial^2}{\partial x^2} u(x, y) - \frac{\partial^2}{\partial y^2} u(x, y) = \lambda u(x, y)$$

- Separation of variables

$$u(x, y) = X(x)Y(y)$$

$$-X''(x)Y(y) - X(x)Y''(y) = \lambda X(x)Y(y)$$

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda = \mu$$

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Example: dim $= 2$, rectangular membrane

- 2D Dirichlet spectral problem $\Longrightarrow$ 1D Dirichlet spectral problems

$$-X''(x) = \mu X(x), \quad X(0) = X(a) = 0$$

$$\mu_m = \left(\frac{\pi m}{a}\right)^2, \quad m = 1, 2, 3 \ldots$$

$$-Y''(y) = (\lambda - \mu_m) Y(y), \quad Y(0) = Y(b) = 0$$

$$\lambda - \mu_m = \left(\frac{\pi n}{b}\right)^2, \quad n = 1, 2, 3 \ldots$$

- Eigenvalues

$$\lambda_{m,n} = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2, \quad m, n = 1, 2, 3, \ldots$$

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Example: dim $= 2$, rectangular membrane

- In particular, if $\Omega = [0, \pi] \times [0, \pi]$ then Dirichlet spectrum is given by formulas

$$\lambda_{m,n} = m^2 + n^2, \quad m, n = 1, 2, 3, \ldots$$

Geometric optimization of eigenvalues

- Eigenvalues are functionals on the "set of domains"

$$\Omega \longmapsto \lambda_i(\Omega, D)$$

$$\Omega \longmapsto \lambda_i(\Omega, N)$$

- Naïve question: can we find

$$\min_{\Omega} \lambda_i(\Omega, D), \quad \max_{\Omega} \lambda_i(\Omega, D)?$$

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Example: rectangular membranes

- Let us consider now only λ_1 and $\Omega = [0, a] \times [0, b]$
- Then
$$\lambda_1 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$
- Naïve question
$$\min_{a,b} \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2, \quad \max_{a,b} \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2?$$
- Boring answer: no min or max, but $\inf = 0$, $\sup = +\infty$

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Rescaling

- What happens if $a \longmapsto ka$, $b \longmapsto kb$?
- Then

$$\lambda_1(ka, kb) = \left(\frac{\pi}{ka}\right)^2 + \left(\frac{\pi}{kb}\right)^2 =$$

$$= \frac{1}{k^2}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right] = \frac{1}{k^2}\lambda_1(a, b)$$

- One should fix the area! Let Area $= 1 \iff b = \frac{1}{a}$, then

$$\lambda_1(a) = \left(\frac{\pi}{a}\right)^2 + (\pi a)^2$$

- If Area $= 1$ then

$$\min_a \lambda_1(a) = \lambda_1(1) = 2\pi^2, \quad \sup_a \lambda_1(a) = +\infty$$

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Example: rectangular membranes

- $\min_a \lambda_1(a) = \lambda_1(1)$ means that *a drumhead of square shape produces the lowest possible sound among all rectangular drumheads of given area*

Right question

- Find
$$\inf_{\substack{\Omega\subset\mathbb{R}^n \\ \mathrm{Vol}(\Omega)=c}} \lambda_i(\Omega, D)$$
- In the $i = 1$ 2D case this means *"A drumhead of which shape produces the lowest possible sound among all drumheads of given area?"*

Rayleigh-Faber-Krahn theorem

- If $i = 1$ then the minimum is reached on a ball of given volume, i.e.

$$\min_{\substack{\Omega \subset \mathbb{R}^n \\ \mathrm{Vol}(\Omega) = c}} \lambda_1(\Omega, D) = \lambda_1(B, D),$$

  where B is the ball of volume c in $\mathbb{R}^n$.
- This means that the optimal drumhead form is the disc.

Krahn-Szegö theorem

- If $i = 2$ then the minimum is reached on the union of two identical balls, i.e.

$$\min_{\substack{\Omega\subset\mathbb{R}^n \\ \mathrm{Vol}(\Omega)=c}} \lambda_2(\Omega, D) = \lambda_2(B \sqcup B, D),$$

where $B \sqcup B$ is the union of two identical balls in $\mathbb{R}^n$ such that $\mathrm{Vol}(B \sqcup B) = c$.

What about $i \geqslant 3$?

- We do not know the answer even in the case of planar domains

No	Optimal union of discs	Computed shapes
3	46.125	46.125
4	64.293	64.293
5	82.462	78.47
6	92.250	88.96
7	110.42	107.47
8	127.88	119.9
9	138.37	133.52
10	154.62	143.45

Laplace-Beltrami operator on manifolds

- Laplace-Beltrami operator on a Riemannian manifold

$$\Delta f = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\right),$$

where g_{ij} is the metric tensor, g^{ij} are the component of the matrix inverse to g_{ij} and $g = \det g$.

