Differential Geometry

Exam for the spring semester 2005

Rules of the game. Solutions in writing (English very much appreciated) are to be submitted by **July 31, 2005**. Difficulty level of problems varies,—some may require quite a bit of efforts, though none is *very* difficult. Email me at `sergei.yakovenko@weizmann.ac.il` in case you have questions.

Problem 1. Let $M \simeq \mathbb{R}^{n^2}$ be the manifold of square $n \times n$-matrices, and $V_a(x) = a \cdot x$ the vector field on it (matrix multiplication between matrices a, x is intended).

(1) Compute the flow of the field V_a. Compute the commutator $[V_a, V_b]$ for two fields defined by two matrices $a, b \in M$.

(2) The same questions for the vector field $W_a(x) = a \cdot x - x \cdot a$.

(3) Find n functions $f_1, \dots, f_n$ with differentials linearly independent almost everywhere, which are *first integrals* of *any field* $W = W_a$, i.e., $L_W f_i = 0$.

Problem 2. A vector field on a manifold is *complete*, if any trajectory of this field can be infinitely continued forward and backward.

(1) Give an example of complete and incomplete vector fields.

(2) Prove that on a *compact* manifold any vector field is complete.

(3) Show that on any manifold[1] M and any vector field X on it, there exists a *positive* function $f \in C^\infty(M)$ such that the field fX is complete.

[1] With a countable base of open neighborhoods.

Problem 3. A compact n-dimensional manifold carries n commuting vector fields $X_1, \dots, X_n$, linear independent at every point. Prove that M is diffeomorphic to the torus $\mathbb{R}^n/\mathbb{Z}^n$.

What can be said in the case when M is non-compact?

Problem 4. Let M be a *Riemannian* manifold with the metric tensor $\langle \bullet, \bullet \rangle$ and the corresponding volume form $\omega \in \Lambda^n(M)$.

Prove that for any $2n$ vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$,

$$\omega(X_1, \dots, X_n) \cdot \omega(Y_1, \dots, Y_n) = \det \| \langle X_i, Y_j \rangle \|_{i,j=1}^n.$$

Problem 5. In an open disk $\{|x| < 1\} \subset \mathbb{R}^n$ any closed k-form is exact for all $k = 1, \dots, n$.

(1) What is the most popular name for this statement?
(2) Prove it or give a reference to your favorite textbook.

Problem 6. Prove that a 2-form ω on the sphere $\mathbb{S}^2$ is exact if and only if $\int_{\mathbb{S}^2} \omega = 0$. Compute the de Rham cohomology $H^k_{\mathrm{dR}}(\mathbb{S}^2, \mathbb{R})$ for $k = 0, 1, 2$.

Problem 7. Compute the de Rham cohomology $H^k_{\mathrm{dR}}(C^2, \mathbb{R})$ of the cylinder $C^2 = \mathbb{S}^1 \times \mathbb{R}^1$ for $k = 0, 1, 2$.

Problem 8. Let X be the vector field of velocity of rotation (1 revolution/24 hours) on the surface of the Earth and Y the vector field of unit length along γ that points always exactly to the North. (Unit=radius of the Earth!)

(1) Compute the covariant derivative $\nabla_X Y$.
(2) Describe the parallel transport along the parallel at a given latitude θ (full revolution).
(3) Describe the parallel transport along the equilateral triangle with the vertex at the North pole and the angle φ between the sides.
(4) Compute the area of such triangle. Guess the relationship between the two answers.

Problem 9. Consider the surface M^2 of the unit cube in $\mathbb{R}^3$. What (very singular) curvature one should assign to it so as to achieve a similarity with a smooth manifold? Feel free to explain what do *you* mean by similarity, i.e., what properties of geodesic curves on the smooth manifolds you wish to preserve.

Problem 10. Construct a polyhedral surface which would have a *negative* (singular) curvature, cf. with the previous problem, and compute this curvature.

Problem 11. Let Ω be a $n \times n$-matrix whose entries are 1-forms $\omega_{ij} = a_{ij}(z)\, dz$ *holomorphic* in a domain $U \subset \mathbb{C} \simeq \mathbb{R}^2$.

(1) Describe a connection in the trivial real $2n$-dimensional vector bundle over U, for which Ω is the connection matrix.

(2) Compute the curvature of this connection.

(3) Assume that U is the punctured disk $\mathbb{C} \smallsetminus \{0\}$ and the forms ω_{ij} are meromorphic at the origin (the functions $a_{ij}(z)$ have a pole there). Describe the parallel transport along a loop circumventing the origin.

Good Luck!