ISRAEL SCIENCE FOUNDATION WORKSHOP ON ORBITS, PRIMITIVE IDEALS AND QUANTUM GROUPS

MARCH 3 – 8

Abstracts

Research Surveys

Florence Fauquant-Millet

Title: Adapted pairs in a biparabolic subalgebra of a simple Lie algebra of type A and regular nilpotent elements.

Abstract: Let $\mathfrak{g}$ be a complex simple Lie algebra of type A_n and $\mathfrak{q}$ a biparabolic subalgebra of $\mathfrak{g}$, that is the intersection of two parabolic subalgebras whose sum is equal to $\mathfrak{g}$. Define the semicentre $Sy(\mathfrak{q})$ of the symmetric algebra $S(\mathfrak{q})$ of $\mathfrak{q}$ to be the vector space generated by the one-dimensional representations of $\mathfrak{q}$ in $S(\mathfrak{q})$. There exists a canonical truncation of $\mathfrak{q}$, denoted by $\mathfrak{q}_\Lambda$, such that the semicentre $Sy(\mathfrak{q})$ of $S(\mathfrak{q})$ is equal to the subalgebra $Y(\mathfrak{q}_\Lambda)$ of polynomials on $\mathfrak{q}_\Lambda^*$ invariant by the adjoint action of $\mathfrak{q}_\Lambda$. A. Joseph has given a lower and an upper bound for $Sy(\mathfrak{q})$, one coming from the Hopf dual of the enveloping algebra of $\mathfrak{q}$ and the other one coming from the Borel case. These two bounds are shown to be polynomial algebras on the same number of generators whose weight eventually differ from a coefficient 1/2. When these weights do not differ (this is the case when $\mathfrak{g}$ is of type A_n or C_n and $\mathfrak{q}$ is any biparabolic subalgebra) both bounds coincide and in this case $Sy(\mathfrak{q})$ is deduced to be also a polynomial algebra. In order to prove that $Sy(\mathfrak{q}) = Y(\mathfrak{q}_\Lambda)$ should be a polynomial algebra outside type A or C, we would like to find a "Weierstrass section", that is an affine space $\eta + V$ (with $\eta \in \mathfrak{q}_\Lambda^*$ and V a vector subspace of $\mathfrak{q}_\Lambda^*$) such that there exists an algebra isomorphism (given by the restriction of functions) between $Y(\mathfrak{q}_\Lambda)$ and the regular functions $\mathbb{C}[\eta+V]$ on $\eta+V$. Then the existence of a Weierstrass section implies the polynomiality of $Y(\mathfrak{q}_\Lambda)$.

But Weierstrass sections are quite difficult to find. That is why we use "adapted pairs". An adapted pair for $\mathfrak{q}_\Lambda$ is a pair $(h, \eta) \in \mathfrak{q}_\Lambda \times \mathfrak{q}_\Lambda^*$ such that η is regular in $\mathfrak{q}_\Lambda^*$ and $(ad\, h)(\eta) = -\eta$. When $\mathfrak{g}$ is of type A_n, A. Joseph has constructed adapted pairs for any biparabolic subalgebra. In this mini-course I will first set the background.

Then I will describe the construction of the second element η of an adapted pair for a truncated biparabolic subalgebra of $\mathfrak{g}$ of type A_n.

Finally I will explain what corresponds to the last work of A. Joseph and myself, that is how we have shown that there exists a regular nilpotent element y of $\mathfrak{g}^* \simeq \mathfrak{g}$ whose restriction to $\mathfrak{q}_\Lambda$ is equal to η.

The description of η and the construction of y can be explained via meanders. I will show how to use the "local" properties of these meanders to obtain a new simple root system of $\mathfrak{sl}_{n+1}(\mathbb{C})$ corresponding to an element y whose restriction to the truncated biparabolic is equal to η.

The motivation of our work is that it gives rise to an element w of the Weyl group of $\mathfrak{g}$ allowing to pass from the standard regular nilpotent element of $\mathfrak{g}$ to the regular nilpotent element y of $\mathfrak{g}$ whose restriction gives the second element of an adapted pair. One could expect that w should take sense in every simple Lie algebra, taking a universal form (not depending on the type of the simple Lie algebra $\mathfrak{g}$) which could give us Weierstrass sections in almost all types of simple Lie algebras even when no adapted pairs exist.

