INTRODUCTION TO MANIFOLDS — IV

Appendix: algebraic language in Geometry

1. Algebras.

♡ **Definition.** A (`commutative associatetive`) `algebra` (`over reals`) is a linear space A over $\mathbb{R}$, endowed with two operations, $+$ and $\cdot$, satisfying the natural axioms of arithmetics: $(A, +)$ is an additive Abelian (=commutative) group with the neutral element denoted by 0, while $(A, \cdot)$ is a commutative semigroup. If there is a $\cdot$-neutral element, then it is denoted by 1, though existence of such an element is not usually assumed.

♢ *Example.* The basic example is that of real numbers. Another elementary examples: matrices $\mathrm{Mat}_n(\mathbb{R})$. Other examples follow.

The Principal Example. **Let M be a smooth n-dimensional manifold, and $A = C^\infty(M)$ the space of al smooth functions on it. Then if one sets**

$$(f + g)(x) = f(x) + g(x),$$
$$(f \cdot g)(x) = f(x)g(x),$$
$$(\lambda f)(x) = \lambda f(x),$$

then al the axioms will be satisfied.

Now the principal wisdom comes.

The main idea of algebraic approach to geometry is to study properties of the manifold via algebraic properties of the algebra (ring) $C^\infty(M)$.

2. Reconstruction of points.

♡ **Definition.** An `ideal` I of an algebra A is a subalgebra with the property

$$A \cdot I \subseteq I,$$

which is to be understood as

$$\forall a \in A, \quad \forall u \in I \qquad au \in I.$$

♣ `Problem 1.` Prove that $1 \in I \implies I = A$.

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♣ `Problem 2.` Prove that $\{0\}$ is always an ideal in any algebra.

The principal example of an ideal is the following one.

♣ `Problem 3.` If $Z \subseteq M$ is a closed subset, then

$$I_Z = \{ f \in C^\infty(M) \colon f|_Z \equiv 0 \}$$

is an ideal in $A = C^\infty(M)$.

♣ `Problem 4.` If $Z \subset M$ is not a closed set, and $\overline{Z}$ is its closure, then

$$I_Z = I_{\overline{Z}}.$$

An ideal is called `maximal one`, if there is no other ideal with the property

$$I \subsetneq I' \subsetneq A.$$

♣ `Problem 5.` Prove that for two subsets $Z \subseteq W \subseteq M$

$$A = I_\varnothing \supseteq I_Z \supseteq I_W \supseteq I_M = \{0\}.$$

♡ **Definition.** With any ideal $I \subseteq A = C^\infty(M)$ one may associate its `zero locus`

$$V(I) = \{ x \in M \colon \forall f \in I \quad f(x) = 0 \}.$$

Theorem. *If an ideal is maximal, then its zero locus is a point.*

Points of a manifold M are in one-to-one correspondence with maximal ideals of the algebra $C^\infty(M)$.

♡ **Definition.** A family $f_1, \dots, f_\alpha, \dots$ is a `basis` of an ideal I (which eventually may coincide with the whole algebra), if

$$\forall f \in I \quad \exists c_1, \dots, c_n \in A \ (n < \infty)\colon \qquad f = \sum_\alpha^n c_\alpha f_\alpha.$$

One says that the ideal is generated by f_α. The most interesting case is when the basis consists of a finite number of elements.

♣ `Problem 6.` An ideal $I_0 \subseteq C^\infty(\mathbb{R}^n)$ is generated by the functions

$$x_1, \dots, x_n \colon \mathbb{R}^n \to \mathbb{R}.$$

3. Maps = automorphisms of algebras. If $F\colon M_1 \to M_2$ is a smooth map between two (different) manifolds, and $A_i = C^\infty(M_i)$ are the corresponding algebras, $i = 1, 2$, then the map

$$F^*\colon A_2 \to A_1$$

is a (homo)morphism of algebras:

$$F^*(\lambda f + \mu g) = \lambda F^* f + \mu F^* g, \qquad F^*(fg) = F^f\, F^* g.$$

♣ `Problem 7.` If a morphism is given, how one can reconstruct the map?

♣ `Problem 8.` If $m\colon A_2 \to A_1$ is a morphism, and $I_1 \subseteq A_1$ is an ideal, then the full m-preimage $m^{-1}(I_1) \subset A_2$ is an ideal. Prove.

♣ `Problem 9.` Is it true, that the preimage of a *maximal* ideal is a maximal ideal again?

♣ `Problem 10.` How the answer to the previous prolem may be interpreted in geometric terms?

