Lecture 4

Commutator & commutation

X, Y two vector fields on M

$\{f^t\}\{g^s\}$ respective flows: $X = \frac{d}{dt}\Big|_{t=0} f^t$, $Y = \frac{d}{ds}\Big|_{s=0} g^s$

$[X,Y] := XY - YX$ again a derivation of $\mathcal{O} = C^\infty(M)$

◁ Direct proof: using the Leibniz rule for X, Y verify that it also holds for $Z = [X,Y]$ ▷

Computation: (from the last time)

$$\lim_{t\to 0} \frac{1}{t}\left(f^{-t*} Y f^{t*} - Y\right) = \pm [X,Y]$$

Obvious observation: ~~f^t commutes with Y $\forall t$ ⟺ $[X,Y]=0$~~

~~g^s commutes with f^t~~

$$f^t \circ g^s = g^s \circ f^t \ \forall s,t \Rightarrow f^t Y = Y f^t \ \forall t \Rightarrow [X,Y] = 0$$

Theorem: $[X,Y]=0 \Longrightarrow f^t$ and g^s ~~commute~~

Proof: (A) $\frac{d}{dt} f^t = X f^t = f^t X$

◁ (group property ▷)

(B) $[X,Y] = 0 \Rightarrow f^t Y = Y f^t$

◁ $\frac{d}{dt}\left(f^t Y f^{-t} - Y\right) = \underset{(A)}{f^t X} Y f^{-t} + f^t Y \underset{(A)}{(-X) f^{-t}} - 0$

$= f^t [X,Y] f^{-t} = 0$

at $t=0$ vanishes $\Rightarrow$ vanished identically

(C) $fY = Yf \Rightarrow \forall s \;\; f g^s = g^s f$

◁ Let $h^s = f^{-1} g^s f$. Then h^s – OPGroup,

and $\frac{d}{ds}\big|_{s=0} h^s = f^{-1}\left(\frac{d}{ds}\big|_{s=0} g^s\right) f = f^{-1} Y f = Y$

$\Rightarrow$ by uniqueness of the exponent (OPG) $h^s = g^s \;\; \forall s \in \mathbb{R}$.

Apply to $f = f^t$ and use (B)

Geometric interpretation:

$X_1, \dots, X_k \in \mathcal{D}~~(M) \text{ commute}~~ \Rightarrow \forall a \in M$

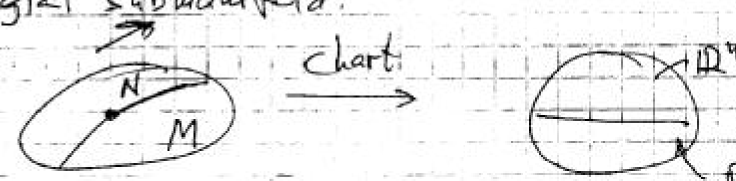
the map

$\mathbb{R}^k \longrightarrow M, \quad (t_1, \dots, t_k) \longmapsto (\exp t_1 X_1 \cdot \ldots \cdot \exp t_k X_k)(a)$

– k-dimensional surface in M passing through a and tangent to the subspace spanned by $(X_1, \dots, X_k)$

– Integral submanifold.

Def



Algebraic description: $\mathfrak{A} = C^\infty(M)$

$\mathfrak{B} = C^\infty(N)$

$i: N \hookrightarrow M$ embedding

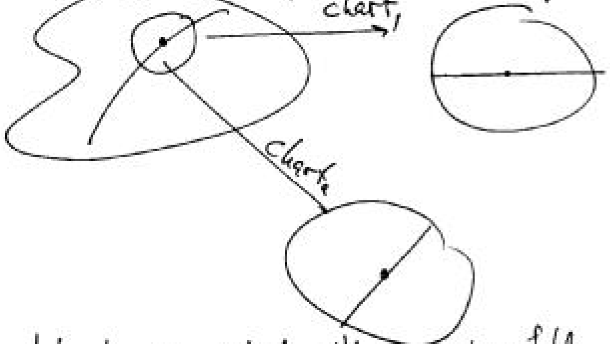
$\mathfrak{B} \to \mathfrak{A}$ surjection (onto)

Kernel: Ideal of functions vanishing on N

Lecture (4) (end) - Lect (5)

<u>Submanifold</u> $N \subseteq M$ (manifold)

‖ Subset that locally looks as a linear subspace
‖ of the linear space.



