Euclidean sections of convex bodies

Series of lectures given in Bedlewo, Poland, July 6-12, 2008
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This is a somewhat expanded form of a four hours course given, with small variations, first at the educational workshop Probabilistic methods in Geometry, Bedlewo, Poland, July 6-12, 2008 and a few weeks later at the Summer school on Fourier analytic and probabilistic methods in geometric functional analysis and convexity, Kent, Ohio, August 13-20, 2008.
The main part of these notes gives yet another exposition of Dvoretzky's theorem on Euclidean sections of convex bodies with a proof based on Milman's. This material is by now quite standard. Towards the end of these notes we discuss issues related to fine estimates in Dvoretzky's theorem and there there are some results that didn't appear in print before. In particular there is an exposition of an unpublished result of Figiel (Claim 3.2) which gives an upper bound on the possible dependence on ϵ in Milman's theorem. We would like to thank Tadek Figiel for allowing us to include it here. There is also a better version of the proof of one of the results from [Sc2] giving a lower bound on the dependence on ϵ in Dvoretzky's theorem. The improvement is in the statement and proof of Proposition 4.2 here which is a stronger version of the corresponding Corollary 1 in [Sc2].

1 Lecture 1

By a convex, symmetric body $K \subset \mathbb{R}^n$ we shall refer to a compact set with non-empty interior which is convex and symmetric about the origin (i.e, $x \in K$ implies that $-x \in K$

This series of lectures will revolve around the following theorem, a somewhat weaker (in terms of the dependence of k on n) version of which was

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proved by Aryeh Dvoretzky [Dv], answering a question of Grothendieck. The question of Grothendieck was asked in [Gr] in relation with a paper of Dvoretzky and Rogers [DR]. [Gr] gives another proof of the main application (the existence, in any infinite dimensional Banach space, of an unconditionally convergent series which is not absolutely convergent) of the result of Dvoretzky and Rogers [DR] a version of which is used bellow (Lemma 2.1).

Theorem 1.1. *(A. Dvoretzky, 1960) For every $\epsilon > 0$ there exists a constant $c = c(\epsilon) > 0$ such that for every $n \in \mathbb{N}$ and every convex symmetric body in $K \subset \mathbb{R}^n$ there exists a subspace $V \subseteq \mathbb{R}^n$ satisfying:*

1. $\dim V = k$*, where* $k \geq c \cdot \log n$.
2. $V \cap K$ *is "ϵ-euclidean", which means that there exists* $r > 0$*, such that:*

$$r \cdot V \cap B_2^n \subset V \cap K \subset (1 + \epsilon) r \cdot V \cap B_2^n.$$

For example the unit ball of ℓ_∞^n - the n-dimensional cube - is far from the Euclidean ball. Its easy to see, that the ratio of radii of the bounding and the bounded ball is $\sqrt{n}$:

$$B_2^n \subset B_\infty^n \subset \sqrt{n} B_2^n$$

and $\sqrt{n}$ is the best constant. Yet, according to Dvoretzky theorem, we can find a subspace of $\mathbb{R}^n$ of dimension proportional to $\log n$ in which the ratio of bounding and bounded balls will be $1 + \epsilon$.

There is a simple correspondence between symmetric convex sets in $\mathbb{R}^n$ and norms on $\mathbb{R}^n$ Given by $\|x\|_K = \inf\{\lambda > 0 \ : \ \frac{x}{\lambda} \in K\}$ The following is an equivalent formulation of Dvoretzky's Theorem in terms of norms

Theorem 1.2. *For every $\epsilon > 0$ there exist a constant $c = c(\epsilon) > 0$ such that for every $n \in \mathbb{N}$ and every norm $\|\cdot\|$ in $\mathbb{R}^n$ ℓ_2^k $(1+\epsilon)$-embeds in $(\mathbb{R}^n, \|\cdot\|)$ for some $k \geq c \cdot \log n$.*

By "X C-embed in Y" I mean: There exists a one to one bounded operator $T : X \to Y$ with $\|T\|\|(T_{|TX})^{-1}\| \leq C$.

Clearly, Theorem 1.1 implies Theorem 1.2. Also, Theorem 1.2 clearly implies a weaker version of Theorem 1.1, with B_2^n replaced by some ellipsoid (which by definition is an invertible linear image of B_2^n). But, since any k-dimensional ellipsoid easily seen to have a $k/2$-dimensional section which is a multiple of the Euclidean ball, we see that also Theorem 1.2 implies Theorem 1.1. this argument also shows that proving Theorem 1.1 for K is equivalent to proving it for some invertible linear image of K.

A Very vague sketch of the proof: Consider the unit sphere of ℓ_2^n, the surface of B_2^n, which we will denote by $S^{n-1} = \{x \in \mathbb{R}^n \ : \ \|x\|_2 = 1\}$. Let $\|x\|$ be some arbitrary norm in $\mathbb{R}^n$. The first task will be to show that there exists a "large" set $S_{\text{good}} \subset S^{n-1}$ satisfying $\forall x \in S_{\text{good}}.\ |\|x\| - M| < \epsilon M$ where M is the average of $\|x\|$ on S^{n-1}. Moreover, we shall see that, dependeing on the Lipschitz constant of $\|\cdot\|$, the set S_{good} is "almost all" the sphere in the measure sense. This phenomenon is called *concentration of measure*.

The next stage will be to pass from the "large" set to a large dimensional subspace of $\mathbb{R}^n$ contained in it. Denote $O(n)$ - the group of orthogonal transformations from $\mathbb{R}^n$ into itself. Choose some subspace V_0 of appropriate dimension k and fix an ϵ-net N on $V_0 \cap S^{n-1}$. For some $x_0 \in N$,"almost all" transformations $U \in O(n)$ will send it into some point in S_{good}. Moreover, if the "almost all" notion is good enough, we will be able to find a transformation that sends all the points of the ϵ-net into S_{good}. Now there is a standard approximation procedure that will let us pass from the ϵ-net to all points in the subspace.

