# On Smooth Sets of Integers

Dedicated to the memory of Professor Shimon Even for his inspiration and encouragement

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**Abstract**

This work studies evenly distributed sets of integers — sets whose quantity within each interval is proportional to the size of the interval, up to a bounded additive deviation. Namely, for $\rho, \Delta \in \mathbb{R}$ a set $A$ of integers is $(\rho, \Delta)$-*smooth* if $\text{abs}(|I| \cdot \rho - |I \cap A|) < \Delta$ for any interval $I$ of integers; a set $A$ is $\Delta$-smooth if it is $(\rho, \Delta)$-smooth for some real number $\rho$. The paper introduces the concept of $\Delta$-smooth sets and studies their mathematical structure. It focuses on tools for constructing smooth sets having certain desirable properties and, in particular, on mathematical operations on these sets. Two additional papers of us build on the work of this paper and present practical applications of smooth sets to common and well studied scheduling problems.

One of those mathematical operations is composition of sets. For two infinite sets $A, B \subseteq \mathbb{N}$, the *composition* of $A$ and $B$ is the subset $D$ of $A$ such that, for all $i$, the $i$-th member of $A$ is in $D$ iff the $i$-th member of $\mathbb{N}$ is in $B$. This operator enables the partition of a $(\rho, \Delta)$-smooth set into two sets that are $(\rho_1, \Delta)$-smooth and $(\rho_2, \Delta)$-smooth, for any $\rho_1, \rho_2$ and $\Delta$ obeying some reasonable restrictions. Another powerful tool for constructing smooth sets is a partial one-to-one function $f$ from the unit interval into the natural numbers having the property that any real interval $X \subseteq [0, 1)$ has a subinterval $Y$ which is 'very close' to $X$ s.t. $f(Y)$ is $(\rho, \Delta)$-smooth, where $\rho$ is the length of $Y$ and $\Delta$ is a small constant.

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# 1 Introduction

## 1.1 Smooth Sets

This work studies evenly distributed sets of integers — sets whose quantity within each interval is proportional to the size of the interval, up to a bounded additive deviation. Namely, for $\rho, \Delta \in \mathbb{R}$ a set $A$ of integers is $(\rho, \Delta)$-*smooth* if $\text{abs}(|I| \cdot \rho - |I \cap A|) < \Delta$ for any interval $I$ of integers; a set $A$ is $\Delta$-smooth if it is $(\rho, \Delta)$-smooth for some real number $\rho$. The paper introduces the concept of $\Delta$-smooth sets and studies their mathematical structure. It focuses on tools for constructing smooth sets having certain desirable properties and, in particular, on mathematical operations on these sets. Two additional papers of us [16, 15] build on the work of this paper and present practical applications of smooth sets to common and well studied scheduling problems. Specifically, we construct schedules based on smooth sets whose characteristic zero-one sequence can be computed in a very efficient manner, in a constant time per integer.

The initial concept of a 1-smooth set, introduced by Lincoln, Even, and Cohn [14], captures the idea of a set of integers which is 'as evenly distributed as possible'. A similar concept of *balanced sets* (or *sequences*[1]) was introduced by Morse and Hedlund in a seminal paper back in 1939 [19]; a set $A$ of integers is *balanced* if $\text{abs}(|I \cap A| - |J \cap A|) \leq 1$ for any two intervals of integers $I$ and $J$ with $|I| = |J|$. Any 1-smooth set is clearly balanced but not the other way around (Subsection 1.2).

A distinctive feature of $(\rho, 1)$-smooth sets, with a rational $\rho$, is their uniqueness. Up to a translation, there is exactly one $(\rho, 1)$-smooth set for any rational[2] $\rho \in [0, 1]$ (Lemma 16). The balanced sets enjoy a similar property. For any such $\rho$, there are three balanced sets with rate[3] $\rho$, except for the case of $\rho \in \{0, 1\}$ in which there are two such sets [19, 7]. This uniqueness, however, gives rise to the major weaknesses of the 1-smooth (or balanced) sets. Unlike our general smooth sets, they are not closed under natural mathematical operations, and there are no systems of 1-smooth (or balanced) sets having certain desirable properties, as indicated in Subsection 1.2.

The general smooth sets are closed under many natural mathematical operations. A simple example is the union of two disjoint sets. If the two sets in question are $\Delta_1$-smooth and $\Delta_2$-smooth then their union is $(\Delta_1 + \Delta_2)$-smooth. (The 1-smooth or balanced sets are not closed under this operation). A very useful operator for constructing smooth sets is set composition. For (finite or infinite) $A, B \subset \mathbb{N}$ this operator produces a subset $D$ of $A$ such that, for all $i$, the $i$-th member of $A$ is in $D$ iff the $i$-th member of $\mathbb{N}$ is in $B$ and $A$ has an $i$-th member. Namely, the *composition* of two sets $A \subset \mathbb{N}$ and $B \subset \mathbb{N}$ is $A \circ B \triangleq \{n \in A \; : \; |[0, n) \cap A| \in B\}$. As shown by Lemma 22, this operator is associative. The class of the smooth sets is closed under the composition operator and, under some restrictions, $A \circ B$ is as smooth as $A$ (Lemma 24).

Given a $(\rho, \Delta)$-smooth set $A$ and two positive numbers $\beta_1, \beta_2$ with $\beta_1 + \beta_2 = 1$, it is sometimes desirable to partition $A$ into two sets $A_1$ and $A_2$ s.t. each $A_i$ is $(\rho \cdot \beta_i, \Delta)$-smooth. By the following theorem, under some restrictions, this can be achieved via set composition. This partition is the cornerstone of one of our scheduling techniques presented in [16].

**Theorem 1:** Let $A \subset \mathbb{N}$ be $(\rho, \Delta)$-smooth and let $\beta_1 + \beta_2 = 1$ with $\beta_1, \beta_2 \leq (\Delta - 1)/\Delta$. Then there is a partition of $A$ into $\langle A_1, A_2 \rangle$ s.t. each $A_i$ is $(\beta_i \cdot \rho, \Delta)$-smooth. In fact, for any partition of $\mathbb{N}$ into $\langle B_1, B_2 \rangle$ where each $B_i$ is $(\beta_i, 1)$-smooth, the sets $A_1 = A \circ B_1$ and $A_2 = A \circ B_2$ satisfy the above statement.

Another method for constructing smooth sets concerns mapping of real intervals into subsets of $\mathbb{N}$. To that end, we need the following terminology. For a set $A$ of integers, if the limit $\lim_{|I| \to \infty}(|A \cap I|/|I|)$ exists where $I$ ranges over the intervals of integers[4] then this limit is called the *rate* of $A$. Clearly, a $(\rho, \Delta)$-smooth set has rate $\rho$. For a real interval $X$, let $\|X\|$ denote its length. A *smooth shuffle*

[1] A set of of integers is sometimes identified with its characteristic zero-one sequence.

[2] But there are $2^{\aleph_0}$ $(\rho, 1)$-smooth sets for any irrational $\rho$ (Lemma 17).

[3] The rate of a $(\rho, \Delta)$-smooth set of integers is $\rho$ and the general concept of rate is defined shortly.

[4] That is, there is a real number $\rho$ s.t. for any $\epsilon > 0$, for all sufficiently large intervals $I$, we have $\text{abs}(|A \cap I|/|I| - \rho) < \epsilon$.

is a partial one-to-one function from the unit interval into the natural numbers having the following properties:

a. Any real interval $X \subset [0,1)$ is mapped onto a subset of the natural numbers whose rate is $\|X\|$.

b. For some constant $\Delta$, the image of every real interval $X \subset [0,1)$ is $\Delta$-smooth.

A smooth shuffle would be a powerful tool for constructing smooth sets with a variety of desired properties. Unfortunately, we failed to find such a function and we conjecture that there is no smooth shuffle. Fortunately, there exists a (plain) *shuffle* — a partial one-to-one function from the unit interval into the natural numbers obeying only statement (a) above. Moreover, there is a shuffle s.t. the image of any interval can be 'smoothed' into a smooth set by removing a small fraction of the interval, as stated by the next theorem which is the foundation of our scheduling technique presented in [16, 15].

**Theorem 2** There is a shuffle $f$ s.t. for any $\epsilon > 0$ and for any real interval $X \subset [0,1)$, there exists an interval $Y \subset X$ s.t. $\|X\|(1-\epsilon) \leq \|Y\|$ and $f(Y)$ is $O(\log(1/\epsilon))$-smooth.

## 1.2 Related Work

**Balanced Sets.** The concept of balanced sets (defined in Subsection 1.1) was introduced by Morse and Hedlund in a seminal paper back in 1939 [19] and has been studied extensively since, e.g., [9, 24, 8, 6, 23, 7]. Balanced sets are used in various fields such as theoretical computer science [9], scheduling [1, 20], and number theory [24].

It is worth mentioning that Morse and Hedlund did not use the term 'balanced' and refer to these sequences as *Sturmian Trajectories*; their motivation for studying these sequences arose, not from even distribution, but from differential equations: "They [Sturmian Trajectories] may be used to characterize the distribution of the zeros of the solutions of a differential equation of the form $y'' + f(x)y = 0$, where $f(x)$ is a periodic function of $x$".

The notion of balanced set does not exactly capture the concept of being 'as evenly distributed as possible' while having a given rate. As mentioned by Morse and Hedlund, the sequence $\cdots 0001000 \cdots$ is Sturmian (balanced) with rate 0 (and characterizes the distribution of the zeros of the function $y(x) \triangleq x$ which is a solution of a differential equation of the above form). Another example is the sequence $\cdots 010101101010 \cdots$ which is balanced with rate 1/2. Clearly, each of these sequences has an irregular area and is not 'perfectly distributed'. Such an irregularity can happen at most in one place and only when the rate is rational [19, 7].

**1-smooth sets.** This irregularity is eliminated in the notion of 1-smooth set introduced in 1969 by Lincoln, Even, and Cohn [14]. These sets are the 'perfectly distributed' subclass of the balanced sets. A geometric manifestation of the 1-smooth sets are *billiard sequences* [2]. Such a sequence is generated by the trajectory of an ideal ball on a billiard table as follows. Hitting the north or south side produces a 1, while hitting the other sides produces a 0. Generalizations of billiard sequences to higher dimensions have also been studied [2, 3].