<u>Not</u> submanifolds:

(a)

(point deleted)

(b)

(irrational winding around the torus)

// Image of the line $y = \alpha x$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$

(c)

(cross)

<u>Ideal</u> associated with a submanifold:

$$I_N = \{ f \in C^\infty(M) : f|_N = 0 \}$$

<u>Def</u>: Vector field X is tangent to a submanifold, if

$$X(I_N) \subseteq I_N \text{, i.e., if } \forall f \in I_N \;\; X(f) \in I_N.$$

In this case N is called an <u>invariant submanifold</u> (by X).

<u>Proposition</u> (elementary) If X, Y are tangent to N, then $[X,Y]$ also is.

The embedding $i : N \hookrightarrow M$ is associated with the homomorphism

$$i^* : C^\infty(M) \to C^\infty(N) \;:\; \text{"restriction" on } N \text{ (of smooth functions)}$$

<u>Prop</u> (above) implies that $\operatorname{Ker} i^* = I_N \subseteq C^\infty(M)$

X, Y tangent to N define two derivations $\tilde{X}, \tilde{Y}$ of $C^\infty(N)$ and, <u>naturally</u>,

$$[X,Y] = [\tilde{X}, \tilde{Y}].$$

In the future we omit tildas!

<u>Distribution</u>: a linear k-subspace in each tangent space

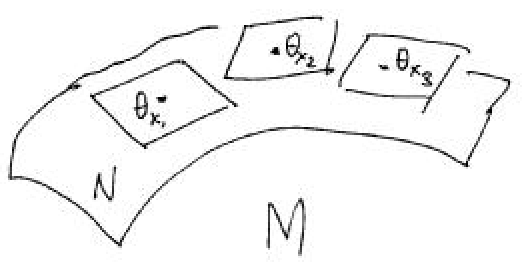
($\dim \theta_x = k$) $\quad \theta_x \subseteq T_x M$, $\quad \theta = \{\theta_x \mid x \in M\}$

smoothly depending on x. ← What does it mean?

"Space without basis" $\quad$ $\theta_x = \mathrm{Span}\,(X_1(x), \dots, X_k(x))$ provided X_i linear independent everywhere

<u>Thm</u> (from Lect. 4): Distribution spanned by commuting vector fields is integrable.

[<u>Def</u> θ_x is integrable, if $\forall x \in M$ there exists a submanifold $N = N_x \subseteq M$, passing through x, and tangent to θ_x (locally!), i.e. any vector field in θ is tangent to N.]



<u>Exercise</u>: Prove that if θ is integrable, then locally
$$M = \bigsqcup_x N_x \quad \text{(disjoint union)},$$
i.e., every two N_x, N_y are either disjoint, or coincide.

Assumption of the thm is not explicit, — nobody knows the hidden basis..

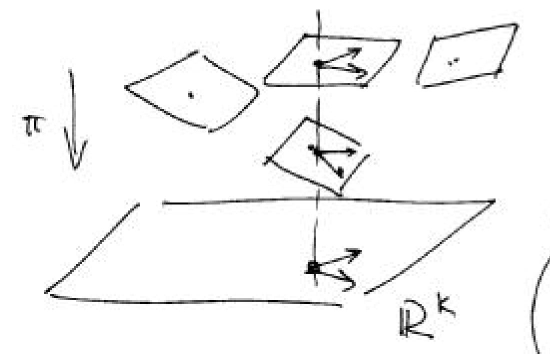
<u>Def</u>. $\{\theta_x\}$ is <u>involutive</u>, if $\forall X, Y$ in θ $\;[X,Y] \subset \theta$
(Closed by the Lie bracket)
– Can be verified in finite terms: $\quad [X_j, X_k] = \sum_1^n c_{jk} X_k$

