On constructing expanders for any number of vertices

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Abstract

While typical constructions of explicit expanders work for certain sizes (i.e., number of vertices), one can obtain construction of about the same complexity by manipulating the original expanders. One way of doing so is detailed and analyzed below.

For any $m \in [0.5n, n]$ (equiv., $n \in [m, 2m]$), given an m-vertex expander, G_m, we construct an n-vertex expander by connecting each of the first $n - m$ to of G_m to an (otherwise isolated) new vertex, and adding edges arbitrarily to regain regularity. The analysis of this construction uses the combinatorial definition of expansion.

1 The story, which can be skipped

Expander graph have numerous applications in the theory of computation (see, e.g., [3]), which explains the extensive interest in constructing these objects. Actually, when talking about expander graphs, one typically refers to families of regular graphs of fixed degree for a varying number of vertices that are $\Omega(1)$-expanding, where the expansion factor is fixed for the entire family. That is, there exisats a constant $c > 0$ such that for every graph $G = (V, E)$ in the family, and every $S \subseteq V$ of size at most $|V|/2$, it holds that

$$|\{v \in V \setminus S : \exists u \in S \text{ s.t. } \{u, v\} \in E\}| \geq c \cdot |S|.$$

While designers of expanders focus on optimizing various parameters, their users tend to care most of having explicit expanders for any number of vertices (i.e., for any size). The most popular notions of being explicit are a minimal notion that requires that the graph be constructed in time that is polynomial in its length, and a stronger notion of expansion that requires that the neighbors of each vertex in each gtraph can be identified in time that is poly-logarithmic in the size of the graph (equiv., polynomial in the size of the description of the vertices, assuming a non-redundent representation).[1]

Unfortunately, typical constructions of explicit expanders work only for certain sizes (i.e., number of vertices). Yet, fortunately, one can obtain constructions of the same level of expliciness (or complexity) by manipulating the original expanders. One such construction was presented recently by Murtagh *et al.* [4]. It reminded me of a different construction, which I heard from Noga Alon many years ago. (In fact, checking something else in [2, Apdx. E.2], I noticed that I used Noga's construction there (see last paragraph in [2, Apdx. E.2.1.2])...)

The starting point in both cases is a construction for a "dense" set of sizes M; that is, for every $n \in \mathbb{N}$ there exists $m \in M$ and an explicit m-vertex expander such that $m \in [0.5n, n]$ (equiv., $n \in [m, 2m]$). The aim is to obtain an explicit n-vertex expander, for any given $n \in \mathbb{N}$.

The construction of Murtagh *et al.* [4] takes an m-vertex graph, where $m \in [0.5n, n]$, designates $n - m$ pairs of vertices in it, joins each such pair to a single vertex (doubling the degree), and adds self-loops on the other $m-(n-m)$ vertices to regain regularity. The analysis of this construction is conducted in terms of the algebraic definition of expansion (i.e., eigenvalues), and is presented in Appendix B of their paper. Assuming that the m-vertex graph has a second eigenvalue smaller (in absolute value) than $\beta < 1/3$, the resulting n-vertex graph has a second eigenvalue smaller than $(1 + 3\beta)/2$.

Noga Alon's construction starts by picking $m_1 \in [0.5n, n]$. Discarding the fortunate case of $m_1 = n$, note that if $m_1 = n/2$ we are done by connecting two copies of the m_1-vertex graph by a matching. The resulting n-vertex graph is shown to be an expander using the combinatorial definition of expansion (i.e., the expansion of vertex-sets). In general, we set $r_1 = n - m_1 \in (0, 0.5n]$, and proceed by picking $m_2 \in [0.5r_1, r_1]$, setting $r_2 = r_1 - m_2$, and so on; that is, in iteration i we pick $m_i \in [0.5r_{i-1}, r_{i-1}]$ and set $r_i = r_{i-1} - m_i$, till we get to $r_t = O(1)$. At this point we connect the vertices of the $t-1$ smaller graphs to $\sum_{i=2}^{t} m_i$ vertices of the m_1-vertex graph by using a matching (and add self-loops to maintain regularity).