**Bracket Sequences.** Another related class of sets (sequences) are the *Bracket* (or *Beatty*) *Sequences* — sets of the form $\lfloor \alpha \cdot \mathbb{Z} + \beta \rfloor$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$ — investigated in [4, 10, 11, 17, 18]. The Bracket Sequences constitute a proper subclass of the 1-smooth sets which is less convenient to work with since this subclass is not closed under the transformations $A \mapsto (\mathbb{Z} \setminus A)$ and $A \mapsto (-A)$, as shown in Section 4; clearly, the class of the $\Delta$-smooth sets, for any fixed $\Delta$, as well as the class of the balanced sets, is closed under these transformations. In fact, the class of the 1-smooth sets is the closure of the class of the Bracket Sequences under any single one of the above transformations (Section 4).

As mentioned above, an important aspect of smooth sets is their closure under natural mathematical operations and, unfortunately, this is not the case with the 1-smooth sets (as well as with the balanced sets). An example is the union of two disjoint sets; e.g., the sets $5 \cdot \mathbb{Z}$ and $(5 \cdot \mathbb{Z} + 1)$ are disjoint and 1-smooth while their union is not 1-smooth. Another example is set composition: e.g., the set $A = (3 \cdot \mathbb{N} \cup (3 \cdot \mathbb{N} + 1))$ is 1-smooth, but $A \circ A$ is not 1-smooth. In addition, as shown shortly, 1-smooth sets are too limited for many applications. For these reasons we generalize the concept of 1-smooth sets and introduce the above concept of $\Delta$-smooth sets.

**Persistent Scheduling.** The subject of scheduling demonstrates several practical applications in which general smooth sets are highly attractive but 1-smooth sets are too limited. Consider the task of scheduling persistent clients on a single slot-oriented resource so that each client is served in a disjoint set of slots (natural numbers) having a pre-defined rate. It is usually desired that these sets are $\Delta$-smooth for a small $\Delta$. In this context, 1-smooth (or balanced) sets are too limited; consider the case of only two clients, one with rate 1/3 and the other with rate 1/2. Now, an $(1/j, 1)$-smooth set for $j \in \mathbb{N}$ is the range of an arithmetic sequence whose difference is $j$. Hence, there are no two disjoint 1-smooth sets having rates 1/2 and 1/3. This problem is overcome by our concept of $\Delta$-smooth sets.

In certain scheduling applications, the scheduling of each slot should be computed extremely fast. Two additional papers of us [16, 15] build on the work of this paper to construct extremely efficient schedules in which the per-slot computation takes $O(1)$ time. Of these two papers, the former one focuses on a distributed environment and the latter focuses on a centralized environment.

**Scheduling of One-time Jobs.** Consider the task of scheduling an infinite sequence of identical one-time jobs on several machines (resources) having (possibly) different processing speeds. It is required that the rate of the set of jobs (natural numbers) assigned to each resource is proportional to its speed. Note that in this context, unlike the previous one of persistent scheduling, it is required that all the natural numbers are assigned. Tijdeman [21] addressed this problem under a different terminology[5] and has shown that for any vector of speeds there is a solution in which the set assigned to each resource is 2-smooth. Altman at el. [1] showed that a schedule is optimal (w.r.t. a natural objective function) when the allocated sets are 1-smooth (or balanced).

**Smooth Partition.** The last scheduling task is related to the following question: 'For what vectors $\langle \rho_1, \cdots \rho_m \rangle$ of rates is there a partition of the natural numbers to 1-smooth sets[6] having these rates?'. The above question has been studied extensively [10, 18, 22, 24, 25, 1]; only partial results are known, but it is already known that such vectors are rare. From a practical point of view, this rareness is a serious deficiency of the 1-smooth sets which is overcome by our concept of $\Delta$-smooth sets. Returning to the question of partitioning $\mathbb{N}$ into 1-smooth sets, in the case of $m > 2$ and under the requirement that the rates are distinct, the following conjecture of Fraenkel is widely accepted:

**Fraenkel's Conjecture:** There is a partition conforming to such a vector $v$ of rates iff the elements of $v$ sum to 1 (of course) and constitute a geometric sequence with ratio 2.

This conjecture has been established for $m = 3$ [18, 22], $m = 4$ [1], $m = 5$ [24] and $m = 6$ [25].

**Discrepancy Theory.** Discrepancy Theory, see e.g. [5], studies various types of sets that are 'evenly distributed'. For example, "How uniformly can $N$ points in the unit cube be distributed relative to a given family of 'nice' sets (e.g., boxes with sides parallel to the coordinate axes, rotated boxes, balls, all convex sets, etc.)." Thus, this theory studies issues which bear relation to the notions of balanced and smooth sets and, in particular, to our open question about a smooth shuffle.

The rest of the paper is organized as follows. Section 2 presents some elementary properties of smooth sets; Section 3 introduces another notion of smoothness; Section 4 presents some special properties of the 1-smooth sets; Section 5 provides a recursive characterization of smooth sets; Section 6 studies composition of sets of naturals numbers and shows that this operation preserves smoothness; and Section 7 presents and studies our shuffle function.

To distinguish between sets of integers and sets of real numbers, we use uppercase letters from the head of the Latin alphabet (e.g. $A, B$) to denote sets of the former type and uppercase letters from the tail of the Latin alphabet (e.g. $X, Y, Z$) to denote sets of the latter type.

[5] Tijdman refers to this problem as the *Chairman Assignment Problem* in which a union of states, each having a positive weight, should select a yearly union chairman in a 'fair' manner.

[6] It has been established [11, 24, 1] that w.r.t. this question, and other similar questions, there is no difference between 1-smooth sets, balanced sequences and Bracket Sequences. Also, it does not matter whether the partition is of the integers or of the natural numbers.

## 2 Smooth Sets

This section presents some elementary properties of smooth sets. Actually, it uses a relative concept of smoothness, as follows. Let $W$ be a set of integers. An *interval of* $W$ is a subset of $W$ of the form $W \cap J$ where $J$ is a (finite) interval of integers. A set $A$ is $(\rho, \Delta)$-*smooth w.r.t.* $W$ if $A \subset W$ and $\mathrm{abs}(|I| \cdot \rho - |I \cap A|) < \Delta$ for any interval $I$ of $W$; such a set is $\Delta$-smooth if it is $(\rho, \Delta)$-smooth for some $\rho$. We are mainly interested in two types of smooth sets: those that are smooth w.r.t. $\mathbb{Z}$ (the integers) and those that are smooth w.r.t. $\mathbb{N}$ (the natural numbers, including zero). To this end, let $\mathbb{W}$ henceforth denote either the set $\mathbb{N}$ or the set $\mathbb{Z}$; the definitions and lemmas referring to $\mathbb{W}$ are valid for both interpretation. When the interpretation of $\mathbb{W}$ is clear from the context we usually omit the 'w.r.t. $\mathbb{W}$' qualifier.

The following lemma was established by Lincoln et al. [14] and is already implicit in the early work of Morse and Hedlund [19]; an alternative proof is provided by our Lemma 9.

**Lemma 1:** A $(\rho, 1)$-smooth set $A \subset \mathbb{W}$ exists for any $0 \leq \rho \leq 1$.

The number 1 in the above '$(\rho, 1)$-smooth set' is tight — replacing it with a larger number makes the lemma weaker and replacing it with a smaller number makes the lemma incorrect, since if $A$ is $(\rho, \Delta)$-smooth and $0 < \rho < 1$ then $\Delta > \rho$ and $\Delta > 1 - \rho$.

The following lemma, whose proof is immediate, states that smoothness is a local property, and is invariant under translation.

**Lemma 2:**

a. A set $A$ is $(\rho, \Delta)$-smooth w.r.t. $\mathbb{W}$ iff $A \cap I$ is $(\rho, \Delta)$-smooth w.r.t. $I$ for any interval $I$ of $\mathbb{W}$.

b. Let $i \in \mathbb{Z}$ and let $D \subset \mathbb{Z}$. Then $A$ is $(\rho, \Delta)$-smooth w.r.t. $D$ iff $A + i$ is $(\rho, \Delta)$-smooth w.r.t. $D + i$.

For $\rho \in \mathbb{R}$, $A \subset \mathbb{W}$ and a finite $D \subset \mathbb{W}$, define $\delta(A, D, \rho) \triangleq |D| \cdot \rho - |D \cap A|$. Recall that the rate of a set of integers $A$ is $\rho$ if for any $\epsilon > 0$, for all sufficiently large intervals $I$, we have $\mathrm{abs}(|A \cap I|/|I| - \rho) < \epsilon$. (Clearly, a set has at most one rate.) Note that if the rate of $A$ is $\rho$ and $I$ is an interval then $\delta(A, I, \rho)$ is the *deviation* of $|A \cap I|$ from its nominal value, $|I| \cdot \rho$. Therefore, this $\delta$ function is handy in the context of smooth sets; for example, $A$ is $(\rho, \Delta)$-smooth w.r.t. $\mathbb{W}$ if and only if $\mathrm{abs}(\delta(A, I, \rho)) < \Delta$ for all intervals $I \subset \mathbb{W}$. The $\delta$ function is 'additive', as stated by the following lemma whose proof is immediate.

**Lemma 3:** Let $\rho, \rho_1, \rho_2 \in \mathbb{R}$ and $A, A_1, A_2, D, D_1, D_2 \subset \mathbb{W}$ s.t. $D, D_1$ and $D_2$ are finite. Then:

a. If $D_1 \cap D_2 = \emptyset$ then $\delta(A, D_1 \cup D_2, \rho) = \delta(A, D_1, \rho) + \delta(A, D_2, \rho)$.

b. If $A_1 \cap A_2 = \emptyset$ then $\delta(A_1 \cup A_2, D, \rho_1 + \rho_2) = \delta(A_1, D, \rho_1) + \delta(A_2, D, \rho_2)$.

By the following lemma, smooth sets are closed under basic set operations.

**Lemma 4:** Let $A_1, A_2 \subset \mathbb{W}$ be $(\rho_1, \Delta_1)$-smooth and $(\rho_2, \Delta_2)$-smooth, respectively. Then:

a. If $A_1 \cap A_2 = \emptyset$ then $A_1 \cup A_2$ is $(\rho_1 + \rho_2, \Delta_1 + \Delta_2)$-smooth.

b. The set $\mathbb{W} \setminus A_1$ is $(1 - \rho_1, \Delta_1)$-smooth.

c. If $A_2 \subset A_1$ then $A_1 \setminus A_2$ is $(\rho_1 - \rho_2, \Delta_1 + \Delta_2)$-smooth.