The original proof of Dvoretzky is very involved. Several simplified proofs were given in the beginning of the 70-s; one by Figiel [Fi], one by Szankowski [Sz] and one, a version of which we'll present here, by Milman [Mi]. This proof is based on the notion of **Concentration of Measure**. Milman was also the first to get the right estimate ($\log n$) of the dimension k in terms on the original dimension n. The dependence of k on ϵ is still wide open and we'll discuss it in detail later in this series.

Denote by μ the normalized Haar measure on S^{n-1} - the unique, probability measure which is invariant under the group of orthogonal transformations. The main tool will be the following concentration of measure theorem of Paul Levy.

Theorem 1.3. *(P. Levy) Let, $f : S^{n-1} \longrightarrow \mathbb{R}$ be a Lipshitz function with a constant L:*

$$\forall x, y \in S^{n-1}.\ |f(x) - f(y)| \leq L \cdot \|x - y\|_2.$$

Then,

$$\mu\{x \in S^{n-1} \ : \ |f(x) - Ef| > \epsilon\} \leq 2e^{-\frac{\epsilon^2 n}{2L^2}}.$$

Remark: The theorem also holds with the expectation of f replaced by its median.

Our next goal is to prove the following theorem of Milman which, gives some lower bound on the dimension of almost Euclidean section in each convex body. It will be the main tool in the proof of Dvoretzky's Theorem.

Theorem 1.4. *(V. Milman) For every $\epsilon > 0$ there exists a constant $c = c(\epsilon) > 0$ such that for every $n \in \mathbb{N}$ and every norm $\|\cdot\|$ in $\mathbb{R}^n$ there exists a subspace $V \subseteq \mathbb{R}^n$ satisfying:*

1. $\dim V = k$, *where* $k \geq c \cdot \left(\frac{E}{b}\right)^2 n$.

2. *For every $x \in V$:*

$$(1-\epsilon)E \cdot \|x\|_2 \leq \|x\| \leq (1+\epsilon)E \cdot \|x\|_2.$$

Here $E = \int_{S^{n-1}} \|x\| d\mu$ and b is the smallest constant satisfying $\|x\| \leq b\|x\|_2$.

The definition of b implies that the function $\|\cdot\|$ is Lipschitz with constant b of S^{n-1}. Applying Theorem 1.3 we get a subset of S^{n-1} of probability very close to one ($\geq 1 - 2e^{-\epsilon^2 E^2 n/2}$), assuming E is not too small, on which

$$(1-\epsilon)E \leq \|x\| \leq (1+\epsilon)E. \tag{1.1}$$

We need to replace this set of large measure with a set which is large in the algebraic sense: A set of the form $V \cap S^{n-1}$ for a subspace V of relatively high dimension. The way to overcome this difficulty is to fix an ϵ-net in $V_0 \cap S^{n-1}$ (i.e., a finite set such that any other point in $V_0 \cap S^{n-1}$ is of distance at most ϵ from one of the points in this set) for some *fixed* subspace V_0 (of dimension k to be decided upon later) and show that we can find an orthogonal transformation U such that $\|Ux\|$ satisfies equation 1.1 for each x in the ϵ-net. A successive approximation argument (the details of which can be found, e.g., in [MS], as all other details which are not explained here), then gives a similar inequality (maybe with 2ϵ replacing ϵ) for all $x \in V_0 \cap S^{n-1}$, showing that $V = UV_0$ can serve as the needed subspace.

To find the required $U \in O(n)$ we need two simple facts. The first is to notice that if we denote by ν the normalized Haar measure on the orthogonal group $O(n)$, then, using the uniqueness of the Haar measure on S^{n-1}, we get that, for each fixed $x \in S^{n-1}$, the distribution of Ux, where U is distributed according to ν, is μ. It follows that, for each fixed $x \in S^{n-1}$, with probability, ν, $\geq 1 - 2e^{-\epsilon^2 E^2 n/2}$,

$$(1-\epsilon)E \leq \|Ux\| \leq (1+\epsilon)E.$$

Using a simple Union bound we get that for any fine set $N \subset S^{n-1}$, with ν-probability $\geq 1 - 2|N|e^{-\epsilon^2 E^2 n/2}$, U satisfies

$$(1-\epsilon)E \leq \|Ux\| \leq (1+\epsilon)E$$

for all $x \in N$ ($|N|$ denotes the cardinality of N).

Lemma 1.5. *For every $0 < \epsilon < 1$ there exists an ϵ-net N on S^{k-1} of cardinality $\leq \left(\frac{3}{\epsilon}\right)^k$.*

So as long as, $2\left(\frac{3}{\epsilon}\right)^k e^{-\epsilon^2 E^2 n/2} < 1$ we can find the required U. This translates into: $k \geq c\frac{\epsilon^2}{\log\frac{3}{\epsilon}}E^2 n$ for some absolute $c > 0$ as is needed in the coclusion of Theorem 1.4.

To prove the lemma, let $N = \{x_i\}_{i=1}^m$ be a maximal set in S^{k-1} such that for all $x, y \in N$ $\|x-y\|_2 \geq \epsilon$. The maximality of N implies that it is an ϵ-net for S^{k-1}. Consider $\{B(x_i, \frac{\epsilon}{2})\}_{i=1}^m$ - the collection of balls of radius $\frac{\epsilon}{2}$ around the x_i-s. They are mutually disjoint and completely contained in $B(0, 1+\frac{\epsilon}{2})$. Hence:

$$mVol\left(B(x_1, \frac{\epsilon}{2})\right) = \sum Vol\left(B(x_i, \frac{\epsilon}{2})\right) = Vol\left(\bigcup B(x_i, \frac{\epsilon}{2})\right) \leq Vol\left(B(0, 1+\frac{\epsilon}{2}\right).$$

The k homogeneity of the Lebesgue measure in $\mathbb{R}^k$ implies now that $m \leq \left(\frac{1+\epsilon/2}{\epsilon/2}\right)^k = \left(1+\frac{2}{\epsilon}\right)^k$.