Ivan Losev

Title: Finite W-algebras.

Abstract: Finite W-algebras were introduced by Premet, [P2], following an earlier work by Kostant, Lynch, Kawanaka, Moeglin, Premet himself and others. Each W-algebra is constructed from a pair $(\mathfrak{g}, \mathbb{O})$, where $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathbb{O}$ is a nilpotent orbit in $\mathfrak{g}$, when $\mathbb{O}$ is zero, we recover the universal enveloping algebra $U(\mathfrak{g})$. At the moment, there are two surveys on the subject, [W],[L5]. One of motivations to consider W-algebras is their connection with the infinite dimensional representation theory of $U(\mathfrak{g})$ that is a manifestation of the orbit method. It is this connection that will be emphasized in my lectures. Time permitting I will also try to describe a connection to the representation theory of semisimple Lie algebras in positive characteristic, [P1],[BMR],[BM]. Here is a preliminary (optimistic) plan for the four lectures. 1) Construction of W-algebras: via quantum slices ([L1],[L2],[L6]) and, briefly, via quantum Hamiltonian reduction, ([P2],[GG]). 2) Functors between categories of modules over $U(\mathfrak{g})$ and over the W-algebras: Harish-Chandra bimodules ([L2], [Gi]), Whittaker modules (for $U(\mathfrak{g})$) and categories $\mathcal{O}$ (for W-algebras), following [P2],[L1],[BGK], [L3]. 3) Classification of finite dimensional irreducible modules over W-algebras, following [L2],[LO],[L4]. 4) Dimensions of irreducible modules over W-algebras and Goldie ranks of primitive ideals, following [L7]. Connections to [BM]. I will try to recall all preliminaries on nilpotent orbits and on the representation theory of $U(\mathfrak{g})$.

References

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Marc Rosso

Title: Quantum quasi shuffles.

Abstract: We shall start by reviewing (classical) shuffle and quasi shuffle algebras (stressing their universal properties) and their occurence in different places in mathematics: iterated integrals, multiple zeta values, Rota-Baxter algebras. We shall then move to the braided (or quantum) framework by introducing the braided tensor category of Hopf bimodules over a Hopf algebra, and consider the associated free cotensor coalgebra. Its universal property allows to construct many Hopf algebra structures on it (compatible with the "deconcatenation coproduct"). The simplest one leads to quantum shuffle algebras, and, in some specific examples the subalgebra generated in degree 1 (alias Nichols algebra) is isomorphic to the plus part of quantum groups. Considering the next case, where the (coinvariants) of the Hopf bimodule have a braided algebra structure, leads to quantum quasi-shuffle algebras. In specific examples, one reconstructs in this way the whole quantum group.

Hans-Juergen Schneider

Title: Hopf algebras and root systems.

Abstract: I will give an introduction to some recent results on the classification of Hopf algebras and the relationship between (a certain class of) Hopf algebras and generalized root systems and Weyl groupoids. The basic notion is the Nichols (or quantum symmetric) algebra of a braided vector space and of a Yetter-Drinfeld module over some Hopf algebra. Important special cases of Nichols algebras are the plus parts of deformed universal enveloping algebras. The Weyl groupoid for diagonal braidings was introduced by I. Heckenberger. The talks are based on joint papers with N. Andruskiewitsch, N. Andruskiewitsch and I. Heckenberger, and with I. Heckenberger.

Research Talks

Lucas Fresse

Title: Partial flag varieties and nilpotent elements.

Abstract: The flag variety of a complex reductive algebraic group G is by definition the quotient G/B by a Borel subgroup. It identifies with the set of Borel subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$. Given a nilpotent element $x \in \mathfrak{g}$, one calls Springer fiber the subvariety formed by the Borel subalgebras which contain x. Springer fibers have in general a quite complicated structure (in general not irreducible, singular). Nevertheless, a theorem of De Concini, Lusztig, and Procesi asserts that, when G is classical, a Springer fiber can always be paved by finitely many subvarieties isomorphic to affine spaces. In the talk, we study varieties generalizing the Springer fibers to the context of partial flag varieties, that is, subvarieties of the quotient G/P by a parabolic subgroup (instead of a Borel subgroup). We propose a generalization of DeConcini-Lusztig-Procesi's theorem to this context.