**Proof:** Statement (a) follows from Lemma 3(b). The same lemma implies that for any interval $I$ of $\mathbb{W}$:

$$\delta(A_1, I, \rho_1) + \delta(\mathbb{W} \setminus A_1, I, 1 - \rho_1) =$$
$$\delta(A_1 \cup (\mathbb{W} \setminus A_1), I, \rho_1 + (1 - \rho_1)) = \delta(\mathbb{W}, I, 1) = 0$$

Hence, $\delta(\mathbb{W} \setminus A_1, I, 1 - \rho_1) = -\delta(A_1, I, \rho_1)$ for any interval $I$, implying statement (b). Statement (c) follows from the facts that $A_1 \setminus A_2 = \mathbb{W} \setminus (A_2 \cup (\mathbb{W} \setminus A_1))$ and $A_2 \cap (\mathbb{W} \setminus A_1) = \emptyset$ and from statements (a) and (b). $\square$

## 3 Semi-smooth Sets

This section introduces another notion of smoothness which applies only to subsets of $\mathbb{N}$. The new notion is less elegant than the regular smooth notion, but is useful for investigating the latter one.

An *initial interval* of $\mathbb{N}$ is an interval of the form $I = [0, n)$ for some $n \in \mathbb{N}$. (Such an interval can be empty). Let $\mathbb{I}_0$ denote the set of the initial intervals of $\mathbb{N}$. For a real interval $Z$, a set $A$ is *$(\rho, Z)$-semi-smooth* if $A \subset \mathbb{N}$ and $\delta(A, I, \rho) \in Z$ for any $I \in \mathbb{I}_0$. This definition, for $I = \emptyset$, implies that if $A$ is $(\rho, Z)$-semi-smooth then $0 \in Z$. The following lemma is analogous to Lemma 4(b); its proof is straightforward.

**Lemma 5:** If $A$ is $(\rho, Z)$-semi-smooth then $\mathbb{N} \setminus A$ is $(1 - \rho, -Z)$-semi-smooth

Recall that $\|Z\|$ denotes the length of a real interval $Z$. A real interval is *semi-open* if it is open on one side and closed on the other; i.e., it is of the form of $[x, y)$ or $(x, y]$ for some $x, y \in \mathbb{R}$. Let $\mathcal{R}_\Delta \triangleq \{Z \mid Z \text{ is a semi-open real interval, } 0 \in Z, \text{ and } \|Z\| = \Delta\}$. Let $[\mathcal{R}_\Delta)$ and $(\mathcal{R}_\Delta]$ denote the sets of intervals of $\mathcal{R}_\Delta$ that are closed at the left and at the right sides, respectively. The following lemma establishes the connection between semi-smooth sets and smooth sets.

**Lemma 6:**

a. A set $A \subset \mathbb{N}$ is $(\rho, \Delta)$-smooth iff $A$ is $(\rho, Z)$-semi-smooth, for some $Z \in \mathcal{R}_\Delta$.

b. A set $A \subset \mathbb{Z}$ is $(\rho, \Delta)$-smooth iff $A \cap \mathbb{N}$ is $(\rho, Z)$-semi-smooth and $(-A - 1) \cap \mathbb{N}$ is[7] $(\rho, -Z)$-semi-smooth for some $Z \in \mathcal{R}_\Delta$.

**Proof:** Consider statement (a). The right to left direction follows from the fact that any interval of $\mathbb{N}$ is the difference between two initial intervals, and from Lemma 3(a). For the other direction, assume $A$ is $(\rho, \Delta)$-smooth and let $\overline{Z} = \{\delta(A, I, \rho) \mid I \in \mathbb{I}_0\}$. It follows from Lemma 3(a) that $\text{abs}(z_1 - z_2) < \Delta$ for any $z_1, z_2 \in \overline{Z}$. Thus, there exists an interval $Z \in \mathcal{R}_\Delta$ s.t. $\overline{Z} \subset Z$. Clearly, $A$ is $(\rho, Z)$-semi-smooth. The proof of statement (b) is similar, and is omitted. $\square$

The *cardinality function* of a set $A \subset \mathbb{N}$ is the function $\psi_A : \mathbb{I}_0 \to \mathbb{N}$ defined by $\psi_A(I) = |A \cap I|$. For a nonempty interval $I$, define $I^+ \triangleq I \cup \{\max(I) + 1\}$. The following lemma is straightforward.

**Lemma 7:** Let $f : \mathbb{I}_0 \to \mathbb{N}$. Then $f = \psi_A$ for some $A$ iff the following holds:

1. $f(\emptyset) = 0$.
2. $0 \le f(I^+) - f(I) \le 1$ for any $I \in \mathbb{I}_0$.

Clearly, $\psi_A$ determines $A$. The following lemma establishes the existence and the uniqueness of certain semi-smooth sets.

**Lemma 8:** Let $0 \le \rho \le 1$ and let $Z \in \mathcal{R}_1$. Then there is exactly one $(\rho, Z)$-semi-smooth set.

**Proof:** For $I \in \mathbb{I}_0$ define the following interval: $Q_I \triangleq |I| \cdot \rho - Z$. Clearly, $Q_I$ is a semi-open interval of size 1, implying that there is exactly one integer in $Q_I$. Denote this integer by $f(I)$. By definition, a set $A$ is $(\rho, Z)$-semi-smooth iff $\psi_A = f$. The function $f$ meets the requirements of Lemma 7, and is therefore the cardinality function of a unique set $A$. $\square$

An *infinite interval of* $\mathbb{W}$ is an infinite subset of $\mathbb{W}$ s.t. $\forall x, y, z \in \mathbb{W} : (x, y \in I \wedge x < z < y) \to z \in I$. A non-qualified 'interval' of $\mathbb{Z}$ or $\mathbb{N}$ means a finite interval. The following lemma shows that any set which is smooth w.r.t. a finite or infinite interval of $\mathbb{Z}$ can be extended to a set which is smooth w.r.t. $\mathbb{Z}$.

[7] This strange expression $(-A - 1)$ is due to the fact that the transformation $i \mapsto (-i - 1)$ is a bijection of the negative integers onto $\mathbb{N}$.

**Lemma 9:** Let $\Delta \geq 1$, let $0 \leq \rho \leq 1$, let $I$ be a finite or infinite interval of $\mathbb{Z}$ and let $A \subset I$. Then $A$ is $(\rho, \Delta)$-smooth w.r.t. $I$ iff $A = B \cap I$ for some set $B$ which is $(\rho, \Delta)$-smooth w.r.t. $\mathbb{Z}$.

Clearly, $\emptyset$ is $(\rho, 1)$-smooth w.r.t. $\emptyset$ for any $\rho$, hence Lemma 9 implies Lemma 1.

**Proof:** The right to left direction of the lemma follows from Lemma 2. Consider the other direction. By a straightforward application of (a trivial form of) König's infinity lemma, it suffices to show that for any two finite or infinite intervals $I$ and $J$ with $I \subset J$ and $|J \setminus I| = 1$ and for any set $A \subset I$ that is $(\rho, \Delta)$-smooth w.r.t. $I$ there is a set $B \subset J$ s.t. $B \cap I = A$ and $B$ is $(\rho, \Delta)$-smooth w.r.t. $J$.

By symmetry, we may assume that $I \subset \mathbb{N}$ and that $J \setminus I = \{-1\}$. Let $\overline{Z} = \{\delta(A, I', \rho) \mid I' \in \mathbb{I}_0 \ \& \ I' \subset I\}$. It follows from Lemma 3(a) that $\mathrm{abs}(z_1 - z_2) < \Delta$ for any $z_1, z_2 \in \overline{Z}$. Thus, there exists an interval $Z \in \mathcal{R}_\Delta$ s.t. $\overline{Z} \subset Z$. Since $\Delta \geq 1$, there is a set $B' \subset \{-1\}$ s.t. $\delta(B', \{-1\}, \rho) \in (-Z)$; let $B = A \cup B'$. To show that $B$ is $(\rho, \Delta)$-smooth w.r.t. $J$, let $J'$ be an interval of $J$. The hard case is when $(-1) \in J'$. By Lemma 3, $\delta(B, J', \rho) = \delta(A, (J' \setminus \{-1\}), \rho) + \delta(B', \{-1\}, \rho) \in Z - Z$. Since $Z \in R_\Delta$, $\mathrm{abs}(\delta(B, J', \rho)) < \Delta$. $\square$

Due to the last lemma, sets that are smooth w.r.t. $\mathbb{N}$ share many properties with those that are smooth w.r.t. $\mathbb{Z}$.

# 4 1-smooth sets

The 1-smooth sets have special properties that are not shared by other smooth sets and several of these properties are presented here. Some part of the following results, in particular Lemmas 16 and 17, are already known (in a different terminology) [14, 19, 7], but we prove all of them for the sake of completeness. Lemma 10 below provides an arithmetic characterization of the 1-smooth sets. To that end, for $\rho, x \in \mathbb{R}$ with $0 < \rho \leq 1$, define $A'_{\rho,x} \triangleq \lfloor \mathbb{Z} \cdot (1/\rho) + x \rfloor$ and $A''_{\rho,x} \triangleq \lceil \mathbb{Z} \cdot (1/\rho) + x \rceil$. Sets of the former type are called *Bracket Sequences* or *Beatty Sequences* and have been studied in [4, 10, 11, 17, 18].

**Lemma 10:** Let $0 < \rho \leq 1$. Then:

a. $A'_{\rho,x}$ is $(\rho, 1)$-smooth for any $x$.

b. $A''_{\rho,x}$ is $(\rho, 1)$-smooth for any $x$.

c. Any $(\rho, 1)$-smooth set is either of the form of Statement (a) or of the form of Statement (b).

d. Any $(\rho, 1)$-smooth set with $\rho \in \mathbb{Q}$ is both of the form of Statement (a) and of the form of Statement (b).

e. If $\rho$ is irrational then there is a $(\rho, 1)$-smooth set which is not of the form of Statement (b) and there is such a set which is not of the form of Statement (a).

f. The above statements holds when $A'_{\rho,x}$ and $A''_{\rho,x}$ are replaced with $A'_{\rho,x} \cap \mathbb{N}$ and $A''_{\rho,x} \cap \mathbb{N}$ and the smoothness is w.r.t. $\mathbb{N}$ instead of $\mathbb{Z}$.

To prove Lemmas 10, we use the following auxiliary lemmas. The first one follows from the fact that for any real $x$, $-A'_{\rho,x} = A''_{\rho,-x}$.

**Lemma 11:** The transformation $A \mapsto (-A)$ is a bijection from $\left\{A'_{\rho,x} \mid \rho, x \in \mathbb{R},\ 0 < \rho \leq 1\right\}$ onto $\left\{A''_{\rho,x} \mid \rho, x \in \mathbb{R},\ 0 < \rho \leq 1\right\}$ and vice versa.

The following fact is well known [12].

