Algorithmic Game Theory – Practice Questions

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Comment: Please take into account these are draft questions, and might contain a typo or some ambiguity. The structure of the exam will be similar, but with some choice. You will have 3 hours. No books, notes, or other external material will be allowed.

Part I

For each statement below, determine if it is TRUE or FALSE, and explain your answer in 1-3 sentences (i.e. sketch of the proof or a convincing argument why it is correct/wrong).

a. In the well known game tic-tac-toe, the first player has a dominant strategy.

   (Note: if you never played tic-tac-toe, it suffices to say that when unexperienced players play the game, it may happen that either player wins, but when experienced players play, the game always ends in a draw.)

b. For every undirected graph $G(V, E)$ and vertex $v_0 \in V$, the following function $c : 2^E \to \mathbb{R}$ is submodular:

   $$c(S) = \text{size of the connected component of } v_0 \text{ in the subgraph } G_S = (V, E \setminus S) \ .$$

   (In other words, $c(S)$ is the number of vertices that are connected to v_0 in G_S.)

c. Let (G, r, c) be a selfish-routing instance, i.e., G is a digraph with and k source-sink pairs (s_i, t_i) whose flow requirements are $r_i \geq 0$. Assume $c_e(f)$ is an affine cost function for edge e. Then a flow f that is optimal in the sense of minimizing the cost function $C(f) = \sum_{e \in E} c_e(f_e) \cdot f_e$, is (always) a unique Wardrop equilibrium.

d. Recall the **Clark Pivot Rule**: Let A be the set of alternatives and $v_i : A \mapsto R$ be the valuation functions of the n players. Let f be a *social choice function* that maximize welfare (that is maximizes over $a \in A$ the value $\sum_{i=1}^{n} v_i(a)$). When $a = f(v_1, v_2, \ldots v_n)$ the payment to player is $p_i(v_1, v_2, \ldots v_n) = max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{i \neq j} v_i(a)$.

   Is the Clark Pivot Rule incentive compatible when there may be collusions between the players?

Part II

Answer the following questions.

1. There are n bins. At each round "nature" throws a peanut into one of the bins. If you (the player) are standing at the chosen bin you catch the peanut, and otherwise you miss it. Nature is completely oblivious to your location when it throws the peanut. You may choose to move to any bin before any round. The game ends as soon as d peanuts were thrown at any single bin, *independently of whether they were caught or not.*

   Goal: to catch as many peanuts as possible.

   What guarantee can you have as a function of d and n? Describe the algorithm that will achieve this guarantee.

2. Show an algorithm for stable matching and prove that it produces a stable matching and works in polynomial time.

3. Show that if a multiplayer game is given in standard form, then there is a polynomial time algorithm that decides whether the game has a correlated equilibrium in which every player has expected payoff that is nonnegative.

   Recall that a correlated equilibrium in a multi-player game is a probability distribution over profiles of pure strategies such that if

   (a) a player tries to maximize his expected payoff, and

   (b) there is a trusted party that draws a profile strategy from the probability distribution and shows each player only his part of the profile, and

   (c) a player believes that all other players will follow the recommendations that they receive,

   then following the recommendation is an optimal strategy.

4. Recall the following cost-sharing game called *the set cover game*: There is a set A of n agents, and a collection $\mathcal{E}$ of subsets of A (that together covers all of A i.e. $\cup_{T\in\mathcal{E}}T = A$). For every subset of agents $S \subseteq A$, the cost $c(S)$ is minimum number of sets from $\mathcal{E}$ that covers all of S, namely the minimum k such that there are k sets $T_1, \ldots, T_k \in \mathcal{E}$ with $S \subseteq \cup_{l=1}^{k} T_l$.

   Show that for the special case of this game where every agent $j \in A$ belongs to (can be covered by) at most two sets in $\mathcal{E}$, there are instances with an empty γ-core for $\gamma \approx 1/2$.

   Hint: Consider first the instance where $n = 3$ and there is a set for every pair of agents.

5. Let (A, c) be a cost-sharing game. *Shapley's cost-allocation* is defined as follows: Let $\pi : 1, 2, ..., n \mapsto A$ be a random permutation of the agents A, and define the cost-share of each agent j to be

$$\alpha_j = \mathbb{E}_\pi[c(\pi(\{1, 2, \ldots, \pi^{-1}(j)\})) - c(\pi(\{1, 2, \ldots, \pi^{-1}(j) - 1\}))].$$

(In words, α_j is the average, over all permutations, of the increase in cost when adding the agent j to the serviced-set.) Prove for submodular games, Shapley's cost-allocation is in the core (i.e. 1-core).

Recall: a game is submodular if

$$\forall S, T \subseteq A, \quad c(S) + c(T) \geq c(S \cup T) + c(S \cap T).$$

Hint: Assume first that π is a fixed permutation, say the identity.

THE END.