Complexity Theory

Homework Set No. 2

Date Due: Nov 11th 2004

Please turn in the homework on time. Preferably - printed.

1. show that for every Boolean function $f : X \times Y \mapsto \{0,1\}$ and $z \in 0,1$ we have $N^z(f) \geq logD(f)$

2. for $x, y \in 0, 1^n$ let $GT(x,y)$ be 1 iff $x > y$ (when x and y represent integers between 0 and $2^n - 1$). Show that $N^1(GT)$ and $N^0(GT)$ are both $\Theta(n)$.

3. Let $f : 0, 1^n \mapsto \{0,1\}$. A decision tree for computing f is a labeled rooted binary tree. Each internal node is labeled with an index $1 \leq i \leq n$ and we think of the computation as branching according to the ith bit of the input. Each leaf is labeled with 0 or 1 and for input $x \in \{0,1\}^n$ the resulting path from the root should lead to a leaf labeled with $f(x)$. The complexity of a tree is the number of the queries along its longest path, or its depth. The deterministic decision tree complexity of f ($D_T(f)$) is the depth of the shallowest tree computing f.

A non deterministic decision tree for f and $z \in \{0,1\}$ (for verifying that $f(x) = z$ has, in addition to the query nodes, nondeterministic nodes where the computation can go either way. These nodes are not considered for the depth. The leafs are labeled either with z or with 'failure'. Each input x corresponds now to a set of paths from the root to a leaf. For $x \in \{0,1\}^n$ such that $f(x) = z$ there should be at least one path leading to a leaf labeled z and for $x \in \{0,1\}^n$ such that $f(x) \neq z$ there should be no path leading to a leaf labeled z. Let $N_T^z(f)$ denote the depth of the shallowest tree for verifying that $f(x) = z$.

Prove that for any $f : \{0,1\}^n \mapsto \{0,1\}$ we have

$$D_T(f) \leq N_T^0(f) N_T^1(f).$$

Hint: think of the proof for the communication compleixity case

Can you give an example where the result is tight?