**Lemma 12:** For any irrational $\sigma$, the set $(\sigma \cdot \mathbb{N}(\mathrm{mod}\ 1))$ is dense in $[0, 1)$.

**Lemma 13:** Let $0 < \rho \leq 1$. Then:

a. $A = A'_{\rho,x} \cap \mathbb{N}$ for some $x \in \mathbb{R}$ iff $A$ is $(\rho, Z)$-semi-smooth for some $Z \in (\mathcal{R}_1]$.

a'. If $\rho$ is irrational then for any $A'_{\rho,x}$ there is a unique $Z \in (\mathcal{R}_1]$ satisfying the right side of statement (a).

b. $A = A'_{\rho,x}$ for some $x \in \mathbb{R}$ iff $A \cap \mathbb{N}$ is $(\rho, Z)$-semi-smooth and $(-A-1) \cap \mathbb{N}$ is $(\rho, -Z)$-semi-smooth for some $Z \in (\mathcal{R}_1]$.

b'. If $\rho$ is irrational and $(\mathbb{Z} \cdot (1/\rho) + x) \cap \mathbb{Z} \neq \emptyset$ then there is a unique $Z \in \mathcal{R}_1$ (sic) satisfying the right side of statement (b).

c. All the above statements hold when $A'_{\rho,x}$ is replaced with $A''_{\rho,x}$ and $(\mathcal{R}_1]$ is replaced with $[\mathcal{R}_1)$.

**Proof:** We use the following fact: for any $0 < \rho \leq 1$, $y = x \cdot \rho (\text{mod } 1)$ and $I \in \mathbb{I}_0$ the following holds:

$$\begin{aligned} |I \cap A'_{\rho,x}| &= \min\{j \mid \lfloor (j+y)/\rho \rfloor \notin I\} \\ &= \min\{j \mid |I| \leq \lfloor (j+y)/\rho \rfloor\} \\ &= \min\{j \mid |I| \leq (j+y)/\rho\} \\ &= \min\{j \mid j \geq |I| \cdot \rho - y\} \\ &= \lceil |I| \cdot \rho - y \rceil \end{aligned}$$

Thus,

$$\delta(A'_{\rho,x}, I, \rho) = y + ((|I| \cdot \rho - y) - \lceil |I| \cdot \rho - y \rceil) \tag{1}$$

We prove a stronger version of statements (a) and (b) in which, not only the required objects $Z$ or $x$ exists, but they satisfy $Z = (x \cdot \rho (\text{mod } 1) - 1,\ x \cdot \rho (\text{mod } 1)]$. We refer to these enhanced versions as statements (â) and ($\hat{\text{b}}$). Clearly, $\{r - \lceil r \rceil \mid r \in \mathbb{R}\} = (-1, 0]$; this and Eq. (1) imply the left to right direction of statement (â). The other direction of statement (â) follows from the previous direction for $x = \max(Z)/\rho$ and from Lemma 8. Consider statement (a'). By Lemma 12, the set $((\rho \cdot \mathbb{N} - y)(\text{mod } 1))$ is dense in $[0, 1)$; hence, by Eq. (1), $\{\delta(A, I, \rho) \mid I \in \mathbb{I}_0\}$ is dense in $Z$. Thus, $A'_{\rho,x}$ is not $(\rho, Z')$-semi-smooth for any other $Z' \in (\mathcal{R}_1]$.

Consider the left to right direction of statement ($\hat{\text{b}}$). By statement (â), $A = A'_{\rho,x} \cap \mathbb{N}$ is $(\rho, Z)$-semi-smooth, and thus it suffices to prove that for any $I \in \mathbb{I}_0$, $\delta(-A-1, I, \rho) \in (-Z)$. Let $I \in \mathbb{I}_0$, $i = |I|$ and $\delta = \delta(-A-1, I, \rho)$. It is easy to check that $\delta = \delta(A+i, I, \rho)$ and $A + i = A'_{\rho,x+i}$. By statement (â), $(A+i) \cap \mathbb{N}$ is $(\rho, Z')$-semi-smooth with $Z' = Z + i \cdot \rho + k$ for some $k \in \mathbb{Z}$. By definition, $\delta \equiv i \cdot \rho (\text{mod } 1)$, implying that $Z' = Z + \delta + k'$ for some $k' \in \mathbb{Z}$. Since $0 \in Z$, $\delta \in Z'$ and $\|Z'\| = 1$, we have $k' = 0$. The fact that $0 \in Z'$ implies that $-\delta \in Z$. The other direction of statement ($\hat{\text{b}}$) follows from the previous direction for $x = \max(Z)/\rho$ and from Lemma 8.

Consider statement (b'). By previous arguments, $Q = \left\{\delta(A'_{\rho,x}, I, \rho) \mid I \in \mathbb{I}_0\right\}$ is dense in $Z$. Let $n \in (\mathbb{Z} \cdot (1/\rho) + x) \cap \mathbb{Z})$. Then $n \cdot \rho - \rho \cdot x \in \mathbb{Z}$. We consider only the case of $n \in \mathbb{N}$, the other case is similar. By Eq. (1), $\delta(A'_{\rho,x}, [0, n), \rho) = y = \max(Z)$. Hence, $Z$ is the unique interval of $\mathcal{R}_1$ with $Q \subset Z$. Statement (c) follows from the following variant of Lemma 11. The transformation $A \mapsto (-A-1)$ is a bijection from the class of the $A'_{\rho,x}$ sets onto the class of the $A''_{\rho,x}$ sets and is the inverse of itself. □

**Lemma 14:** The transformation $A \mapsto (\mathbb{Z} \setminus A)$ is a bijection from $\left\{A'_{\rho,x} \mid \rho, x \in \mathbb{R},\ 0 < \rho < 1\right\}$ onto $\left\{A''_{\rho,x} \mid \rho, x \in \mathbb{R},\ 0 < \rho < 1\right\}$ and vice versa.

**Proof:** The transformation in question is one-to-one and is the inverse of itself; hence it suffice to show that it maps the first class onto the second one. The fact that it maps the first class into the second class is implied by the left to right direction of Lemma 13(b), Lemma 5 and the right to left direction of Lemma 13(b,c). The fact that the mapping is onto is implied by the left to right direction of Lemma 13(b,c), Lemma 5 and the right to left direction of Lemma 13(b). □

Lemma 13(b') implies that the two classes, $\left\{A'_{\rho,x} \mid \rho, x \in \mathbb{R},\ 0 < \rho < 1\right\}$ and $\left\{A''_{\rho,x} \mid \rho, x \in \mathbb{R},\ 0 < \rho < 1\right\}$, are distinct. Hence, lemmas 11 and 14 imply that the class of Bracket Sequences is not closed under each of the transformations $A \mapsto (-A)$ and $A \mapsto (\mathbb{Z} \setminus A)$.

**Proof of Lemma 10:** Statements (a), (b) and (c) follow from Lemma 13(b,c) and from Lemma 6. Statement (d) follows from Lemma 15 below and statement (e) follows from Lemma 13(b',c). Statement (f) w.r.t. statements (a) to (d) follows from these statements and from Lemma 9. Consider statement (f) w.r.t. statement (e). Let $x \in \mathbb{R}$ with $(\mathbb{Z} \cdot (1/\rho) + x) \cap \mathbb{Z} \neq \emptyset$; by Lemma 13(b'), $A'_{\rho,x}$ is not of the form of statement (b) (of the current lemma). Let $i \in (\mathbb{Z} \cdot (1/\rho) + x) \cap \mathbb{Z}$. If $i \in \mathbb{N}$ then it is easily verified that $A'_{\rho,x} \cap \mathbb{N}$ is not of the form of statement (b); otherwise, $-i \in (\mathbb{Z} \cdot (1/\rho) - x) \cap \mathbb{N}$ and $A'_{\rho,-x} \cap \mathbb{N}$ is not of the form of statement (b). The second part of statement (e), concerning a set which is not of the form of statement (a), follows from Lemmas 14 and 5. $\square$

It is not hard to verify that the two sets provided by Lemma 10(e) are unique, up to a translation, and the transformation $A \mapsto (-A)$ swaps these sets.

**Lemma 15:** Let $A$ be $(\rho, 1)$-smooth w.r.t. $\mathbb{W}$ and let $0 < \rho = p/q$ and $p, q \in \mathbb{N}$. Then $A = A'_{\rho,i'/p} \cap \mathbb{W} = A''_{\rho,i''/p} \cap \mathbb{W}$ for some integers $0 \leq i', i'' < q$.

**Proof:** By Lemma 9, we can assume $\mathbb{W} = \mathbb{Z}$. Since $\rho \in \mathbb{Q}$, for any $x' \in \mathbb{R}$ there is a $x'' \in \mathbb{R}$, and vice versa, s.t. $A'_{\rho,x'} = A''_{\rho,x''}$. Thus, by Lemma 10, $A = A'_{\rho,x'} = A''_{\rho,x''}$ for some $x', x'' \in \mathbb{R}$. Consider $A'_{\rho,x'} = \lfloor \mathbb{Z} \cdot (1/\rho) + x' \rfloor$ and assume, w.l.o.g., that $0 \leq x' < q/p$. Let $i = \lfloor x' \cdot p \rfloor$. Clearly, $0 \leq i < q$ and $0 \leq x' - i/p < 1/p$; thus $\lfloor n \cdot q/p + x' \rfloor = \lfloor n \cdot q/p + i/p \rfloor$ for any $n \in \mathbb{Z}$. Hence, $A = A'_{\rho,i/p}$ as required. The other part of the proof is similar and is omitted. $\square$

The following two lemmas study the number of $(\rho, 1)$-smooth sets and show that in this regard there is a dramatic difference between the cases of a rational and irrational rate.

**Lemma 16:** Let $p/q = \rho \leq 1$, $p, q \in \mathbb{N}$, and $\gcd(p, q) = 1$. Then:

a. Up to a translation, there is exactly one $(\rho, 1)$-smooth set w.r.t. $\mathbb{Z}$.

b. There are exactly $q$ sets that are $(\rho, 1)$-smooth w.r.t. $\mathbb{W}$.

**Proof:** The case $\rho \in \{0, 1\}$ is trivial and thus we assume that $\rho \in (0, 1)$. Consider statement (a). By Lemma 15, it suffices to show that any $A'_{\rho,i/p}$ with $i \in \mathbb{Z}$ is a translation of $A'_{\rho,0}$. By elementary number theory, $i = k \cdot p + j \cdot q$ for some $k, j \in \mathbb{Z}$. We have:

$$A'_{\rho,0} + k = \lfloor \mathbb{Z} \cdot (q/p) + k \rfloor = \lfloor \mathbb{Z} \cdot (q/p) + kp/p \rfloor$$

$$= \lfloor \mathbb{Z} \cdot (q/p) + (i - jq)/p \rfloor = \lfloor (\mathbb{Z} - j) \cdot (q/p) + i/p \rfloor = A'_{\rho,i/p}.$$

Consider statement (b). By Lemma 9, we can assume $\mathbb{W} = \mathbb{Z}$. By Lemma 15, there are at most $q$ such sets. To show that there are at least $q$ such sets it suffices to show that the sets $A'_{\rho,0} + i$, $i = 0, 1, \cdots, q-1$ are all distinct. Assume otherwise. Then $A'_{\rho,0}$ is periodic with period $j < q$. Hence, the rate of $A'_{\rho,0}$ is a rational of the form $n/j$, contradicting the fact that $p$ and $q$ are relatively prime. $\square$

The above lemma implies that, for $\rho, p$ and $q$ as above, any $(\rho, 1)$-smooth set is periodic, and its minimal period is $q$. The following lemma is implied by Lemmas 6 and 13(a,a').

