DIFFERENTIAL AND INTEGRAL GEOMETRY (1^{rst} PART)

PROBLEMS FOR THE EXAM

The assignment is given for doing it at home, and submitted no later than **July 21, 1996**. *Most of the problems are very simple, in fact one step away from definitions. In order for this step not to be degenerate, such definitions are supplied when necessary. The textbooks to be consulted, among your other preferred ones, are given below. In case of problems do not hesitate to ask me or Dmitri Novikov* `dmitri@wisdom`.

References

1. N. J. Hicks, *Notes on Differential Geometry*, Van Nostrand.
2. B. Dubrovin, S. Novikov, A. Fomenko, *Modern Geometry*, vol. 1.
3. J. Milnor, *Morse Theory*.

Good Luck!

Everywhere below M is a smooth manifold, $\mathcal{X}(\mathcal{M})$ stands for the space of vector fields on M, and $\Lambda^p(M)$ denotes the space of differential p-forms on M.

The Christoffel symbols are defined as $\nabla_{e_k} e_j = \sum_{i=1}^n \Gamma^i_{jk} e_i$, where $\{e_i\}$ are the coordinate fields.

By L_X is denoted the Lie derivation, by i_X the inner antiderivation taking a $(p+1)$-form ω into p form $i_X\omega$ as follows: $(i_X\omega)(Y_1,\dots,Y_p) = \omega(X,Y_1,\dots,Y_p)$

Problem 1. With any vector field in $\mathbb{R}^{\nVdash}$ one can associate 1-form A_X (action) and 2-form F_X (flow) as follows:

$$A_X(Y_1) = \langle X, Y_1\rangle\,, \qquad F_X(Y_1,Y_2) = \text{3-dim (mixed) volume of } (X,Y_1,Y_2).$$

1. Prove that any 1-form ω can be represented as A_X and any 2-form Ω as F_X for an appropriate vector field X.
2. For two vector fields $X_1, X_2 \in \mathcal{X}(\mathbb{R}^{\nVdash})$ find $Y \in \mathcal{X}(\mathbb{R}^{\nVdash})$ such that $F_Y = A_{X_1} \wedge A_{X_2}$.
3. Find Y such that $F_Y = dA_X$.

Problem 2. Prove that:

1. $[L_X, i_Y] = i_{[X,Y]}$,
2. $[L_X, L_Y] = L_{[X,Y]}$.

Problem 3. Prove that for any p-form ω

$$L_X\{\omega(Y_1,\dots,Y_p)\} = \\ (L_X\omega)(Y_1,\dots,Y_p) + \omega(L_XY_1,Y_2,\dots,Y_p) + \dots + \omega(Y_1,\dots,Y_{p-1},L_XY_p).$$

Date: June 1996.

Problem 4. Let $\omega \in \Lambda^1(\mathbb{R}^{\nvDash})$ be a form. Consider the distribution of 2-planes that are null subspaces of ω. Prove that this distribution is integrable (in the sense of Frobenius), if and only if $\omega \wedge d\omega \equiv 0$.

Problem 5. Let $X_1, \dots, X_n$ be smooth linear independent vector fields on an n=-dimensional manifold, and $\omega_1, \dots, \omega_n$ the dual 1-forms: $\omega_i(X_j) = \delta_i^j$. Define the family of smooth functions $\{c_{ij}^k(x)\}_{i,j,k=1}^n$ by the identities

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k \, X_k.$$

Prove that

$$d\omega_k = -\frac{1}{2} \sum_{i,j=1}^{n} c_{ij}^k \omega_i \wedge \omega_j.$$

Problem 6. Let $H \colon \mathbb{R}^{\ltimes} \to \mathbb{R}$ be a smooth function and $M_0 = \{H = 0\}$ its level surface. Assume that $dH \neq 0$ on M_0 and M_0 is compact. Let $\omega \in \Lambda^{n-1}(\mathbb{R}^{\ltimes})$ be an $(n-1)$-form.