This completes the sketch of the proof of Theorem 1.4.

2 Lecture 2

In order to prove Dvoretzky's theorem (Theorem 1.1) we need to estimate E and b. Since the problem is invariant under invertible linear transformation we may assume that S^{n-1} is included in K, i.e., $b = 1$. In remains to estimate E from below. As we'll see this can be done quite effectively for many interesting examples (we'll show the computation for the ℓ_p^n balls). However in general it may happen that E is very small even if we assume as we may that S^{n-1} touches the boundary of K. This is easy to see.

The way to overcome this difficulty is to assume in addition that S^{n-1} is the ellipsoid of maximal volume inscribed in K. An ellipsoid is just an invertible linear image of the canonical Euclidean ball. Given a convex body one can find by compactness an ellipsoid of maximal volume inscribed in it. It is known that there is a unique such ellipsoid but for the reasoning below we don't need that. The invariance of the problem let us assume that the canonical Euclidean ball is such an ellipsoid. The advantage of this special situation comes from the following Lemma

Lemma 2.1. *(Dvoretzky-Rogers) Let $\|\cdot\|$ be some norm on $\mathbb{R}^n$ and denote its unit ball by $K = B_{\|\cdot\|}$. Assume the Euclidean ball $B_2^n = B_{\|\cdot\|_2}$ is (the) ellipsoid of maximal volume inscribed in K. Then there exist and orthonormal basis $x_1, \dots, x_n$ such that*

$$e^{-1}(1 - \frac{i-1}{n}) \leq \|x_i\| \leq 1, \quad \textit{for all } 1 \leq i \leq n.$$

Remark: This is a weaker version of the original Dvoretzky-Rogers lemma. It shows in particular that half of the x_i-s have norm bounded from below: for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ $\|x_i\| \geq (2e)^{-1}$. This is what will be used in the proof of the main theorem.

Proof. First of all choose an arbitrary $x_1 \in S^{n-1}$ of maximal norm. Of course, $\|x_1\| = 1$. Suppose we have chosen $\{x_1, \dots, x_{i-1}\}$ that are orthonormal. Choose x_i as the one having the maximal norm among all $x \in S^{n-1}$ that are orthogonal to $\{x_1, \dots, x_{i-1}\}$. Define a new ellipsoid which is smaller in some directions and bigger in others:

$$\mathcal{E} = \{\sum_{i=1}^{n} a_i x_i \ : \ \sum_{i=1}^{j-1} \frac{a_i^2}{a^2} + \sum_{i=j}^{n} \frac{a_i^2}{b^2} \leq 1\}.$$

Suppose, $\sum_{i=1}^{n} b_i x_i \in \mathcal{E}$. Then $\sum_{i=1}^{j-1} b_i x_i \in aB_2^n$, hence $\|\sum_{i=1}^{j-1} b_i x_i\| \leq a$. Moreover, for each $x \in span\{x_j, \dots, x_n\} \bigcap B_2^n$ we have $\|x\| \leq \|x_j\|$ and since

$\sum_{i=j}^n b_i x_i \in bB_2^n$, $\|\sum_{i=j}^n b_i x_i\| \le \|x_j\| b$. Thus,

$$\|\sum_{i=1}^n b_i x_i\| \le \|\sum_{i=1}^{j-1} b_i x_i\| + \|\sum_{i=j}^n b_i x_i\| \le a + \|x_j\| \cdot b.$$

The relation between the volumes of $\mathcal{E}$ and B_2^n is $Vol(\mathcal{E}) = a^{j-1} b^{n-j+1} Vol(B_2^n)$. If $a + \|x_j\| \cdot b \le 1$, then $\mathcal{E} \subseteq K$. Using the fact that B_2^n is the ellipsoid of the maximal volume inscribed in K we conclude that

$$\forall a, b, j \text{ s.t. } a + \|x_j\| \cdot b = 1, \quad a^{j-1} b^{n-j+1} \le 1.$$

Substituting $b = \frac{1-a}{\|x_j\|}$ and $a = \frac{j-1}{n}$ it follows that for every $j \ge 2$

$$\|x_j\| \ge a^{\frac{j-1}{n-j+1}}(1-a) = \left(\frac{j-1}{n}\right)^{\frac{j-1}{n-j+1}} \left(1 - \frac{j-1}{n}\right) \ge e^{-1}\left(1 - \frac{j-1}{n}\right).$$

□

We are now ready to prove Dvoretzky's theorem.

As we have indicated, using Theorem 1.4, and assuming as we may that B_2^n is the ellipsoid of maximal volume inscribed in $K = B_{\|\cdot\|}$, it is enough to prove that

$$E = \int_{S^{n-1}} \|x\|_d x \ge c\sqrt{\frac{\log n}{n}}, \tag{2.1}$$

for some absolute constant $c > 0$.

This will show Dvoretzky's theorem with the bound $k \ge c\frac{\epsilon^2}{\log \frac{1}{\epsilon}} \log n$.