**Lemma 17:** For any irrational $\rho \in (0, 1)$: there are $2^{\aleph_0}$ sets that are $(\rho, 1)$-smooth w.r.t. $\mathbb{W}$.

The next lemma shows that, in the case of a rational rate, there is a dramatic difference between the number of 1-smooth sets and the number of $(1 + \epsilon)$-smooth sets.

**Lemma 18:** For any $0 < \rho < 1$ and $\epsilon > 0$, there are $2^{\aleph_0}$ sets that are $(\rho, 1 + \epsilon)$-smooth w.r.t. $\mathbb{W}$.

**Proof:** By Lemma 9, we can assume $\mathbb{W} = \mathbb{N}$. If $\rho$ is irrational then the lemma follows from Lemma 17. Assume that $\rho \in \mathbb{Q}$. Actually, we show that there are $2^{\aleph_0}$ sets that are $(\rho, [0, 1])$-semi-smooth. Let $n \in (\mathbb{N}/\rho) \cap \mathbb{N} \triangleq P$. For any $(\rho, [0, 1])$-semi-smooth set $A$ there are two alternatives for $\delta(A, [0, n), \rho)$: 0 and 1. Moreover, for each member of $P$ this selection is independent of the other selections. Since $P$ is countable, there are indeed $2^{\aleph_0}$ sets that are $(\rho, [0, 1])$-semi-smooth. $\square$

# 5 Neighboring of Smooth Sets

This section provides a recursive characterization of smooth sets. To that end, it measures the distance between two sets of integers as follows. For $A, B \subset \mathbb{W}$ and $n \in \mathbb{N}$, the sets $A$ and $B$ are *$n$-neighbors* (w.r.t. $\mathbb{W}$) if $\text{abs}(|I \cap A| - |I \cap B|) \leq n$ for any interval $I$ of $\mathbb{W}$. The following lemma is straightforward.

**Lemma 19:** Let $A, B, C \subset \mathbb{W}$.

a. The sets $A$ and $B$ are 0-neighbors iff $A = B$.

b. If $A$ and $B$ are $n_1$-neighbors and $B$ and $C$ are $n_2$-neighbors then $A$ and $C$ are $(n_1+n_2)$-neighbors.

c. The sets $A$ and $B$ are $n$-neighbors iff $\mathbb{W} \setminus A$ and $\mathbb{W} \setminus B$ are $n$-neighbors.

Due to Lemma 19(a,b), the neighbor relation establishes a metric, except that the distance between two sets can be infinite. However, for any fixed $\rho$, any two smooth sets with rate $\rho$ are $n$-neighbors for some finite $n$. Hence, the neighbor relation establishes a metric over this class. The following lemma provides a recursive characterization of smooth sets.

**Lemma 20:** For any $\Delta \geq 1$, $A \subset \mathbb{W}$ and $n \in \mathbb{N}$: $A$ is $(\rho, \Delta+n)$-smooth iff there exists a $(\rho, \Delta)$-smooth set $B \subset \mathbb{W}$ s.t. $A$ and $B$ are $n$-neighbors.

The right to left direction of the lemma follows from the definitions. To prove the other direction, we extend the semi-smooth notation as follows. For an initial interval $J$, a set $A$ is *$(\rho, Z)$-semi-smooth w.r.t. $J$* if $\delta(A, I, \rho) \in Z$ for any $I \in \mathbb{I}_0$ with $I \subset J$. (Note that $A$ is not necessarily a subset of $J$.) We use the following auxiliary lemma.

**Lemma 21:** Let $\Delta \geq 1$, $I \in \mathbb{I}_0$, $Z \in \mathcal{R}_{\Delta+1}$, $Z' \in \mathcal{R}_\Delta$, $Z' \subset Z$ and $Z \setminus Z' \in \mathcal{R}_1$ and let $A$ be $(\rho, Z)$-semi-smooth w.r.t. $I$. Then there is a set $B \subset I$ s.t. $B$ is $(\rho, Z')$-smooth w.r.t. $I$ and $B$ and $A \cap I$ are 1-neighbors.

**Proof:** We prove a stronger version of the lemma in which the conclusion is replaced by:

Then there is a set $B \subset I$ s.t. $B$ is $(\rho, Z')$-semi-smooth w.r.t. $I$ and, for any $I' \in \mathbb{I}_0$ with $I' \subset I$,

$$\text{abs}(\delta(A, I', \rho) - \delta(B, I', \rho)) = \begin{cases} 0 & \delta(A, I', \rho) \in Z' \\ 1 & \text{otherwise} \end{cases}$$

First we show that any set $B$ that satisfies the new conclusion also satisfies the original one. Let $B$ satisfy the new conclusion. There is an $e \in \{-1, 1\}$, which depends only on $Z$ and $Z'$ s.t. $\delta(A, I', \rho) - \delta(B, I', \rho) \in \{0, e\}$ for any such $I'$. By Lemma 3(a), $B$ and $A \cap I$ are 1-neighbors.

The proof of the new version is by induction on $|I|$. The case of $I = \emptyset$ is trivial. Assume $I \neq \emptyset$ and let $i = \max(I)$ and $I^- \triangleq I \setminus \{i\}$. Let $A$ be $(\rho, Z)$-semi-smooth w.r.t. $I$ and let $B' \subset I^-$ be the set provided by the induction hypothesis. Define $B \subset I$ by $B \cap I^- = B'$ and

$$(i \in B \Leftrightarrow i \in A) \iff (\delta(A, I, \rho) \in Z' \Leftrightarrow \delta(A, I^-, \rho) \in Z')$$

In other words, $A$ and $B$ agree on the element $i$ iff the predicate " $\delta(A, J, \rho) \in Z'$ " has the same truth value for $J = I$ and $J = I^-$. It is not hard to verify that $B$ satisfies the conclusion of the stronger version. $\square$

**Proof of Lemma 20:** As said, the right to left direction follows from the definitions. Consider the left to right direction. By induction on $n$, augmented with lemma 19(b), it suffices to consider the case $n = 1$. By Lemma 9, we may assume that $\mathbb{W} = \mathbb{Z}$. By a straightforward application of König's infinity lemma, it suffices to show that for any interval $I$ of $\mathbb{Z}$ there is a $B \subset I$ which is $(\rho, \Delta)$-smooth w.r.t. $I$ and is a 1-neighbor of $A \cap I$. By symmetry, it suffice to prove the last assertion for $I \in \mathbb{I}_0$. By Lemma 6(b), $A$ is $(\rho, Z)$-semi-smooth for some $Z \in \mathcal{R}_{\Delta+1}$. Since $\Delta \geq 1$, either $1 \in Z$ or $(-1) \in Z$. By symmetry, we may assume that $(-1) \in Z$; let $Z' = Z \cap (Z + 1)$. The variables $\Delta, Z, Z'$ and $I$ satisfies the premise of Lemma 21. The set $B$ provided by this lemma satisfies the above requirements. $\square$

# 6 Composition of Smooth Sets

This section studies composition of sets of natural numbers and shows that smoothness is preserved under this operation. This provides a way to partition a given smooth set into two smooth sets of given rates.

For $A, B \subset \mathbb{N}$ the *set composition* of $A$ and $B$ is the subset $D$ of $A$ s.t., for all $i$, the $i$-th member of $A$ is in $D$ iff the $i$-th member of $\mathbb{N}$ is in $B$ and $A$ has an $i$-th member. In other words, the composition of $A$ and $B$ is $A \circ B \triangleq \{n \in A \; : \; |[0, n) \cap A| \in B\}$. Note that for any $D \subset A$ there is a set $B \subset \mathbb{N}$ s.t. $D = A \circ B$ and, when $A$ is infinite, this $B$ is unique.

To show that set composition is associative we establish another characterization of this operator, as follows. For a set $A \subset \mathbb{N}$ define its *monotonic function* $\lambda_A$ to be the unique strongly monotonic function from a finite or an infinite initial interval of $\mathbb{N}$ onto $A$. (When $A$ is infinite this initial interval is the entire set $\mathbb{N}$.) Clearly, any strongly monotonic function from an initial interval of $\mathbb{N}$ into $\mathbb{N}$ is the monotonic function of a unique set. For two functions $f$ and $g$, let $f \circ g$ denote the "first $g$ then $f$" composition of $f$ and $g$; i.e., $(f \circ g)(x) = f(g(x))$ and the left side is defined for a certain $x$ iff $g(x)$ and $f(g(x))$ are defined. Note that when $f$ and $g$ are strongly monotonic functions from initial intervals of $\mathbb{N}$ into $\mathbb{N}$ then so is $f \circ g$. It is easy to check that $\lambda_{A \circ B} = \lambda_A \circ \lambda_B$ and the term 'set composition' is due to this equality. This equality and the fact that composition of functions is an associative operator implies the following lemma.

**Lemma 22:** Composition of sets is an associative operator.

Recall that a set $A$ is $(\rho, \Delta)$-*smooth w.r.t.* $W \subseteq \mathbb{Z}$ if $A \subset W$ and $\text{abs}(|I| \cdot \rho - |I \cap A|) < \Delta$ for any interval $I$ of $W$. In the following we shorten the phrase '$A$ is $(\rho, \Delta)$-smooth w.r.t. $W$' to '$A$ is a $(\rho, \Delta)$-subset of $W$'. It is easy to verify the following lemma.

**Lemma 23:** For $A \subset \mathbb{N}$ and $\rho \in \mathbb{R}$: $A'$ is a $(\rho, \Delta)$-subset of $A$ iff $A' = A \circ B$ for some $(\rho, \Delta)$-smooth $B$.

The following lemma shows that smoothness is preserved by composition and moreover, under some restriction, $A \circ B$ is as smooth as $A$.