The analysis of Noga's construction is less trivial than it seems. The source of trouble is that, when analyzing the expansion of sets, one needs to consider sets of size at most $n/2$ and such sets may have more than $m_1/2$ vertices in the large (m_1-vertex) expander. This difficulty can be resolved by using a definition that guarantees expansion also for larger sets (actually, it suffices to guarantee expansion for sets that have density at most $3/4$).

[1] See [3, Def. 2.3] or [2, Apdx. E.2.1.2].

Furthermore, the standard definition of expansion does imply expansion also for larger sets (as needed above).

Thinking a little more about Noga's suggestion, I realized that, if one does not care about the specific parameters, then the smaller expanders play no real role. Hence, the added small expanders can be replaced by isolated vertices; that is, wishing to have an n-vertex expander and given an m-vertex expander such that $m \in [0.5n, n]$, we connect $n-m$ auxiliary vertices (which are otherwise isolated) to $n-m$ vertices of the original expander (and then add edges arbitrarily to recover regularity). The analysis works via the combinatorial definition of expansion, with the aforementioned cavaet.

2 The actual construction and its analysis

While typical constructions of explicit expanders work for certain sizes (i.e., number of vertices), one can obtain construction of about the same complexity by manipulating the original expanders. One way of doing so is detailed and analyzed below.

The construction. For $m \in [0.5n, n]$ (equiv., $n \in [m, 2m]$), given an m-vertex expander, G_m, we construct an n-vertex expander by connecting each of the first $n - m$ to of G_m to an (otherwise isolated) new vertex, and add edges arbitrarily to regain regularity. Hence, we obtain a construction of expanders for all sizes, provided we are given a construction of expanders for a sufficiently dense set of sizes (which is effectively accessible as assumed below).

Construction 1 (padding and matching with isolated vertices): *Let $d \in \mathbb{N}$ and $M \subseteq \mathbb{N}$ be a set such that*

1. *Given any $m \in M$, we can construct an m-vertex d-regular graph $G_m = ([m], E_m)$.*
2. *For every $n \in \mathbb{N}$, we can determine an $m \in M$ such that $m \in [0.5n, n]$* (equiv., $n \in [m, 2m]$).

Then, we construct a d'-regular n-vertex graph $G_n = ([n], E_n)$ by picking $m \in M \cap [0.5n, n]$, constructing $G_m = ([m], E_m)$, and letting

$$E_n = E_m \cup \{\{i, m+i\} : i \in [n-m]\} \cup E_{m,n},$$

where $d' \in \{d+1, d+2\}$ and $E_{m,n}$ is an arbitrary set of $\frac{(d'-d)\cdot n}{2} - (n-m)$ edges that is added so to make G_n be d'-regular. Specifically, $d' = d+2$ must be used if n is odd and d is even, and $d' = d+1$ is used otherwise.

We say that a graph $G = (V, E)$ is (ρ, c)-expanding if for every $S \subset V$ such that $|S| \leq \rho \cdot |V|$ it holds that $|\partial(S)| \geq c \cdot |S|$, where $\partial(S) = \{u \in V \setminus S : \exists v \in S \text{ s.t. } \{v, u\} \in E\}$ is the boundary of S. The standard definition of expansion corresponds to $(0.5, \Omega(1))$-expansion, but it implies $(\rho, \Omega(1))$-expansion for any constant $\rho < 1$.[2] Hence, when showing that G_n is an expander, we may assume that G_m is $(0.75, \Omega(1))$-expanding, rather than $(0.5, \Omega(1))$-expanding.