1. Prove that for all small t the surface $M_t = \{H = t\}$ is also smooth and compact.
2. Prove that the function $I(t) = \oint_{M_t} \Omega$ is differentiable in a small neighborhood of zero.
3. Prove that for any $(n-1)$-form Ω such that $dH \wedge \Omega = d\omega$,

$$\frac{d}{dt} I(t)|_{t=0} = \oint_{M_0} \Omega.$$

Problem 7. Let ∇ be a connexion and ω a p-form. Define the new p-form $\nabla_X \omega$ by the identity

$$\nabla_X \{\omega(Y_1, \dots, Y_p)\} = L_X \{\omega(Y_1, \dots, Y_p)\} =$$
$$(\nabla_X \omega)(Y_1, \dots, Y_p) + \omega(\nabla_X Y_1, Y_2, \dots, Y_p) + \dots + \omega(Y_1, \dots, Y_{p-1}, \nabla_X Y_p).$$

Prove that $\nabla_X \omega$ is indeed a p-form, i.e.

$$\nabla_X \omega(f Y_1, Y_2, \dots, Y_p) = f \cdot \nabla_X \omega(Y_1, Y_2, \dots, Y_p)$$

for any smooth function $f \colon M \to \mathbb{R}$.

Problem 8. Let ∇ be a connexion on the (x_1, x_2)-plane with the Christoffel symbols $\Gamma_{12}^1 = \Gamma_{21}^1 = 1$ and all other zeros.

1. Find all geodesic curves for this connection (by writing the general integral of the corresponding equations).
2. Find a geodesics through the point $(2, 1)$ tangent to the vector $(1, 1)$.

Do the same for the non-symmetric connexion with only $\Gamma_{12}^1 = 1$ and all other zeros.

Problem 9. Let X_1, X_2 be two commuting vector fields on a Riemannian manifold M. Consider the small curvilinear tetragon C_t obtained by travelling along X_1 for time t, then along X_2 for the same time t, then back along X_1 and back along X_2 for the same small time t. Denote by T_t the operator of parallel translation along C_t.

Prove that $T_t = E + t^2 R(X_1, X_2) + O(t^3)$ (where, obviously, R is the curvature tensor that is defined as $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for any two vector fields.

Problem 10. Let ∇, $\overline{\nabla}$ be two connexions on $\mathbb{R}^n$.

1. Prove that their difference is a tensor, i.e. $B(X,Y) = \nabla_X Y - \overline{\nabla}_X Y$ is bilinear.
2. Prove that the connexions ∇ and $\overline{\nabla}$ have the same geodesics if and only if $B(X,X) \equiv 0$, i.e. if B is an *antisymmetric* vector-valued tensor (vector 2-form).
3. Prove that two connexions with the same geodesics and the same torsions coincide (recall that $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$).
4. Prove that for any connexion $\overline{\nabla}$ there exists a *symmetric* (torsionless) connexion with the same geodesics.

Problem 11. Let M be a surface in $\mathbb{R}^3$ with both Gaussian and mean curvatures vanishing. Prove that this surface is a plane.

Note: the mean curvature is defined as the trace of the Weingarten operator $L\colon X \to \overline{\nabla}_X N$, where N is the normal vector field and $\overline{\nabla}$ the flat (Euclidean) connexion in $\mathbb{R}^3$.

Problem 12. Let M be a surface in $\mathbb{R}^3$ with the Gaussian curvature $K\colon M \to \mathbb{R}$ and $N\colon M \to \mathbb{S}^2$ the Gauss normal map. Denote by $D_\rho(m)$ the disc of radius $\rho > 0$ about a point $m \in M$. Prove that

$$\lim_{\rho\to 0} \frac{\text{Area of } N(D_\rho(m))}{\text{Area of } D_\rho(m)} = K(m).$$

Problem 13. Let M be a two-dimensional manifold with the metric $dx_1^2 + f(x_1)\,dx_2^2$ with some smooth positive f. Construct a new coordinates (u_1, u_2) in which the metric will be conformal, i.e. of the form $g(u_1,u_2)(du_1^2 + du_2^2)$ and compute g.

Problem 14. If M is a Riemannian manifold and γ a smooth curve parametrized by the arclength and tangent to a vector field X, the *geodesic curvature* of γ is defined as $\langle \nabla_X X, \nabla_X X\rangle^{1/2}$.

Compute the geodesic curvature of:

1. an equator on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$,
2. the "Tropic of Cancer" (assuming the latter be at $30°$ latitude for simplicity).
3. Compute the integral (against the arclength) curvature of the circle at a constant latitude φ and compare it with the area of the northern spherical cap bounded by this circle.