**Lemma 24:** Let $A$ and $B$ be $(\rho_1, \Delta_1)$-smooth and $(\rho_2, \Delta_2)$-smooth subsets of $\mathbb{N}$. Then:

a. $A \circ B$ is $(\rho_1 \cdot \rho_2, \rho_2 \cdot \Delta_1 + \Delta_2)$-smooth.

b. If $\Delta_2 = 1$ and $\rho_2 \leq (\Delta_1 - 1)/\Delta_1$ then $A \circ B$ is $(\rho_1 \cdot \rho_2, \Delta_1)$-smooth.

**Proof:** Statement (b) follows directly from statement (a). To prove statement (a), let $I$ be an interval of $\mathbb{N}$.

$$
\begin{array}{rcll}
|I \cap (A \circ B)| & = & |(I \cap A) \cap (A \circ B)| & \text{since } A \circ B \subset A. \\
 & & & \text{Since } A \circ B \text{ is } (\rho_2, \Delta_2)\text{-subset of } A, \\
 & = & |I \cap A| \cdot \rho_2 + \Delta_2' & \text{for some } \Delta_2' \text{ with abs}(\Delta_2') < \Delta_2. \\
 & & & \text{Since } A \text{ is } (\rho_1, \Delta_1)\text{-smooth}, \\
 & = & (|I| \cdot \rho_1 + \Delta_1') \cdot \rho_2 + \Delta_2' & \text{for some } \Delta_1' \text{ with abs}(\Delta_1') < \Delta_1. \\
 & = & |I| \cdot \rho_1 \cdot \rho_2 + \Delta_1' \cdot \rho_2 + \Delta_2' &
\end{array}
$$

Thus, $|I \cap (A \circ B)| - |I| \cdot \rho_1 \cdot \rho_2 = \Delta_1' \cdot \rho_2 + \Delta_2'$. Clearly, $\text{abs}(\Delta_1' \cdot \rho_2 + \Delta_2') \leq \text{abs}(\Delta_1') \cdot \rho_2 + \text{abs}(\Delta_2') < (\Delta_1 \cdot \rho_2 + \Delta_2)$, and this implies the required smoothness. $\square$

Given a $(\rho, \Delta)$-smooth set $A$ and two positive numbers $\beta_1$ and $\beta_2$ with $\beta_1 + \beta_2 = 1$, it is sometimes desirable to partition $A$ into two sets $A_1$ and $A_2$ s.t. each $A_i$ is $(\rho \cdot \beta_i, \Delta)$-smooth. By the following theorem, which follows from Lemmas 1 and 24(b), this is possible under some restrictions on $\beta_1$ and $\beta_2$.

**Theorem 1:** Let $A \subset \mathbb{N}$ be $(\rho, \Delta)$-smooth and let $\beta_1 + \beta_2 = 1$ with $\beta_1, \beta_2 \leq (\Delta - 1)/\Delta$. Then there is a partition of $A$ into $\langle A_1, A_2 \rangle$ s.t. each $A_i$ is $(\beta_i \cdot \rho, \Delta)$-smooth. In fact, for any partition of $\mathbb{N}$ into $\langle B_1, B_2 \rangle$ where each $B_i$ is $(\beta_i, 1)$-smooth, the sets $A_1 = A \circ B_1$ and $A_2 = A \circ B_2$ satisfy the above statement.

The above requirement '$\beta_1, \beta_2 \leq (\Delta - 1)/\Delta$' is mandatory, as stated by the next lemma.

**Lemma 25:** Let $0 < \rho < 1$ and $\Delta \geq 1$. Then there is a $(\rho, \Delta)$-smooth set $A$ s.t. for any $\beta < 1$ which is close enough to 1 there is no subset of $A$ which is $(\rho \cdot \beta, \Delta)$-smooth.

Actually, we establish a stronger result — the existence of isolated smooth sets. Such a set is a $(\rho, \Delta)$-smooth set which is "substantially different" from any $(\rho', \Delta)$-smooth set with $\rho' \neq \rho$. To quantify the difference between sets we proceed as follows. Define the *infimum rate* of a set $A \subset \mathbb{N}$ by:

$$\mathrm{rate}_{\mathrm{inf}}(A) = \lim_{n \to \infty} (\inf \{|A \cap I|/|I| \ : \ I \text{ is an interval } \& \ |I| \geq n\})$$

Since every bounded, non-decreasing sequence has a limit, $\mathrm{rate}_{\mathrm{inf}}(A)$ is defined for any $A \subset \mathbb{N}$. Let the *distance* between two sets $A, A' \subset \mathbb{N}$ be defined by: $D(A, A') \triangleq \mathrm{rate}_{\mathrm{inf}}(A \oplus A')$, where $A \oplus A'$ is the symmetric difference between $A$ and $A'$. Note that the distance $D$ is a pseudo-metric, that is, it satisfies all the requirements of a metric except that $D(A, A') = 0$ for some distinct $A$ and $A'$. A set $A \subset \mathbb{N}$ is an *isolated* $(\rho, \Delta)$*-smooth set* if:

1. $A$ is $(\rho, \Delta)$-smooth.
2. There is an $\epsilon > 0$ s.t. $D(A, A') > \epsilon$ for any $(\rho', \Delta)$-smooth set $A'$ with $\rho \neq \rho'$.

Lemma 25 is clearly implied by the following Lemma, whose proof is deferred to the appendix.

**Lemma 26:** There is an isolated $(\rho, \Delta)$-smooth set for any $0 < \rho < 1$ and $1 \leq \Delta$.

# 7 A Shuffle

This section provides a powerful tool for constructing disjoint smooth sets via a certain mapping from the unit interval into the natural numbers. To that end, we henceforth extend any function $f$ to be defined on any set $Z$ (not necessarily a subset of the domain of $f$) by $f(Z) \triangleq \{f(z) \mid z \in Z \ \wedge \ f(z) \text{ is defined}\}$. Recall that $\|X\|$ denote the length of a real interval $X$ and that the rate of a set of integers $A$ is $\rho$ if for any $\epsilon > 0$, for all sufficiently large intervals $I$, we have $\mathrm{abs}(|A \cap I|/|I| - \rho) < \epsilon$. A *smooth shuffle* is a partial one-to-one function $f$ from the unit interval into the natural numbers having the following properties:

a. For any real interval $X \subset [0, 1)$, $f(X)$ has rate $\|X\|$.

b. For some constant $\Delta$, the image of every real interval is $\Delta$-smooth.

A smooth shuffle would be a powerful tool for construction smooth sets with a variety of desirable properties. Unfortunately, we failed to find such a function and we conjecture:

**Conjecture 1:** There is no smooth shuffle.

Fortunately, there is a (plain) *shuffle* — a partial one-to-one function from the unit interval into the natural numbers obeying only statement (a) above. Moreover, there is a shuffle s.t. the image of any interval can be 'smoothed' into a smooth set by removing a small fraction of the interval, as stated by the next theorem.

**Theorem 2:** There is a shuffle $f$ s.t. for any $\epsilon > 0$ and for any real interval $X \subset [0, 1)$, there exists an interval $Y \subset X$ s.t. $\|X\|(1 - \epsilon) \leq \|Y\|$ and $f(Y)$ is $O(\log(1/\epsilon))$-smooth.

Note that the length of the above $X$ intervals, as well as their endpoints, are not necessarily rational.

Our shuffle, denoted $\mu$ and called the *infinite bit reversal function*, is related to the (finite) bit reversal permutation. (The latter permutation was used for a somewhat similar objective by Iwana and Miyano [13] in the context of packet routing on the mesh.) The function $\mu$ is defined only on *binary-fractions* — numbers of the form $l/2^j$ with $j, l \in \mathbb{N}$ — in the unit interval. Let $x \in [0,1)$ be a binary-fraction and let $0.\alpha_0\alpha_1\alpha_2\cdots$ be its finite binary expansion. Then $\mu(x) \in \mathbb{N}$ is the number whose binary expansion is $\cdots\alpha_2\alpha_1\alpha_0$. This definition is meaningful since $\alpha_i \neq 0$ for finitely many $i$. We now prove that $\mu$ is indeed a shuffle; to that end we need the following terminology and lemmas.

Let $n, j \in \mathbb{N}$. An $(n \times j)$-*interval* is an interval $X \subset [0,1)$ of the form $X = [l/2^j, (l+n)/2^j)$ for some $l \in \mathbb{N}$. We allow one or two of the parameters, $n$ and $j$, to be replaced with a $*$ which stands for any natural number. For example, a $(* \times j)$-*interval* is an interval that is an $(n \times j)$-interval for some $n$ (including the case of $n = 0$). The following lemma is straightforward.

**Lemma 27:** For any interval $X \subset [0,1)$ and $j \in \mathbb{N}$, there are two $(* \times j)$-intervals, $X'$ and $X''$, s.t. $X' \subset X \subset X''$ and $\|X''\| - \|X'\| < 2 \cdot 2^{-j}$.

The following lemma follows from the fact that for any $(1 \times j)$-interval $X$, $\mu(X)$ is the range of an arithmetic sequence whose difference is $2^j$ and whose first element is less than $2^j$.

**Lemma 28:** The set $\mu(X)$ is $(\|X\|, 1)$-smooth for any $(1 \times *)$-interval $X$.

By elementary calculus, the rate function is additive (but not $\sigma$-additive). Thus, by Lemma 28, for any $(* \times *)$-interval $X$, $\mu(X)$ has rate $\|X\|$. Elementary calculus and Lemma 27 imply that $\mu$ is a shuffle.

The rest of Theorem 2, namely that the shuffle $\mu$ has the required 'smoothning' property, is proved in two stages. The first stage proves that the image of any $(n \times *)$-interval is $(\lceil \log n \rceil + 1)$-smooth and the second stage proves that any interval $X \subset [0,1)$ can be truncated into an $(n \times *)$-interval $Y$ which is 'very close' to $X$ while $n$ is not 'too big'. We start with the first stage, as follows. An *extreme interval* is either the empty interval or a $(* \times *)$-interval $X$ s.t. for some $(1 \times *)$-interval $Y$, $X \subset Y$ and $X$ shares an endpoint with $Y$.

**Lemma 29:** Let $X$ be a non-empty $(n \times *)$-interval. Then:

a. The set $\mu(X)$ is $(\lceil \log n \rceil + 1)$-smooth.

b. Either $X$ is an extreme interval or there are $\Delta_1, \Delta_2 \in \mathbb{R}$ and two disjoint extreme intervals $X_1$ and $X_2$ s.t. $X_1 \cup X_2 = X$, $\mu(X_1)$ is $\Delta_1$-smooth, $\mu(X_2)$ is $\Delta_2$-smooth and $\Delta_1 + \Delta_2 = (\lceil \log n \rceil + 1)$.+

The proof of this lemma employs the following two auxiliary lemmas.