We now turn to prove the inequality 2.1. According to the Dvoretzky-Rogers lemma 2.1 there are orthonormal vectors $x_1, \ldots, x_n$ such that for all

$1 \leq i \leq \lfloor \frac{n}{2} \rfloor \quad \|x_i\| \geq 1/2e.$

$$\begin{aligned}\int_{S^{n-1}} \|x\| d\mu(x) = \int_{S^{n-1}} \|\sum_{i=1}^{n} a_i x_i\| d\mu(a) = \\ = \int_{S^{n-1}} \frac{1}{2}(\|\sum_{i=1}^{n-1} a_i x_i + a_n x_n\| + \|\sum_{i=1}^{n-1} a_i x_i - a_n x_n\|) d\mu(a) \geq \\ \geq \int_{S^{n-1}} \max\{\|\sum_{i=1}^{n-1} a_i x_i\|, \|a_n x_n\|\} d\mu(a) \geq \\ \geq \int_{S^{n-1}} \max\{\|\sum_{i=1}^{n-2} a_i x_i\|, \|a_{n-1} x_{n-1}\|, \|a_n x_n\|\} d\mu(a) \geq \cdots \geq \\ \geq \int_{S^{n-1}} \max_{1 \leq i \leq n} \|a_i x_i\| d\mu(a) \geq \frac{1}{2e} \int_{S^{n-1}} \max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |a_i| d\mu(a)\end{aligned}$$

To Evaluate the last integral we notice that because of the invariance of the canonical Gaussian distribution in $\mathbb{R}^n$ under orthogonal transformation and (again!) the uniqueness of the Haar measure on S^{n-1}, The vector $(\sum g_i^2)^{-1/2}(g_1, g_2, \ldots, g_n)$ is distributed μ. Here $g_1, g_2, \ldots, g_n$ arre i.i.d. $N(0,1)$ variables. Thus

$$\int_{S^{n-1}} \max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |a_i| d\mu(a) = \mathbb{E} \frac{\max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |g_i|}{(\sum_{i=1}^{n} g_i^2)^{1/2}} = \frac{\mathbb{E} \max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |g_i|}{\mathbb{E}(\sum_{i=1}^{n} g_i^2)^{1/2}} \tag{2.2}$$

(The last equation is not a mistake.)

To evaluate the denominator from above note that by Jensen's inequality:

$$\mathbb{E}(\sum_{i=1}^{n} g_i^2)^{1/2} \leq (\mathbb{E} \sum_{i=1}^{n} g_i^2)^{1/2} = \sqrt{n}.$$

The numerator is known to be of order $\sqrt{\log n}$ (estimate the tail behavior of $\max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |g_i|$.)

This gives the required estimate and concludes the proof of Dvoretzky's theorem.

As another application of Theorem 1.4 we'll estimate the almost Euclidean sections of the ℓ_p^n balls $B_p^n = \{x \in \mathbb{R}^n; \|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \leq 1\}$.

Using the connection between the Gaussian distribution and μ we can write

$$E_p = \int_{S^{n-1}} \|x\|_p d\mu = \mathbb{E}\frac{(\sum g_i^p)^{1/p}}{(\sum g_i^2)^{1/2}} = \frac{\mathbb{E}(\sum g_i^p)^{1/p}}{\mathbb{E}(\sum g_i^2)^{1/2}}.$$

To bound the last quantity from below we will use the following inequality:

$$\sqrt{2/\pi} \cdot n^{1/r} = (\sum(\mathbb{E}|g_i|)^r)^{1/r} \leq \mathbb{E}(\sum g_i^r)^{1/r} \leq (\mathbb{E}\sum g_i^r)^{1/r} = c_r \cdot n^{1/r}$$

Hence:

$$E_p \geq c_p \cdot n^{\frac{1}{p}-\frac{1}{2}}.$$

For $p > 2$ we have $\|x\|_p \leq \|x\|_2$. For $1 \leq p < 2$ we have $\|x\|_p \leq n^{\frac{1}{p}-\frac{1}{2}} \cdot \|x\|_2$. It now follows from Theorem 1.4 that the dimension of the largest ϵ Euclidean section of the ℓ_p^n ball is

$$k \geq \begin{cases} c_p(\epsilon)n^{\frac{2}{p}}, & 2 < p < \infty \\ c(\epsilon)n, & 1 \leq p < 2. \end{cases}$$

3 Lecture 3

In this lecture we'll mostly be concerned with the question of how good are the estimates we got. We begin with the last result on the last lecture concerning the dimension of almost euclidean sections of the ℓ_p^n balls.

Clearly, for $1 \leq p < 2$ the dependence of k on n is best possible. The following proposition of Bennett, Dor, Goodman, Johnson and Newman [BDGJN] shows that this is the case also for $2 < p < \infty$.

Proposition 3.1. *Let $2 < p < \infty$ and suppose that ℓ_2^k C-embeds into ℓ_p^n, meaning that there exists a linear operator $T : \mathbb{R}^k \to \mathbb{R}^n$ such that*

$$\|x\|_2 \leq \|Tx\|_p \leq C\|x\|_2,$$

then $k \leq c(p, C)n^{2/p}$.