Spectral problem for the Laplace-Beltrami operator

- Spectral problem for the Laplace-Beltrami operator on a Riemannian manifold M without boundary

$$\Delta f = \lambda f$$

- The spectrum consists only of eigenvalues

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leqslant \lambda_2(M, g) \leqslant \dots$$

Geometric optimization of eigenvalues

- Let us fix M. Then $\lambda_i(M,g)$ is a functional on the space of Riemannian metrics on M

$$g \longmapsto \lambda_i(M,g)$$

- What is a natural optimization problem?
- Find

$$\sup_g \lambda_i(M,g),$$

where g belongs to the the space of Riemannian metrics on M such that $\mathrm{Vol}(M) = 1$
- This is a good question only for surfaces

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Upper bounds

- In 1980 it was proven by Yang and Yau that for an orientable surface M of genus γ the following inequality holds,
$$\lambda_1(M,g) \leqslant 8\pi(\gamma+1).$$
- A generalization of this result for an arbitrary λ_i was found in 1993 by Korevaar. He proved that there exists a constant C such that for any $i > 0$ and any compact surface M of genus γ the functional $\lambda_i(M,g)$ is bounded,
$$\lambda_i(M,g) \leqslant C(\gamma+1)i.$$

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Beware

- It should be remarked that in 1994 Colbois and Dodziuk proved that for a manifold M of dimension $\dim M \geqslant 3$ the functional $\lambda_i(M, g)$ is not bounded on the space of Riemannian metrics g on M.

Eigenvalues as functions of a metric

- The functional $\lambda_i(M, g)$ depends continuously on the metric g, but this functional is not differentiable.
- However, it was shown in 1973 by Berger that for analytic deformations g_t the left and right derivatives of the functional $\lambda_i(M, g_t)$ with respect to t exist.

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Eigenvalues as functions of a metric

- This led to the following definition by Nadirashvili (1986) and by El Soufi and Ilias (2000).
- **Definition.** A Riemannian metric g on a closed surface M is called *extremal metric* for the functional $\lambda_i(M,g)$ if for any analytic deformation g_t such that $g_0 = g$ the following inequality holds,

$$\frac{d}{dt}\lambda_i(M,g_t)\,\Big|_{t=0+} \leqslant 0 \leqslant \frac{d}{dt}\lambda_i(M,g_t)\,\Big|_{t=0-}.$$

What can we say about particular surfaces?

- The list of surfaces M and values of index i such that the maximal or at least extremal metrics for the functional $\lambda_i(M, g)$ are known is quite short.

What can we say about particular surfaces?

- $\lambda_1(\mathbb{S}^2, g)$. Hersch proved in 1970 that $\sup \lambda_1(\mathbb{S}^2, g) = 8\pi$ and the maximum is reached on the canonical metric on $\mathbb{S}^2$. This metric is the unique extremal metric.
- $\lambda_1(\mathbb{R}P^2, g)$. Li and Yau proved in 1982 that $\sup \lambda_1(\mathbb{R}P^2, g) = 12\pi$ and the maximum is reached on the canonical metric on $\mathbb{R}P^2$. This metric is the unique extremal metric.
- $\lambda_1(\mathbb{T}^2, g)$. Nadirashvili proved in 1996 that $\sup \lambda_1(\mathbb{T}^2, g) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is reached on the flat equilateral torus. El Soufi and Ilias proved in 2000 that the only extremal metric for $\lambda_1(\mathbb{T}^2, g)$ different from the maximal one is the metric on the Clifford torus.

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What can we say about particular surfaces?

- $\lambda_1(\mathbb{K}, g)$. Jakobson, Nadirashvili and I. Polterovich proved in 2006 that the metric on a Klein bottle realized as the Lawson bipolar surface $\tilde{\tau}_{3,1}$ is extremal. El Soufi, Giacomini and Jazar proved in the same year that this metric is the unique extremal metric and the maximal one. Here $\sup \lambda_1(\mathbb{K}, g) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where E is a complete elliptic integral of the second kind,

$$E(k) = \int_0^1 \frac{\sqrt{1 - k^2\alpha^2}}{\sqrt{1 - \alpha^2}} d\alpha.$$

What can we say about particular surfaces?

- $\lambda_2(\mathbb{S}^2, g)$. Nadirashvili proved in 2002 that $\sup \lambda_2(\mathbb{S}^2, g) = 16\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of two spheres of equal radius with canonical metric glued together. The proof contained some gaps filled later by Petrides (2012).

What can we say about particular surfaces?