**Lemma 30:** For any non-empty extreme $(n \times *)$-interval $X$, $\mu(X)$ is $\lceil \log_4 2n \rceil$-smooth.

**Proof:** We prove the lemma by induction on $n$. If $n = 1$ or $n = 2$ then $X$ is an $(1 \times *)$-interval, and $\mu(X)$ is 1-smooth by Lemma 28. Let $X$ be an $(n \times j)$-interval with $n > 2$ and let $Y$ be the minimal $(1 \times *)$-interval satisfying the definition of extreme interval for $X$. Without loss of generality assume $\min(X) = \min(Y)$. Let $\overline{n} = 2^{\lceil \log n \rceil}$; clearly, $Y$ is an $(\overline{n}, j)$-interval. Let $Y_1, Y_2, Y_3$ and $Y_4$ be the four consecutive $(1 \times *)$-intervals of equal size s.t. $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. Since $n > 2$, $\bar{n} \geq 4$ and each $Y_i$ is an $(\overline{n}/4, j)$-interval. Let

$$Z \triangleq \begin{cases} X \cap Y_3 & \|X\| < 3/4 \cdot \|Y\| \\ Y_4 \setminus X & \text{otherwise} \end{cases}$$

In both cases the interval $Z$ is an extreme interval and is an $(n' \times j)$-interval for some $n' \leq \overline{n}/4 < n$. Thus, either $Z$ is empty or, by the induction hypothesis, $\mu(Z)$ is $\lceil \log_4(2n') \rceil$-smooth. Since for any $x \in \mathbb{R}$, $\lceil x/2 \rceil = \lceil \lceil x \rceil / 2 \rceil$, we have:

$$\lceil \log_4(2n') \rceil + 1 = \lceil \log_4 2 \cdot 4 \cdot n' \rceil \leq \lceil \log_4 2\overline{n} \rceil = \lceil (\log_2 2\overline{n})/2 \rceil$$

$$= \lceil \lceil \log_2 2n \rceil /2 \rceil = \lceil (\log_2 2n)/2 \rceil = \lceil \log_4 2n \rceil$$

Thus, $\mu(Z)$ is $(\lceil \log_4 2n \rceil - 1)$-smooth, and this holds also when $Z$ is empty (recall that $n > 2$). Since $Y$ is an $(1 \times *)$-interval, $Y_1 \cup Y_2$ is also an $(1 \times *)$-interval. By Lemma 28, both $\mu(Y)$ and $\mu(Y_1 \cup Y_2)$ are 1-smooth. By our construction, either $X = Z \cup (Y_1 \cup Y_2)$ and $Z$ and $Y_1 \cup Y_2$ are disjoint or $X = Y \setminus Z$ and $Z$ is a subset of $Y$. Thus, By Lemma 4, $\mu(X)$ is $\lceil \log_4 2n \rceil$-smooth. □

**Lemma 31:** Any $(n \times *)$-interval is the union of two disjoint extreme intervals, an $(n_1 \times *)$-interval and an $(n_2 \times *)$-interval with $n_1 + n_2 = n$.

**Proof:** Let $X$ be an $(n \times *)$-interval. The case of $X = \emptyset$ is trivial so assume $X \neq \emptyset$. Let $\overline{X}$ be the closure of $X$ — the interval $X$ together with its endpoints. The binary-fractions are dense in the unit interval, and therefore some of them are members of $\overline{X}$. Let $j \in \mathbb{N}$ be the minimal one s.t. $\overline{X}$ contains a $j$-bit fraction. Since one of any two consecutive $j$-bit fractions is a $(j-1)$-bit fraction, $\overline{X}$ contains exactly one $j$-bit fraction, lets call it $z$. The intervals $X_1 = X \cap (-\infty, z)$ and $X_2 = X \cap [z, \infty)$ satisfy the conclusion of the lemma. □

**Proof of Lemma 29:** Let $X_1$ and $X_2$ be the extreme $(n_1 \times *)$-interval and $(n_2 \times *)$-interval with $n_1 + n_2 = n$ provided by Lemma 31 for $X$. If $X_1$ or $X_2$ is the empty interval then the lemma follows from Lemma 30 and from the fact that $\lceil \log_4 2n \rceil \leq \lceil \log_2 n \rceil + 1$ for any $n \in (\mathbb{N}+1)$. Assume $X_1, X_2 \neq \emptyset$. By Lemma 30, $\mu(X_1)$ and $\mu(X_2)$ are $\lceil \log_4 2n_1 \rceil$-smooth and $\lceil \log_4 2n_2 \rceil$-smooth, respectively, and by Lemma 4(a), $\mu(X)$ is $(\lceil \log_4 2n_1 \rceil + \lceil \log_4 2n_2 \rceil)$-smooth. Thus, to prove statements (a) and (b) it suffice to show that $\lceil \log_4 2n_1 \rceil + \lceil \log_4 2n_2 \rceil \leq \lceil \log_2 n \rceil + 1$. By elementary calculus, $xy \leq ((x+y)/2)^2$ for any $x, y \in \mathbb{R}$, and thus:

$$\begin{aligned} \lceil \log_4 2n_1 \rceil + \lceil \log_4 2n_2 \rceil &< \log_4(2n_1) + \log_4(2n_2) + 2 \leq \log_4(4n_1 \cdot n_2) + 2 \\ &\leq \log_4(2(n_1+n_2)/2)^2 + 2 = \log_2 n + 2 \leq \lceil \log_2 n \rceil + 2. \end{aligned}$$

Since the values of both sides of the inequality are integers, $\lceil \log_4 2n_1 \rceil + \lceil \log_4 2n_2 \rceil \leq \lceil \log_2 n \rceil + 1$. □

This concludes the first stage of the proof that the shuffle $\mu$ has the required 'smoothning' property. The second stage proves that any interval $X \subset [0,1)$ can be truncated into an $(n \times *)$-interval $Y$ which is 'very close' to $X$ while $n$ is not 'too big'. This stage is established by Lemma 27 and by the following lemma.

**Lemma 32:** For any $0 < \epsilon$ and for any interval $X \subset [0,1)$ there is a $(n \times *)$-interval $Y \subset X$ s.t. $n < 4/\epsilon$ and $\|X\|(1-\epsilon) \leq \|Y\|$.

**Proof:** Assume, w.l.o.g., that $\epsilon \leq 1$ and that $X \neq \emptyset$. Let $j \in \mathbb{N}$ be s.t. $2 \cdot 2^{-j} \leq \|X\| \cdot \epsilon < 4 \cdot 2^{-j}$. By Lemma 27, there exists an $(n \times j)$-interval $Y \subset X$ s.t. $\|X\| \leq \|Y\| + 2 \cdot 2^{-j}$. We have:

$$\begin{array}{rcll} \|Y\| & \geq & \|X\| - 2 \cdot 2^{-j} & \\ & \geq & \|X\| - \|X\| \cdot \epsilon & \text{Since } \|X\| \cdot \epsilon \geq 2 \cdot 2^{-j} \\ & = & \|X\| \cdot (1-\epsilon) & \end{array}$$

Clearly, $2^{-j} \cdot n = \|Y\| \leq \|X\| < 1/\epsilon \cdot 4 \cdot 2^{-j}$; thus, $n < 4/\epsilon$. □

**Proof of Theorem 2:** Let $f = \mu$. It was already proven that $\mu$ is a shuffle. Let $0 < \epsilon$, let $X$ be a sub-interval of the unit interval and let $Y$ be the interval provided by Lemma 32 for $X$ and $\epsilon$. By Lemma 29, the image of this interval is $(\lceil \log(4/\epsilon) \rceil + 1)$-smooth. □

The function $\mu$ does not contradict Conjecture 1; that is, $\mu$ is not a smooth shuffle. Moreover, $\mu$ produces non-smooth sets, as follows.

**Lemma 33:** There is an interval $U \subset [0,1)$ s.t. $\mu(U)$ is not $\Delta$-smooth for any finite $\Delta$.

The proof of the lemma is deferred to the appendix.

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# Appendix

## A Isolated Smooth Sets

This subsection proves lemma 26, repeated here.

**Lemma 26:** There is an isolated $(\rho, \Delta)$-smooth set for any $0 < \rho < 1$ and $1 \leq \Delta$.

Lemma 26 is proved via several lemmas and notations. Let an *odd interval* denote an interval of $\mathbb{Z}$ whose two endpoints are odd numbers.

**Lemma 34:** Let $\ell', \ell'' \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{Z}$ such that:

1. For any interval $J \subset \mathbb{N}$: $|J| \geq 2\ell' - 1$ implies $\sum_{i \in J} f(i) > 0$.
2. For any odd interval $J \subset \mathbb{N}$: $\sum_{i \in J} f(i) > 0$ implies $|J| \geq 2(\ell'' + 1)$.

Then $\text{rate}_{\text{inf}}(\{i |\ f(i) \neq 0\}) \geq \ell''/(2\ell')$.

**Proof:** We prove the lemma by showing that $|\{i \in J :\ f(i) \neq 0\}| \geq \ell''$ for any interval $J$ with $|J| = 2\ell'$. Let $J$ be such an interval and let $J'$ be the maximal odd interval which is a sub-interval of $J$. Clearly, $|J'| = 2\ell' - 1$. By requirement (1), $\sum_{i \in J'} f(i) > 0$.

Let a *refuse* denote an odd integer $i$ with $f(i) = 0$. Let $v = \langle v_0, v_1, \cdots, v_{n+1} \rangle$ be the strongly monotonic sequence s.t. $v_0 = \min(J')$, $v_{n+1} = \max(J')$, all the internal elements of $v$ are refuses and $v$ contains all the refuses of $J'$. We have: $0 < \sum_{i \in J'} f(i) = \sum_{j=0}^{n} \sum_{i \in [v_j,\ v_{j+1}]} f(i)$. Hence, for some $J'' = [v_j,\ v_{j+1}]$, $0 < \sum_{i \in J''} f(i)$. By requirement (2), $|J''| \geq 2(\ell'' + 1)$. None of the internal members of $J''$ is a refuse and $J''$ has, at least, $\ell''$ internal members that are odd. Hence, $|\{i \in J'' :\ f(i) \neq 0\}| \geq \ell''$. $\square$

The definition of smoothness implies the following lemma.

**Lemma 35:** Let $A_i$ be $(\rho_i, \Delta_i)$-smooth, for $i \in \{1,2\}$, and let $I$ be an interval of $\mathbb{N}$. Then $|(A_1 \cap I)| - |(A_2 \cap I)| > (\rho_1 - \rho_2)|I| - \Delta_1 - \Delta_2$.