Proof. Let $T : \mathbb{R}^k \to \mathbb{R}^n$, $T = (a_{ij})_{i=1}^{n}{}_{j=1}^{k}$ be the linear operator from the statement of the claim. Then for every $x \in \mathbb{R}^k$:

$$(\sum_{j=1}^k x_j^2)^{1/2} \leq (\sum_{i=1}^n |\sum_{j=1}^k a_{ij}x_j|^p)^{1/p} \leq C(\sum_{j=1}^k x_j^2)^{1/2}. \tag{3.1}$$

In particular, for every $1 \le l \le n$, substituting instead of x the l-th row of T we get:

$$(\sum_{j=1}^{k} a_{lj}^2)^p \le \sum_{i=1}^{n} |\sum_{j=1}^{k} a_{ij} a_{lj}|^p \le C^p (\sum_{j=1}^{k} a_{lj}^2)^{p/2}.$$

Hence, for every $1 \le l \le n$:

$$(\sum_{j=1}^{k} a_{lj}^2)^{p/2} \le C^p.$$

Let $g_1, \dots, g_k$ be independent standard normal random variables. Then using the fact that $\sum_{j=1}^{k} g_i a_j$ has the same distribution as $(\sum_{j=1}^{k} a_j^2)^{1/2} g_1$ and the left hand side of the inequality (3.1) we have

$$\mathbb{E}(\sum_{j=1}^{k} g_j^2)^{p/2} \le \mathbb{E}(\sum_{i=1}^{n} |\sum_{j=1}^{k} g_j a_{ij}|^p) = \sum_{i=1}^{n} \mathbb{E}(|g_1|^p (\sum_{j=1}^{k} a_{ij}^2)^{p/2}) \le C^p \mathbb{E}|g_1|^p n.$$

On the other hand we can evaluate $\mathbb{E}(\sum_{j=1}^{k} g_j^2)^{p/2}$ from below using the convexity of the exponent function for $p/2 > 1$:

$$\mathbb{E}(\sum_{j=1}^{k} g_j^2)^{p/2} \ge (\mathbb{E}\sum_{j=1}^{k} g_j^2)^{p/2} = k^{p/2}.$$

Combining the last two inequalities we get an upper bound for k:

$$k \le C^2 (\mathbb{E}|g_1|^p)^{2/p} n^{2/p}.$$

□

Remarks:

1. There exist absolute constants $0 < \alpha \le A < \infty$ such that $\alpha\sqrt{p} \le (\mathbb{E}|g_1|^p)^{1/p} \le A\sqrt{p}$. Hence the estimate we get for $c(p, C)$ is $c(p, C) \le ApC^2$. In particular, for $p = \log n$, we have

   $$k \le AC^2 \log(n)$$

   for an absolute A. $\ell_{\log n}^n$ is e-isomorphic to ℓ_∞^n. Hence, if we C-embed ℓ_2^k into ℓ_∞^n, then $k \le Ac^2 \log(n)$, which means that the $\log n$ bound in the Dvoretzky's theorem is sharp.

2. However, the exact dependence on ϵ is an open question. From the proof of Dvoretzky's theorem we got an estimation $k \geq \frac{c\epsilon^2}{\log(1/\epsilon)} \log n$. We'll speak more about this issue below.

Although the last result doesn't directly give good results concerning the dependence on ϵ in Dvoretzky's theorem it can be used to show that one can't expect any better beahiour on ϵ than ϵ^2 in Milman's theorem. This was observed by Tadek Figiel and didn't appear in print before. We thank Figiel for permitting us to include it here.

Claim 3.2 (Figiel)**.** *For any $0 < \epsilon < 1$ and n large enough ($n > \epsilon^{-4}$ will do), there is a 1-symmetric norm, $\|\cdot\|$, on $\mathbb{R}^n$ which is 2-equivalent to the ℓ_2 norm and such that if V is a subspace of $\mathbb{R}^n$ on which the $\|\cdot\|$ and $\|\cdot\|_2$ are $(1+\epsilon)$-equivalent then $\dim V \leq C\epsilon^2 n$ (C is an absolute constant).*

Proof. Given ϵ and $n > \epsilon^{-4}$ (say) let $2 < p < 4$ be such that $n^{\frac{1}{p}-\frac{1}{2}} = 2\epsilon$. Put

$$\|x\| = \|x\|_2 + \|x\|_p$$

on $\mathbb{R}^n$. Assume that for some A and all $x \in V$,

$$A\|x\|_2 \leq \|x\| \leq (1+\epsilon)A\|x\|_2.$$

Clearly, $1 + \frac{\epsilon}{2} \leq \frac{1+n^{\frac{1}{p}-\frac{1}{2}}}{1+\epsilon} \leq A \leq 2$ and be get that for all $x \in V$,

$$(A-1)\|x\|_2 \leq \|x\|_p \leq ((1+\epsilon)A - 1)\|x\|_2 = (A - 1 + \epsilon A)\|x\|_2.$$

Since $\epsilon A \leq n^{\frac{1}{p}-\frac{1}{2}} \leq 4(A-1)$, we get that, for $B = A - 1$,

$$B\|x\|_2 \leq \|x\|_p \leq 5B\|x\|_2.$$

It follows from [BDGJN] that for some absolute C,

$$\dim V \leq Cn^{2/p} = C(n^{\frac{1}{p}-\frac{1}{2}})^2 n = 4C\epsilon^2 n.$$

□

Next we will see another relatively simple way of obtaining an upper bound on k in Dvoretzky's theorem, which, unlike the estimate in Remark 1, tend to 0 as $\epsilon \to 0$. It still leaves a big gap with the lower bound above.

Claim 3.3. *If ℓ_2^k $(1+\epsilon)$-embeds into ℓ_∞^n, then*

$$k \le \frac{C \log n}{\log(1/c\epsilon)} \quad ,$$

for some absolute constants $0 < c, C < \infty$.

Proof. Assume we have $(1-\epsilon)^{-1}$-embedding of ℓ_2^k into ℓ_∞^n, i.e., we have a operator $T = (a_{ij})_{i=1}^{n}{}_{j=1}^{k}$ satisfying, for every $x \in \mathbb{R}^k$,

$$(1-\epsilon)(\sum_{j=1}^{k} x_j^2)^{1/2} \le \max_{1\le i\le n} |\sum_{j=1}^{k} a_{ij}x_j| \le (\sum_{j=1}^{k} x_j^2)^{1/2}. \tag{3.2}$$

This means that there exist vectors $v_1, \dots, v_n \in \mathbb{R}^k$ such that for every $x \in \mathbb{R}^k$:

$$(1-\epsilon)\|x\|_2 \le \max_{1\le i\le n} < v_i, x > \le \|x\|_2. \tag{3.3}$$

In particular, $\|v_i\|_2 \le 1$ for every $1 \le i \le n$.