Theorem 2 (analysis of Construction 1): *If G_m is $(0.75, c)$-expanding, then G_n is $(0.5, c/2)$-expanding.*

The proof does not use the edges in $E_{m,n}$, which makes sense given their arbitrary choice. Yet, it is quite likely that a more careful analysis of other aspects will yield a stronger result. In particular, assuming that G_m is $(0.5, c)$-expanding, we only conclude that G_n is $(0.5, c/12)$-expanding (so the real challenged is to establish a higher expansion bound for G_n, when assuming that G_m is $(0.5, c)$-expanding).

Proof: Recall that $0 \leq n - m \leq m$. For an arbitrary set $S \subset [n]$ of size at most $0.5n$, we consider the following four disjoint subsets of S:

$$\begin{aligned}
S' &\stackrel{\text{def}}{=} \{i \in [n-m] : i \in S \ \&\ m + i \in S\} \\
S'' &\stackrel{\text{def}}{=} \{i \in ([m] \setminus [n-m]) : i \in S\} \\
S''' &\stackrel{\text{def}}{=} \{i \in [n-m] : i \in S \ \&\ m + i \notin S\} \\
R &\stackrel{\text{def}}{=} \{m + i \in S : i \notin S\}
\end{aligned}$$

Note that (S', S'', S''') is a partition of $S \cap [m]$ whereas $(m + S', R)$ is a partition of $S \setminus [m]$. We may assume, without loss of generality, that $S''' = \emptyset$, because moving $i \in S'''$ to $m + i$ (i.e., replacing S by $(S \setminus \{i\}) \cup \{m + i\}$) can only decrease the $\partial(\cdot)$-value.[3]

[2] Assume that the graph is $(0.5, c)$-expanding, and let $S \subset V$ be an arbitrarty set such that $0.5 \cdot |V| < |S| \leq \rho \cdot |V|$. Then, $R \stackrel{\text{def}}{=} V \setminus (S \cup \partial(S))$ has cardinality smaller than $0.5 \cdot |V|$, and it follows that $|\partial(R)| \geq c \cdot |R|$. On the other hand, $\partial(R) \subseteq \partial(S)$, and so $|\partial(S)| \geq c \cdot |R| = c \cdot (|V| - |S| - |\partial(S)|)$. Hence, $|\partial(S)| \geq \frac{c}{1+c} \cdot (|V| - |S|) \geq \frac{c}{1+c} \cdot \frac{1-\rho}{\rho} \cdot |S|$, and it follows that the graph is (ρ, c')-expanding for $c' = \frac{c \cdot (1-\rho)}{(1+c) \cdot \rho}$.

[3] Suppose that $i \in [m] \cap S$ and $m + i \in [n] \setminus S$, and let $T = (S \setminus \{i\}) \cup \{m + i\}$. Then, $i \notin \partial(S)$ and $m + i \in \partial(S)$, whereas $i \in \partial(T)$ and $m + i \notin \partial(T)$, which means that $|\partial(T) \cap \{i, m+i\}| = 1 = |\partial(S) \cap \{i, m+i\}|$. However, $\partial(T) \setminus \{i, m+i\} \subseteq \partial(S) \setminus \{i, m+i\}$, since the move may only eliminate a contribution of i to $\partial(S) \setminus \{i, m+i\}$ (whereas $m + i$ does not contribute to $\partial(T) \setminus \{i, m+i\}$).

Next, we show that $|S| \leq n/2$ implies $|S' \cup S''| \leq 0.75 \cdot m$. This holds because $|S''| \leq 2m - n$, which implies

$$\begin{aligned} |S'| + |S''| &\leq \frac{|S| - |S''|}{2} + |S''| \\ &\leq \max_{s \leq 2m-n} \left\{ \frac{|S| - s}{2} + s \right\} \\ &= \frac{|S| + 2m - n}{2} \\ &\leq \frac{2m - 0.5n}{2} \\ &\leq 0.75 \cdot m \end{aligned}$$