- $\lambda_i(\mathbb{T}^2, g), \lambda_i(\mathbb{K}, g)$. Let $r, k \in \mathbb{N}$, $0 < k < r$, $(r, k) = 1$. Lapointe studied bipolar surfaces $\tilde{\tau}_{r,k}$ of Lawson τ-surfaces $\tau_{r,k}$ and proved the following result published in 2008.
  - If $rk \equiv 0 \mod 2$ then $\tilde{\tau}_{r,k}$ is a torus and it carries an extremal metric for $\lambda_{4r-2}(\mathbb{T}^2, g)$.
  - If $rk \equiv 1 \mod 4$ then $\tilde{\tau}_{r,k}$ is a torus and it carries an extremal metric for $\lambda_{2r-2}(\mathbb{T}^2, g)$.
  - If $rk \equiv 3 \mod 4$ then $\tilde{\tau}_{r,k}$ is a Klein bottle and it carries an extremal metric for $\lambda_{r-2}(\mathbb{K}, g)$.

What can we say about particular surfaces?

- We should also mention the paper in 2005 by Jakobson, Levitin, Nadirashvili, Nigam and I. Polterovich. It is shown in this paper using a combination of analytic and numerical tools that the maximal metric for the first eigenvalue on the surface of genus two is the metric on the Bolza surface $\mathcal{P}$ induced from the canonical metric on the sphere using the standard covering $\mathcal{P} \longrightarrow \mathbb{S}^2$. In fact, the authors state this result as a conjecture, because a part of the argument is based on a numerical calculation.

Lawson τ-surfaces

A.P., *Extremal spectral properties of Lawson tau-surfaces and the Lamé equation*, Moscow Math. J. 12 (2012), 173-192. Preprint `arXiv:math/1009.0285`

- 1. Let $\tau_{m,k}$ be a Lawson torus. We can assume that $m, k \equiv 1 \mod 2$, $(m, k) = 1$. Then the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\lambda_j(\mathbb{T}^2, g)$, where where

$$j = 2\left[\frac{\sqrt{m^2+k^2}}{2}\right] + m + k - 1.$$

The corresponding value of the functional is

$$\lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

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- 2. Let $\tau_{m,k}$ be a Lawson Klein bottle. We can assume that $m \equiv 0 \mod 2$, $k \equiv 1 \mod 2$, $(m,k) = 1$. Then the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\lambda_j(\mathbb{K}, g)$, where

$$j = 2\left[\frac{\sqrt{m^2+k^2}}{2}\right] + m + k - 1.$$

The corresponding value of the functional is

$$\lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

Otsuki tori

A.P., *Extremal spectral properties of Otsuki tori*, to appear in Mathematische Nachrichten, Preprint `arXiv:math/1108.5160`

The metric on an Otsuki torus $O_{\frac{p}{q}} \subset \mathbb{S}^3$ is extremal for the functional $\lambda_{2p-1}(\mathbb{T}^2, g)$.

Bipolar Otsuki tori

M.Karpukhin, *Spectral properties of bipolar surfaces to Otsuki tori*, Preprint `arXiv:math/1205.6316`

The metric on an bipolar Otsuki torus $\tilde{O}_{\frac{p}{q}} \subset \mathbb{S}^4$ is extremal for the functional $\lambda_{2q+4p-2}(\mathbb{T}^2, g)$ for odd q and $\lambda_{q+2p-2}(\mathbb{T}^2, g)$ for even q.

All known extremal metrics on the torus and the Klein bottle are non-maximal

M.Karpukhin, *On maximality of known extremal metrics on torus and Klein bottle*, Preprint `arXiv:1210.8122`

All known extremal metrics on tori are non-maximal. The only exception are the metric on the equilateral torus (its metric is maximal for $\lambda_1(\mathbb{T}^2, g)$) and the metric on the Lawson Klein bottle $\tilde{\tau}_{3,1}$ (its metric is maximal for $\lambda_1(\mathbb{K}^2, g)$.)

A classical theorem

- Let N be a d-dimensional minimal submanifold of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ of radius R. Let Δ be the Laplace-Beltrami operator on N equipped with the induced metric.
- **Theorem.** The restrictions $x^1|_N, \ldots, x^{n+1}|_N$ on N of the standard coordinate functions of $\mathbb{R}^{n+1}$ are eigenfunctions of Δ with eigenvalue $\frac{d}{R^2}$.

A recent theorem by El Soufi and Ilias (2008)

- Let us numerate the eigenvalues of Δ counting them with multiplicities

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \dots$$

- The above mentioned theorem implies that there exists at least one index i such that $\lambda_i = \frac{d}{R^2}$. Let j denotes the minimal number i such that $\lambda_i = \frac{d}{R^2}$.
- Let us introduce the eigenvalues counting function

$$N(\lambda) = \#\{\lambda_i | \lambda_i < \lambda\}.$$

We see that $j = N(\frac{d}{R^2})$.