Suppose $x \in S^{k-1}$, then the left hand side of 3.3 states that there exists an $1 \le i \le n$ such that $< v_i, x > \ge (1-\epsilon)$, hence:

$$\|x - v_i\|_2^2 = \|x\|_2^2 + \|v_i\|_2^2 - 2 < v_i, x > \le 2 - 2(1-\epsilon) = 2\epsilon.$$

Thus, the vectors $v_1, \dots, v_n$ form a $\sqrt{2\epsilon}$-net on the S^{k-1}, which means that n is much larger (exponentially) then k.
Indeed, we have

$$\begin{aligned} & \bigcup_{i=1}^{n} B(v_i, 2\sqrt{2\epsilon}) \supseteq B_2^k \setminus (1-\sqrt{2\epsilon})B_2^k \\ \Rightarrow \quad & nVolB(0, 2\sqrt{2\epsilon}) \ge VolB(0,1) - VolB(0, 1-\sqrt{2\epsilon}) \\ \Rightarrow \quad & n(2\sqrt{2\epsilon})^k \ge 1 - (1-\sqrt{2\epsilon})^k \ge \sqrt{2\epsilon}k(1-\sqrt{2\epsilon})^{k-1}. \end{aligned}$$

This gives for $\epsilon < \frac{1}{32}$ and $k \ge 12$

$$n \ge \frac{k}{2}(\frac{1}{4\sqrt{2\epsilon}})^{k-1} \ge (\frac{1}{4\sqrt{2\epsilon}})^{k/2},$$

or

$$k \le \frac{4 \log n}{\log \frac{1}{32\epsilon}}.$$

□

This shows that the $c(\epsilon)$ in the statement of Dvoretzky's theorem can't be larger than $\frac{C}{\log(1/c\epsilon)}$.

Our last objective in this series of lectures is to improve somewhat the lower estimate on $c(\epsilon)$ in the version of Dvoretzky's theorem we proved. For that we'll need the inverse to Claim 3.3.

Claim 3.4. *ℓ_2^k $(1+\epsilon)$-embeds into ℓ_∞^n for*

$$k = \frac{c \log n}{\log(1/c\epsilon)} \quad ,$$

for some absolute constants $0 < c, C < \infty$.

The proof is very simple and we only state the embedding. Use Lemma 1.5 to find an ϵ-net $\{x_i\}_{i=1}^n$ on s^{k-1} where k and n are related as in the statement of the claim. The embedding of ℓ_2^k into ℓ_∞^n is given by $x \to \{\langle x, x_i \rangle\}_{i=1}^n$.

4 Lecture 4

In this lecture we'll prove a somewhat improved version of Dvoretzky's theorem, replacing the ϵ^2 dependence by ϵ (except for a log factor).

Theorem 4.1. *There is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $\epsilon > 0$, every n-dimensional normed space ℓ_2^k $(1+\epsilon)$-embeds in $(\mathbb{R}^n, \|\cdot\|)$ for some $k \geq \frac{c\epsilon}{(\log \frac{1}{\epsilon})^2} \log n$.*

The idea of the proof is the following: We start as in the proof of Milman's theorem 1.4, assuming S^{n-1} is the ellipsoid of maximal volume inscribed in the unit ball of $B_{\|\cdot\|}$. If E is large enough (so that $\epsilon^2 E^2 n \geq \frac{\epsilon}{(\log \frac{1}{\epsilon})^2} \log n$) we get the result from Milman's theorem. If not, we'll show that the space actually contains a relatively high dimensional ℓ_∞^m and then use Claim 3.4 to get an estimate on the dimension of the embedded ℓ_2^k.

The main proposition is the following one which improves the main proposition of [Sc2]:

Proposition 4.2. *Let $(X, \|\cdot\|)$ be a normed space and let $x_1, \dots, x_n$ be a sequence in X satisfying $\|x_i\| \geq 1/10$ for all i and*

$$E\Big(\|\sum_{i=1}^n g_i x_i\|\Big) \leq L\sqrt{\log n}. \tag{4.1}$$

Then, there is a subspace of X of dimension $k \geq \frac{n^{1/4}}{CL}$ which is CL-isomorphic to ℓ_∞^k. C is a universal constant.

Let us assume the proposition and continue with the

Proof of Theorem 4.1. We start as in the proof of Dvoretzky's theorem, assuming B_2^n is the ellipsoid of maximal volume inscribed in the unit ball of $(\mathbb{R}^n, \|\cdot\|)$. As we already said we may assume $\epsilon^2 E^2 n \leq \frac{\epsilon}{(\log\frac{1}{\epsilon})^2}\log n$ or $E\sqrt{n} \leq \frac{\sqrt{\log n}}{\sqrt{\epsilon}\log\frac{1}{\epsilon}}$. Let $x_1, \ldots, x_n$ be the orthonormal basis given by the Dvoretzky–Rogers Lemma, so that in particular $\|x_i\| \geq 1/10$ for $i = 1, \ldots, n/2$. As we have seen before

$$E\Big(\|\sum_{i=1}^{n/2} g_i x_i\|\Big) \leq E\Big(\|\sum_{i=1}^{n} g_i x_i\|\Big) \leq CE\sqrt{n}$$