Sarah Kitchen

Title: Localization of Generalized Harish-Chandra Modules.

Abstract: Localization in representation theory is well known to be a powerful tool which allows us to apply geometric methods to representation theoretic problems. In this talk, I will discuss the localization of generalized Harish-Chandra Modules and a geometric approach to a conjecture of Penkov and Zuckerman.

Polyxeny Lamprou

Title: Quantum generalized Harish-Chandra isomorphisms.

Abstract: Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. As it is well-known, the Harish-Chandra projection $p : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ maps the $\mathfrak{g}$-invariants in $U(\mathfrak{g})$ isomorphically to the W-invariants (under the translated action of W) in $U(\mathfrak{h})$. Recently, Khoroshkin-Nazarov and Vinberg extended this result and described the image of the $\mathfrak{g}$-invariants (under the adjoint action of $\mathfrak{g}$) in $V \otimes U(\mathfrak{g})$, where V is a finite dimensional $\mathfrak{g}$-module; it involves the Zhelobenko operators. Joseph gave a different proof of their result; the key point of his proof was the computation of certain determinants similar to the Parthasarathy-Ranga Rao-Varadarajan determinants. Later, Balagovic obtained an analogue of this in the quantum case. We will give alternative proofs of the Harish-Chandra isomorphisms in the semisimple and quantum case; these proofs are direct and involve simple $\mathfrak{sl}_2$-calculations.

Kyo Nishiyama

Title: Double flag varieties and spherical actions.

Abstract: Let G be a reductive algebraic group and K its symmetric subgroup, i.e., a subgroup fixed by an involution. Let $P \subset G$ and $Q \subset K$ be parabolic subgroups respectively. We consider the diagonal K action on the product $G/P \times K/Q$, which is our double flag variety. In appropriate settings, it contains usual double flag variety $G/B \times G/B$ and also contains triple flag varieties $G/P_1 \times G/P_2 \times G/P_3$. We discuss finiteness of K-orbits, moment maps, nilpotent variety and orbital varieties.

The talk is based on joint works with Xuhua He, Hiroyuki Ocahi, Yoshiki Oshima and Lucas Fresse.

Vladimir Popov

Title: Orbit closures.

Abstract: Let G be a connected linear algebraic group, let V be a finite dimensional algebraic G-module, and let O_1 and O_2 be two G-orbits in V. I shall describe a constructive way to find out whether or not O_1 lies in the closure of O_2. This yields a constructive way to find out whether given two points of V lie in the same orbit or not. Several classical problems in algebra and algebraic geometry are reduced to this problem.

Dmitry Timashev

Title: On quotients of affine spherical varieties by unipotent subgroups.

Abstract: Let G be a connected complex reductive algebraic group and X be an affine spherical G-variety. Then any Borel subgroup $B \subset G$ acts on X with finitely many orbits. Now suppose that $H \subset B$ is a normal unipotent subgroup such that the algebra of regular functions $\mathbb{C}[G/H]$ is finitely generated. Then the algebra of invariant functions $\mathbb{C}[X]^H$ is finitely generated, too, and one may consider the categorical quotient map $\pi : X \to X//H := \mathrm{Spec}\mathbb{C}[X]^H$. Clearly, B/H acts on $X//H$. Recently Panyushev posed a question whether the number of orbits for this action is always finite. (The point is that π is not always surjective.) An affirmative answer would provide us with new natural examples of algebraic varieties having a nice stratification and therefore nice homological and intersection-theoretic properties. Panyushev conjectured that the answer is affirmative for $H = [U, U]$, the commutator of the maximal unipotent subgroup $U \subset B$. We discuss an approach to answering the question, give a affirmative answer in some cases, and provide a counterexample to the above conjecture.