For $0 < \rho$ define: $\zeta(\rho) \triangleq \max(\{1/q|\ q \in (\mathbb{N}+1)\ \&\ q\rho \in \mathbb{N}\} \cup \{0\})$. Recall that for an initial interval $J$, a set $A$ is $(\rho, Z)$-*semi-smooth w.r.t.* $J$ if $\delta(A, I, \rho) \in Z$ for any $I \in \mathbb{I}_0$ with $I \subset J$. The proof of the following lemma is not hard and is omitted.

**Lemma 36:** Let $0 < \rho < 1$, $1 \leq \Delta$, $Z \in \mathcal{R}_\Delta$, $e \in \{+1, -1\}$, $\xi > 0$ and $\xi \geq \zeta(\rho)$. Then there is a number $m$ which is independent of $Z$ (but may depend on the other variables), an initial interval $I$ and a set $A$ s.t. $|I| < m$, $A$ is $(\rho, Z)$-semi-smooth w.r.t. $I$ and $\delta(A, I, \rho) \in Z \setminus (Z + e \cdot \xi)$.

**Lemma 37:** Let $1 \leq \Delta$ and $0 < \rho < 1$. Then there is a $(\rho, \Delta)$-smooth set $A \subset \mathbb{N}$, a sequence $b = \langle b_i \mid i \in \mathbb{N} \rangle$ of integers and an $m \in \mathbb{N}$ such that:

a. $b_0 = 0$ and $0 < b_{i+1} - b_i \leq m$ for any $i$.

b. $\Delta - 1/2 \leq (-1)^j \cdot \delta(A, [b_j, b_{j+2i+1}), \rho)$ for every $j, i \in \mathbb{N}$.

In other words, the set $A$ provided by Lemma 37 is $(\rho, \Delta)$-smooth, but only marginally so. For an interval of the form $I = [b_j, b_{j+2i+1})$, $|A \cap I|$ deviates by at least $(\Delta - 1/2)$ from its nominal value; it is too large when $j$ is odd and too small when $j$ is even. We later show that this fact, combined with the bound of statement (a), implies that $A$ is an isolated smooth set.

**Proof:** Let $\xi = \max(1/4, \zeta(\rho))$, $Z = [0, \Delta)$ and let $m$ be the bound provided by Lemma 36 for the above parameters and for both $e = +1$ and $e = -1$. Since $\rho \notin \mathbb{N}$, $\xi \leq 1/2$. We construct a sequence $b$ satisfying requirement (a) and a $(\rho, Z)$-semi-smooth set $A$ satisfying the following requirement (b').

b'. $\delta(A, [0, b_k), \rho) \in Z \setminus (Z + (-1)^k \cdot \xi)$ for any $k \in \mathbb{N}$.

By Lemma 6, such an $A$ is $(\rho, \Delta)$-smooth. Requirement (b') implies that either $\xi \leq 1/4$ or $\delta(A, [0, b_k), \rho) = 0$ for any even $k$. (Actually, the last condition holds whenever $\zeta(\rho) \neq 0$). This and Lemma 3 imply that $A$ satisfies requirement (b).

The above $b$ and $A$ are constructed in an inductive manner. Namely, we construct $b$ and a sequence of sets $\langle A_i \subset \mathbb{N} \mid i \in \mathbb{N} \rangle$ s.t. each $A_n$ satisfies the lemma in the following limited manner:

1. $A_n$ is $(\rho, Z)$-semi-smooth w.r.t. $[0, b_n)$.

2. $A_n$ satisfies statement (b') for $k = n$.

3. $A_n \subset [0, b_n)$ and $A_{n+1} \cap [0, b_n) = A_n$ for any $n$.

Giving such sequences, the lemma holds for $A = \cup_{i=0}^{\infty} A_i$ and $b$.

We start with $b_0 = 0$ and $A_0 = \emptyset$, which clearly meet the above requirements. Assume we have constructed $\langle b_0, \ldots b_n \rangle$ and $\langle A_0 \ldots A_n \rangle$ meeting the above requirements. We consider only the case where $n$ is odd; the other case is similar. By Lemma 36 for $Z' = Z - \delta(A_n, [0, b_n), \rho)$, $e = +1$ and for the above $\rho$ and $\xi$, there is an initial interval $I'$ and a set $A' \subset I'$ s.t. $|I'| \leq m$, $A'$ is $(\rho, Z')$-semi-smooth w.r.t. $I'$ and $\delta(A', I', \rho) \in Z' \setminus (Z' + \xi)$. By Lemma 3, the above requirements 1 to 3 hold for $A_{n+1} = A_n \cup (A' + b_n)$ and $b_{n+1} = b_n + |I'|$. □

**Proof of Lemma 26:** Let $A$, $b$ and $m$ be those provided by Lemma 37 for the given $\rho$ and $\Delta$. We show that $A$ is an isolated $(\rho, \Delta)$-smooth set. In fact, we show the following stronger result: for any $(\rho', \Delta + 1/4)$-smooth set $A'$ with $\rho' \neq \rho$, $D(A, A')$ is bounded from zero. We consider only the case where $\rho' > \rho$; the other case is similar. Let $\bar{\rho} \triangleq \rho' - \rho$. Clearly, we may assume that $\bar{\rho}$ is arbitrary small.

Define $f : \mathbb{N} \to \mathbb{Z}$ by $f(i) \triangleq |A' \cap [b_i, b_{i+1})| - |A \cap [b_i, b_{i+1})|$. It suffices to show that the premise of Lemma 34 holds for some $\ell'$ and $\ell''$ s.t. $\ell''/\ell'$ is bounded form zero, since Lemma 34 implies that in this case $D(A, A') \geq \ell''/(2\ell' m)$.

By Lemma 35, $|(A' \cap I)| > |(A \cap I)|$ for any interval $I$ with $|I| \geq (2\Delta + 1/4)/\bar{\rho}$. Hence, $\ell' = \lceil (2\Delta + 1/4)/\bar{\rho} \rceil$ satisfies requirment (1) of Lemma 34. We now show that $\ell'' = \lfloor 1/(8\bar{\rho}m) - 1 \rfloor$ satisfies reuirement (2) of that lemma. For this end, let $J = [j,\ j + 2i + 1)$ be an odd interval and let $I = [b_j, b_{j+2i+1})$. By definition, $\sum_{i \in J} f(i) = |(A' \cap I)| - |(A \cap I)|$. Assume that $\sum_{i \in J} f(i) > 0$. Then:

$$\begin{array}{rcll} -(\Delta + 1/4) & < & \delta(A', I, \rho') & \text{since } A' \text{ is } (\rho', \Delta + 1/4)\text{-smooth} \\ & = & \delta(A', I, \rho) + |I| \cdot \bar{\rho} & \text{since } \rho + \bar{\rho} = \rho' \\ & \leq & \delta(A, I, \rho) - 1 + |I| \cdot \bar{\rho} & \text{since } |A \cap I| < |A' \cap I| \\ & \leq & -(\Delta - 1/2) - 1 + |I| \cdot \bar{\rho} & \text{by Lemma 37.} \end{array}$$

That is, $1/(4\bar{\rho}) < |I|$. Thus we have, $2(\ell'' + 1) \leq 2/(8\bar{\rho}m) = 1/(4\bar{\rho}m) < |I|/m \leq |J|$; i.e., $\ell''$ satisfies requrment (2) of Lemma 34. By Lemma 34, $D(A, A') \geq \ell''/(2\ell' m)$ and, for $\bar{\rho}$ small enough, $\ell''/\ell' = \lfloor 1/(8\bar{\rho}m) - 1 \rfloor \,/\, \lceil (2\Delta + 1/4)/\bar{\rho} \rceil$ is bounded from zero. □

# B The Shuffle $\mu$ Is Not Smooth

In this section we prove that $\mu$ is not a smooth shuffle. Actually we prove a stronger claim — Lemma 33, repeated here.

**Lemma 33:** There is an interval $U \subset [0, 1)$ s.t. $\mu(U)$ is not $\Delta$-smooth for any finite $\Delta$.

In this section, to distinguish between sets of integers and sets of real numbers, we use uppercase letters from the head of the Latin alphabet (e.g. $I$) to denote sets of the former type and uppercase letters with a bar over them (e.g. $\bar{I}$) to denote sets of the latter type.

We show that Lemma 33 holds for $U = \bar{H} = [0, 1/3)$. Moreover, let $I_n \triangleq [0, \lfloor 4^n/3 \rfloor)$ and let $H = \mu(\bar{H})$; we prove the following lemma, which clearly implies Lemma 33.

**Lemma 38:** $\delta(H, I_n, 1/3) = -2/3 \cdot n$ for any $n$.

Let $J_n \triangleq [0, 4^n)$ and let $\bar{J}_n \triangleq \mu^{-1}(J_n)$. We prove Lemma 38 via the following lemmas.

**Lemma 39:** $I_0 = \emptyset$ and $I_{n+1} = J_n \cup (I_n + |J_n|)$.

**Proof:** We have:

$$\lfloor 4^{n+1}/3 \rfloor = \lfloor 4 \cdot 4^n/3 \rfloor = \lfloor (3 \cdot 4^n + 4^n)/3 \rfloor = 4^n + \lfloor 4^n/3 \rfloor .$$

□

**Lemma 40:** $\delta(H, J_n, 1/3) = -2/3$ for any $n$.

**Proof:** We have:

$$H \cap J_n = \mu(\bar{H} \cap \bar{J}_n) = \mu(\{l/4^n \mid l \in \mathbb{N} \;\&\; l/4^n < 1/3\}).$$

Thus,

$$|H \cap J_n| = \lceil 4^n/3 \rceil = 4^n/3 + 2/3.$$

The last equation is due to the fact that $(4^n \bmod 3) = 1$. □

**Lemma 41:** Let $m \in \mathbb{N}$ and $m < 2^n$. Then $m \in H$ iff $(m + 2^{n+1}) \in H$.

**Proof:** Let $\bar{m} = \mu^{-1}(m)$. Clearly, $\bar{m} \cdot 2^n \in \mathbb{N}$ and thus $\mu(\bar{m} + 2^{-n-2}) = m + 2^{n+1}$. It remains to show that $\bar{m} \in \bar{H}$ iff $\bar{m} + 2^{-n-2} \in \bar{H}$. This is derived as follows:

$$\bar{m} < 1/3 \Longleftrightarrow \bar{m} \cdot 2^n < 2^n/3 \Longleftrightarrow \bar{m} \cdot 2^n + 1/4 < 2^n/3 \Longleftrightarrow \bar{m} + 2^{-n-2} < 1/3.$$

□

The previous lemma implies the following one.

**Lemma 42:** $\delta(H, (I_n + |J_n|), 1/3) = \delta(H, I_n, 1/3)$ for any $n$.

Lemma 38 is implied by induction using Lemmas 39, 40 and 42.