So,

$$E\Big(\|\sum_{i=1}^{n/2} g_i x_i\|\Big) \leq \frac{\sqrt{\log n}}{\sqrt{\epsilon}\log\frac{1}{\epsilon}}.$$

and by the Proposition 4.2 there is a subspace of $(\mathbb{R}^n, \|\cdot\|)$ of dimension $k \geq \frac{n^{1/4}}{CL}$ which is CL-isomorphic to ℓ_∞^k where $L = \frac{1}{\sqrt{\epsilon}\log\frac{1}{\epsilon}}$. It now follows from an iteration result of James (see Lemma 4.3 below and Corollary 4.4 following it) that for any $0 < \epsilon < 1$ there is a subspace of $(\mathbb{R}^n, \|\cdot\|)$ of dimension $k \geq cn^{\frac{c\epsilon}{\log L}}$ which is $1+\epsilon$ - isomorphic to ℓ_∞^k. $c > 0$ is a universal constant. We now use Claim 3.4 to conclude that ℓ_2^k embeds in our space for some $k \geq \frac{c\log(cn^{\frac{c\epsilon}{\log L}})}{\log(1/c\epsilon)} = \frac{c'\epsilon\log n}{(\log(1/c\epsilon))^2}$. □

The following simple Lemma is due to R. C. James

Lemma 4.3. *let $x_1, \ldots, x_m$ be vectors in some normed space X such that $\|x_i\| \geq 1$ for all i and*

$$\|\sum_{i=1}^{m} a_i x_i\| \leq L \max_{1\leq i\leq m} |a_i|$$

for all sequences of coefficients $a_1, \ldots, a_m \in \mathbb{R}$. Then X contains a sequence $y_1, \ldots, y_{\lfloor\sqrt{m}\rfloor}$ satisfying $\|y_i\| \geq 1$ for all i and

$$\|\sum_{i=1}^{\lfloor\sqrt{m}\rfloor} a_i y_i\| \leq \sqrt{L} \max_{1\leq i\leq\lfloor\sqrt{m}\rfloor} |a_i|$$

for all sequences of coefficients $a_1, \ldots, a_{\lfloor\sqrt{m}\rfloor} \in \mathbb{R}$.

Proof. Let σ_j, $j = 1, \ldots, \lfloor\sqrt{m}\rfloor$ be disjoint subsets of $\{1, \ldots, m\}$. if for some j

$$\|\sum_{i\in\sigma_j} a_i x_i\| \le \sqrt{L} \max_{i\in\sigma_j} |a_i|$$

for all sequences of coefficients, we are done. Otherwise, for each j we can find a vector $y_j = \sum_{i\in\sigma_j} a_i x_i$ such that $\|y_j\| = 1$ and $\sqrt{L}\max_{i\in\sigma_j}|a_i| < 1$. But then,

$$\|\sum_{j=1}^{\lfloor\sqrt{m}\rfloor} b_j y_j\| \le L \max_{j,\ i\in\sigma_j} |b_j a_i| \le L \max_j |b_j| \sqrt{L^{-1}} = \sqrt{L}\max_j |b_j|.$$

□

Corollary 4.4. *If* ℓ_∞^m L *embeds into a normed space* X*, then for all* $0 < \epsilon < 1$, ℓ_∞^k $\frac{1+\epsilon}{1-\epsilon}$ *embeds into* X *for* $k \sim m^{\epsilon/\log L}$.

Proof. By iterating the Lemma (with a small cheating concerning the $\lfloor\cdot\rfloor$ notation), for all positive integer t there is a sequence of length $k = m^{2^{-t}}$ of norm one vectors $x_1, \ldots, x_k$ in X satisfying

$$\|\sum_{i=1}^{k} a_i x_i\| \le L^{2^{-t}} \max |a_i|$$

for all coefficients. Pick a t such that $L^{2^{-t}} = 1 + \epsilon$ (approximately); i.e., $2^{-t} = \frac{\log 1+\epsilon}{\log L} \sim \frac{\epsilon}{\log L}$ (and thus $k \sim m^{\epsilon/\log L}$). To get a similar lower bound on $\|\sum_{i=1}^k a_i x_i\|$, assume without loss of generality that $\max|a_i| = a_1$. Then

$$\begin{aligned} \|\textstyle\sum_{i=1}^k a_i x_i\| = & \ \|2a_1x_1 - (a_1x_1 - \textstyle\sum_{i=2}^k a_i x_i)\| \ge 2a_1 - \|a_1x_1 - \textstyle\sum_{i=2}^k a_i x_i\| \\ \ge & \ 2a_1 - (1+\epsilon)a_1 = (1-\epsilon)\max|a_i|. \end{aligned}$$

□

We are left with the task of proving Proposition 4.2. We begin with

Claim 4.5. *Let* $x_1, \ldots, x_n$ *be normalized vectors in a normed space. Then for all real* $a_1, \ldots, a_n$,

$$\mathrm{Prob}_{\epsilon_i=\pm1}(\|\sum_{i=1}^{n} \epsilon_i a_i x_i\| < \max_{1\le i\le n} |a_i|) \le 1/2.$$

Proof. Assume as we may $a_1 = \max_{1\leq i\leq n}|a_i|$. If $\|a_1x_1 + \sum_{i=2}^n \epsilon_i a_i x_i\| < a_1$ then

$$\|a_1x_1 - \sum_{i=2}^n \epsilon_i a_i x_i\| \geq 2a_1 - \|a_1x_1 + \sum_{i=2}^n \epsilon_i a_i x_i\| > a_1$$

and thus

$$P(\|\sum_{i=1}^n \epsilon_i a_i x_i\| > a_1) \geq P(\|\sum_{i=1}^n \epsilon_i a_i x_i\| < a_1).$$

So,

$$\begin{aligned} 1 \geq \;& P(\|\textstyle\sum_{i=1}^n \epsilon_i a_i x_i\| \neq \max|a_i|) \\ = \;& P(\|\textstyle\sum_{i=1}^n \epsilon_i a_i x_i\| < a_1) + P(\|\sum_{i=1}^n \epsilon_i a_i x_i\| > a_1) \\ \geq \;& 2P(\|\textstyle\sum_{i=1}^n \epsilon_i a_i x_i\| < a_1). \end{aligned}$$

□

Remark: If $x_1 = x_2$, $a_1 = a_2 = 1$ and $a_3 = \cdots = a_n = 0$ then the 1/2 in the statement of Claim 4.5 cannot be replaced by any smaller constant.