♣ `Problem 11.` Prove that a map F is a diffeomorphism if and only if the morphism F^* is an isomorphism (that is, bijective and invertible).

♣ `Problem 12.` Formulate the properties of a map F being injective and surjective in terms of the morphism F^*.

♡ **Definition.** A vector field is a linear operator (not a morphism!) of the algebra $A = C^\infty(M)$

$$D\colon C^\infty(M) \to C^\infty(M)$$

which satisfies the Leibnitz identiy:

$$D(fg) = f\,Dg + g\,Df.$$

♡ **Definition.** An operator with the above property is called `differentiation` of the algebra A. The set of all such operators is denoted by $\operatorname{Der}(A)$.

Tautology. *If* $A = C^\infty(M)$, *then* $\operatorname{Der}(A) \simeq X(M)$.

As you already know, a vector field v generates a one-parameter subgroup of the group $\operatorname{Diff}^\infty(M)$ of diffeomorphisms of M (the flow maps), which corresponds to a one-parameter group $m^t\colon A \to A$ of isomorphisms, $m^t = (v^t)^*$. It turns out that the following formula makes sense:

$$m^t = \exp(tD) = \operatorname{id} + tD + \frac{t^2}{2!}D^2 + \cdots + \frac{t^n}{n!}D^n + \cdots,$$

which produces the same result.

♣ `Problem 13.` How do you understand such a mystic formula?

♢ *Example.* If $M = \mathbb{R}^1$, and $D = \frac{\partial}{\partial x}$, then the associated vector field is constant, the corresponding flow maps are shifts, $(v^t f)(a) = f(a+t)$, and the boxed formula means that

$$f(a+t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}.$$

Apparently, this is "true"!

"Theorem". *If D is a differentiation (Leibnitz holds!) then the morphism m^t is a homomorphism:*

$$\begin{aligned}\exp(tD)(f+g) &= \exp(tD)f + \exp(tD)g,\\ \exp(tD)(fg) &= (\exp(tD)f)\cdot(\exp(tD)g)\,.\end{aligned}$$

♣ `Problem 14.` Give a correct formulation of the above "Theorem".

On the contrary, if m^t is a one-parameter group of automorphisms of an algebra A, then the operator

$$D = \left.\frac{d}{dt}\right|_{t=0} m^t$$

appears (the value Df is defined as the derivative of the map $t \mapsto f(t) = m^t f$ at $t = 0$).

♣ `Problem 15.` Prove that if everything is well defined, then the above formula yields an element from $\mathrm{Der}(A)$.

4. Action of morphisms on vector fields. If $F\colon M_1 \to M_2$ is a diffeomorphism, and $v \in X(M_1)$ a vector field, then the push forward of such a field is defined (see above). How to make this definition algebraic?

If $A_i = C^\infty(M_i)$, $i = 1, 2$, are two algebras, and $m\colon A_2 \to A_1$ a morphism, then The natural idea would be to define for any $D \in \mathrm{Der}(A)$ the push forward $m_* D$ by the identity

$$(m_* D)f = D(m(f)).$$

♣ `Problem 16.` What is wrong with such a definition?

If $D \in \mathrm{Der}(A_1)$, and $m\colon A_2 \to A_1$ is an isomorphism of algebras, then the result of conjugation,

$$D \mapsto \mathrm{ad}_m D = m^{-1} \circ D \circ m,$$

is a differentiation of A_1. If $A_i = C^\infty(M_i)$, $D \sim v$, $v \in X(M_1)$, and $m = F^*$, where $F\colon M_1 \to M_2$ is a diffeomorphism, then $\mathrm{ad}_m D \sim F_* v$.

♣ `Problem 17.` Prove the above wisdom.

The problem of integrating ordinary differential equations also acquires within this framework a purely algebraic nature. Let $D \in \mathrm{Der}(A)$.

Find an epimorphism $m\colon A \to C^\infty(\mathbb{R})$ **such that**

$$\mathrm{ad}_m D = \frac{\partial}{\partial x}.$$

Another way to prove the boxed exponential formula is to apply the existence result for ODE's to the scalar case, wor which the formula is trivially "true": it holds for $D = \frac{\partial}{\partial x}$, therefore it must "hold" in the same sense for any other differentiation.

Theorem. *If* $D_1, D_2 \in \mathrm{Der}(A)$, *and* $m^t = \exp(tD_2)$, *then*

$$\lim_{t\to 0} \frac{m^{-t} D_1 m^t - D_1}{t} = \left.\frac{d}{dt}\right|_{t=0} m^{-t} D_1 m^t = D_2 D_1 - D_1 D_2 \in \mathrm{Der}(A).$$

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