<u>Thm (Frobenius)</u>: $\quad \theta$ integrable $\Longleftrightarrow$ θ involutive

"$\Rightarrow$" Obvious : $X, Y \in \theta \Rightarrow X, Y$ tangent to $N \Rightarrow [X,Y]$ also

"$\Leftarrow$" Construct the commuting basis.

Proof: (Locally all is in $\mathbb{R}^n$)



a) Start with k commuting vector fields $Y_1, \dots, Y_k$ on the base

b) Lift them to the vector fields in θ, which project onto Y_j

Can be done for any θ

$\pi_* X_j$ are well defined and equal to Y_j by construction.

c) $\pi_* [X_1, X_2] = [\pi_* X_1, \pi_* X_2] = [Y_1, Y_2] = 0$

$\Downarrow$

Commutator is tangent to the „vertical" direction

d) Involutivity: tangent also to the „horizontal" planes $\Rightarrow 0$.

$f : M \longrightarrow N$

$f_* : \mathcal{D}(M) \longrightarrow \mathcal{D}(N)$ vect. fields

If f non-diffeomorphism, it may be not ~~always~~ defined.

always

Dom f_* = „fields constant along preimages $f^{-1}(\cdot)$"

$f_* X = Y$ – two fields are f-related, if

$$\forall \varphi \in C^\infty(N) \quad \underbrace{f^*(Y\varphi)}_{\substack{\text{constant}\\ \text{along } f^{-1}(\cdot)}} = X \underbrace{f^*\varphi}_{\substack{\text{constant along}\\ f^{-1}(\cdot)}},$$

should preserve the constancy along $f^{-1}(\cdot)$

Now (c) becomes obvious: this is a computation in the class of functions on M, constant along fibers $f^{-1}(\cdot)$

Alternative proof: (algebraic) Inductive ~~[illegible]~~ "abelianization."

Algebra of derivations.

Denote by $L_X : \mathcal{D}(M) \longrightarrow \mathcal{D}(M)$ the action of X on vector fields, as usual; If $\{f^t\}$ is the flow of X, then

$$L_X = \frac{d}{dt}\Big|_{t=0} f^t_* \quad ; \quad L_X(Y) = [X, Y].$$

(computed earlier)

Why extra notation? Since L_X (read: Lie derivative) acts on all objects:

a) functions $C^\infty(M)$ $\qquad L_X f = Xf$

b) Vector fields $\mathcal{D}(M)$ $\qquad L_X Y = [X,Y] = -L_Y X$

c) Differential forms, polyvectors, ...

$$L_X L_Y - L_Y L_X = L_{[X,Y]}$$

a) by Definition

b) Nontrivial computation called Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

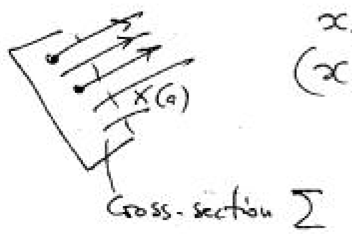
(cyclically symmetric form)

Rectification theorems (Normal forms of vector fields)

Thm 1. If $X(a) \neq 0$, then one can find a local chart on M near a such that

$$X = \frac{\partial}{\partial x_1}$$

Proof:



x_1 = time from ~~Σ~~ Σ to the variable point

$(x_2, \dots, x_{n-1})$ chart (arbitrary) on Σ.

Thm 2. If $X_1, \dots, X_k \in \mathcal{D}(M)$ commute and are linear independent at a, then there exists a local chart such that

$$X_j = \frac{\partial}{\partial x_j}, \quad j = 1 \dots k.$$

◁ Use Frobenius ▷