where the third (resp., last) inequality is due to $|S| \leq n/2$ (resp., $m \leq n$). Having established $|S' \cup S''| \leq 0.75 \cdot m$ and using the $(0.75, c)$-expansion of G_m, we get $|\partial(S' \cup S'')| \geq c \cdot (|S'| + |S''|)$. Turning to R, and using the matching edges (i.e., theset $\{\{i, m{+}i\} : i \in [n{-}m]\}$), we have $|\partial(R)| = |R| \geq c \cdot |R|$, since $c \leq 1/3$. Note that $\partial(S' \cup S'') \cap R = \emptyset$ and $\partial(R) \cap (S' \cup S'') = \emptyset$. since the vertices in R are matched to vertices in $[m] \setminus (S' \cup S'')$. Hence, $|\partial(S' \cup S'' \cup R)| \geq c \cdot (|S'| + |S''| + |R|)/2$, since each vertex may contribute at most twice to the sum $|\partial(S' \cup S'')| + |\partial(R)|$. Noting that $|S'| + |S''| + |R| \geq |S|/2$, we infer that $|\partial(S)| \geq c \cdot |S|/4$.

Using a more careful anlaysis, we note that $|\partial(S' \cup S'' \cup R)| \geq c \cdot (|S'| + |S''|) + 0.5 \cdot |R|$, since $|\partial(R)| = |R|$ and the double contribution may occur only on elements of $\partial(R)$. Using $|S| = 2 \cdot |S'| + |S''| + |R|$, we get

$$\begin{aligned} |\partial(S' \cup S'' \cup R)| &\geq c \cdot (|S'| + |S''|) + 0.5 \cdot |R| \\ &= c \cdot \left(\frac{|S| - (|S''| + |R|)}{2} + |S''| \right) + 0.5 \cdot |R| \\ &= c \cdot \frac{|S| + |S''|}{2} + \frac{1 - c}{2} \cdot |R|, \end{aligned}$$

and the claim follows. ■

3 Postscript

It turns out that I did mention Noga Alon's construction in [2, Apdx. E.2.1.2] (but forgot of this). Also, it seems that Noga has mentioned the construction (and/or variants of it) in some old papers of his. Asking him about this in October 2019, he suggested a few alternative constructions, which are aimed at better expansion parameters [1]. My favorite one, starts with an m-vertex d-regular graph, G_m, for $n \in [m, m+o(m)]$, and obtains an n-regular d'-regular graph by connecting each of the $n-m$ new vertices to d' different old vertices.

A combinatorial analysis of the resulting graph, G_n, maintains much of the expansion features of G_m. Specifically, assume that in G_m, for some monotone non-decreasing function $\mathtt{X} : [m] \to [m]$, every s-subset of $[m]$ has at least $\mathtt{X}(s)$ neighbors outside it (e.g., $\mathtt{X}(s) = \Omega(d \cdot s)$ for $s < m/2d$). Consider an arbitrary set $S \subset [n]$ of vertices in G_n, and let $S' \stackrel{\rm def}{=} S \cap [m]$ and $S'' = S \setminus S'$. If $|S''| > \mathtt{X}(|S|)/2d'$, then $|\partial(S)| \geq |\partial(S'') \setminus S'| \geq d' \cdot |S''| - |S| > 0.5\mathtt{X}(|S|) - |S|$. Otherwise, $|\partial(S)| \geq |\partial(S') \cap [m]| \geq \mathtt{X}(|S'|) \geq \mathtt{X}(|S| - (\mathtt{X}(|S|)/2d')) \geq \mathtt{X}(|S|/2)$.

Noga plans to write a note with a spectral analysis of some of these alternative construction.

References

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[2] O. Goldreich. *Computational Complexity: A Conceptual Perspective.* Cambridge University Press, 2008.

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[4] J. Murtagh, O. Reingold, A. Sidford, and S. Vadhan. Deterministic Approximation of Random Walks in Small Space. In *23rd RANDOM*, LIPIcs 145, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019.