Proposition 4.6. *Let $x_1, \ldots, x_n$ be vectors in a normed space with $\|x_i\| \geq 1/10$ for all i. Then, for n large enough,*

$$P(\|\sum_{i=1}^n g_i x_i\| < \frac{\sqrt{\log n}}{100}) \leq 2/3.$$

Proof.

$$\begin{aligned} & P(\|\textstyle\sum_{i=1}^n g_i x_i\| < \frac{\sqrt{\log n}}{100}) \\ & \quad \leq P(\|\textstyle\sum_{i=1}^n g_i x_i\| < \frac{\sqrt{\log n}}{100} \;\&\; \frac{\sqrt{\log n}}{100} < \max_{1\leq i\leq n}|g_i|\|x_i\|) \\ & \qquad + P(\max_{1\leq i\leq n}|g_i|\|x_i\| \leq \tfrac{\sqrt{\log n}}{100}) \\ & \quad \leq P(\|\textstyle\sum_{i=1}^n g_i x_i\| < \max_{1\leq i\leq n}|g_i|\|x_i\|) + P(\max_{1\leq i\leq n}|g_i| \leq \frac{\sqrt{\log n}}{10}) \\ & \quad \leq \tfrac{1}{2} + (1 - e^{-c\log n})^n \qquad \text{for } n \text{ large enough, by Claim 4.5} \\ & \quad = \tfrac{1}{2} + e^{-n^{1-c}} \end{aligned}$$

□

In the proof of Proposition 4.2 we shall use a theorem of Alon and Milman [AM] (see [Ta] for a simpler proof) which have a very similar statement: Gaussians are replaced by random signs and $\sqrt{\log n}$ by a constant.

Theorem 4.7. *(Alon and Milman) Let $(X, \|\cdot\|)$ be a normed space and let $x_1, \dots, x_n$ be a sequence in X satisfying $\|x_i\| \geq 1$ for all i and*

$$E_{\epsilon_i=\pm 1}\Big(\|\sum_{i=1}^{n} \epsilon_i x_i\|\Big) \leq L. \tag{4.2}$$

Then, there is a subspace of X of dimension $k \geq \frac{n^{1/2}}{CL}$ which is CL-isomorphic to ℓ_∞^k. C is a universal constant.

Proof of Proposition 4.2. Let $\sigma_1, \dots, \sigma_{\lfloor\sqrt{n}\rfloor} \subset \{1, \dots, n\}$ be disjoint with $|\sigma_j| = \lfloor\sqrt{n}\rfloor$ for all j. We'll show that there is a subset $J \subset \{1, \dots, \lfloor\sqrt{n}\rfloor\}$ of cardinality at least $\frac{\sqrt{n}}{4}$ and there are $\{y_j\}_{j\in J}$ with y_j supported on σ_j such that $\|y_j\| = 1$ for all $j \in J$ and

$$E_{\epsilon_i=\pm 1}\Big(\|\sum_{j\in J} \epsilon_j y_j\|\Big) \leq 80L.$$

We then apply the theorem above.

To show this notice that the events $\|\sum_{i\in\sigma_j} g_i x_i\| < \frac{\sqrt{\log n}}{200}$, $j = 1, \dots, \lfloor\sqrt{n}\rfloor$, are independent and by Proposition 4.6 have probability at most 2/3 each. So with probability at least 1/2 there is a subset $J \subset \{1, \dots, \lfloor\sqrt{n}\rfloor\}$ with $|J| \geq \frac{\lfloor\sqrt{n}\rfloor}{4}$ such that $\|\sum_{i\in\sigma_j} g_i x_i\| > \frac{1}{200}\sqrt{\log n}$ for all $j \in J$. Denote the event that such a J exists by A. Let $\{r_j\}_{j=1}^{\lfloor\sqrt{n}\rfloor}$ be a Rademacher sequence independent of the original Gaussian sequence. We get that

$$\begin{aligned} L\sqrt{\log n} \geq\ & E_g\Big(\|\textstyle\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}\sum_{i\in\sigma_j} g_i x_i\|\Big) = E_r E_g\Big(\|\textstyle\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} r_j \sum_{i\in\sigma_j} g_i x_i\|\Big) \\ \geq\ & E_r E_g\Big(\|\textstyle\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} r_j \sum_{i\in\sigma_j} g_i x_i\| \mathbf{1}_A\Big) \\ \geq\ & \tfrac{1}{2} E_g\Big(\Big(E_r\|\textstyle\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} r_j \sum_{i\in\sigma_j} g_i e_i\|\Big)\Big/ A\Big). \end{aligned}$$

It follows that for some $\omega \in A$, there exists a $J \subset \{1, \dots, \lfloor\sqrt{n}\rfloor\}$ with $|J| \geq \frac{\lfloor\sqrt{n}\rfloor}{4}$ such that putting $\bar{y}_j = \sum_{i\in\sigma_j} g_i(\omega) x_i$, one has $\|\bar{y}_j\| > \frac{1}{200}\sqrt{\log n}$ for all $j \in J$ and

$$E_r\Big(\|\sum_{j\in J} r_j \bar{y}_j\|\Big) \leq 2L\sqrt{\log n}.$$

Take $y_j = \bar{y}_j / \|\bar{y}_j\|$. □

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