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A recent theorem by El Soufi and Ilias (2008)

- **Theorem.** The metric g_0 induced on N by minimal immersion $N \subset \mathbb{S}^n$ is an extremal metric for the functional $\lambda_{N\left(\frac{d}{R^2}\right)}(N, g)$.

How to find extremal metrics?

- Find a minimally immersed surface Σ in a unit sphere
- Find $N(2)$
- Then the induced metric on Σ is extremal for $\lambda_{N(2)}$

Lawson τ-surfaces

- **Definition** A Lawson tau-surface $\tau_{m,k} \subset \mathbb{S}^3$ is defined by the doubly-periodic immersion $\Psi_{m,k} : \mathbb{R}^2 \longrightarrow \mathbb{S}^3 \subset \mathbb{R}^4$ given by the following explicit formula,

$$\Psi_{m,k}(x,y) =$$
$$= (\cos mx \cos y, \sin mx \cos y, \cos kx \sin y, \sin kx \sin y).$$

Lawson τ-surfaces

- This family of surfaces is introduced in 1970 by Lawson. He proved that for each unordered pair of positive integers (m, k) with $(m, k) = 1$ the surface $\tau_{m,k}$ is a distinct compact minimal surface in $\mathbb{S}^3$. Let us impose the condition $(m, k) = 1$. If both integers m and k are odd then $\tau_{m,k}$ is a torus. We call it a Lawson torus. If one of integers m and k is even then $\tau_{m,k}$ is a Klein bottle. We call it a Lawson Klein bottle. The torus $\tau_{1,1}$ is the Clifford torus.

Lawson τ-surfaces

- The proof is done by reduction to a periodic Sturm-Liouville problem and applying theory of Magnus-Winkler-Ince equation and the Lamé equation.

Hsiang-Lawson reduction theorem

- Let M be a Riemannian manifold with a metric g' and $I(M)$ its full isometry group. Let $G \subset I(M)$ be an isometry group. Let us denote by π the natural projection onto the space of orbits $\pi : M \longrightarrow M/G$.
- The union M^* of all orbits of principal type is an open dense submanifold of M. The subset M^*/G of M/G is a manifold carrying a natural Riemannian structure g induced from the metric g' on M.

Hsiang-Lawson reduction theorem

- Let us define a volume function $V : M/G \longrightarrow \mathbb{R}$: if $x \in M^*/G$ then $V(x) = Vol(\pi^{-1}(x))$
- Let $f : N \longrightarrow M$ be a G-invariant submanifold, i.e. G acts on N and f commutes with the actions of G on N and M.
- A cohomogeneity of a G-invariant submanifold $f : N \longrightarrow M$ in M is the integer $\dim N - \nu$, where $\dim N$ is the dimension of N and ν is the common dimension of the principal orbits.
- Let us define for each integer $k \geqslant 1$ a metric $g_k = V^{\frac{2}{k}} g$.

Hsiang-Lawson reduction theorem

- Theorem (Hsiang-Lawson). Let $f : N \longrightarrow M$ be a G-invariant submanifold of cohomogeneity k, and let M/G be given the metric g_k. Then $f : N \longrightarrow M$ is minimal is and only if $\bar{f} : N^*/G \longrightarrow M^*/G$ is minimal.
- Corollary. If $M = \mathbb{S}^n$, $G = \mathbb{S}^1$ and $\tilde{N} \subset M^*/G$ is a closed geodesic w.r.t. the metric g_1 then $\pi^{-1}(\tilde{N})$ is a minimal torus in $\mathbb{S}^n$.

Otsuki tori

- Let us consider $M = \mathbb{S}^3$ and $G = \mathbb{S}^1$ acting as

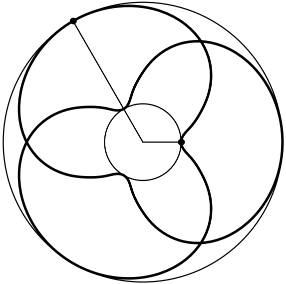
$$\alpha \cdot (x, y, z, t) = (\cos\alpha x + \sin\alpha y, -\sin\alpha x + \cos\alpha y, z, t).$$

- Minimal tori obtained in this case by the described construction are called Otsuki tori.
- Except one particular case (this is a Clifford torus), Otsuki tori are in one-to-one correspondence with rational numbers $\frac{p}{q}$ such that

$$\frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}, \quad p, q > 0, \quad (p, q) = 1.$$

Otsuki tori

- We denote these tori by $O_{\frac{p}{q}}$.
- Example: the geodesic corresponding to $O_{\frac{2}{3}}$.



The Otsuki tori

- Here the proof also use the reduction to a periodic Sturm-Liouville problem but it does not require complicated classical ODEs.