Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 25, S. 419–422 (1979)

CHARACTERIZING SECOND ORDER LOGIC WITH FIRST ORDER QUANTIFIERS

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1. Introduction

In [1] and [2] it is shown that the language consisting of formulae of the form QM, where Q is a *partially ordered quantifier prefix* (Henkin prefix, abbreviated poq) and M is a quantifier-free matrix, is equal in expressive power to Σ_1^1 (notation from ROGERS [3]). Extending the language to allow the attachment of poq's to formulae as an additional formation rule (together with, say, $\wedge$ and $\neg$), yields Δ_2^1 (see [1]). This extension seems, however, to destroy the natural character of the semantics of poq's which existed in the case QM. We view the semantics differently in the extended case, giving rise to an extension $\boldsymbol{Q}$ consisting of formulae of the form PM, where the prefix P is a well formed string of alternating poq's, and M is a quantifier-free formula. The semantics of formulae of $\boldsymbol{Q}$ is given in terms of conventional second order logic. It is then shown that in fact $\boldsymbol{Q}$ is equal in expressive power to full second order logic, by establishing a correspondence between alternating second order quantifiers and alternating poq's. This result supplies an alternative characteristic of second order logic using only (partially ordered) first-order quantification.

2. Definitions

We assume throughout that a fixed second order language L is given, and we freely use $x, x_1, x_2, \ldots, y, \ldots, u, \ldots, v, \ldots$ to stand for variables, and $f, f_1, f_2, \ldots, g, h, \ldots$ to stand for function symbols.

We define the language $\boldsymbol{Q}$ as follows:

A *partially ordered quantifier prefix* (poq) is a tuple of the form

(*) $\quad (x_1, \ldots, x_n; y_1, \ldots, y_m; \beta)$

where β is a function which associates with each y_i for $1 \leqq i \leqq m$, a tuple, with elements taken from $\{x_1, \ldots, x_n\}$. Intuitively, for a poq Q, we will be using $\langle Q \rangle$ to mean that the x's are universally quantified and the y's existentially, but that each y_i depends only on the elements of $\beta(y_i)$.

A *prefix* is defined recursively as follows: $\langle Q \rangle$ is a prefix for any poq Q, and $\neg P_1$ and $P_1 P_2$ are prefixes for any prefixes P_1 and P_2 such that P_1 and P_2 have no variables in common.

A *matrix* is a quantifier-free formula of L.

A *well formed formula* of $\boldsymbol{Q}$ is a formula of the form PM, where P is a prefix and M a matrix.

[1]) The author is indebted to W. J. WALKOE, A. R. MEYER, A. SHAMIR and a referee for comments on previous versions. The idea for this paper was motivated by work with V. R. PRATT related to program semantics. This research was partially supported by the National Science Foundation under contract No. MCS 76-18461.

The semantics of $\boldsymbol{Q}$ will be defined by gathering that part of a prefix P which essentially quantifies over second order variables, on the left, and attaching the other (first order) part of P to the matrix M. For the reader familiar with the standard semantics given in [1] and [2], this step can be seen to be a natural one, once he is willing to admit that the x_i's in (*) are artificial constructs which serve to help define the existential second order character of a single poq.

The *second order part of* P ($sop(P)$) and *Skolem form of* P *and* w ($sf(P, w)$) are defined recursively for any prefix P and wff w in L as follows: If Q is a poq of the form (*) then

$$sop(\langle Q\rangle) = \exists f_1^Q \ldots \exists f_m^Q$$

where the f_i^Q are new function symbols.

$$sop(\neg P) = dual\,(sop(P))$$

where $dual\,(\exists f\pi) = \forall f\, dual\,(\pi)$ and $dual\,(\forall f\pi) = \exists f\, dual\,(\pi)$ for any second order prefix π, and $dual$ of the empty prefix is defined to be empty.

$$sop(P_1P_2) = sop(P_1)\, sop(P_2).$$

Similarly, if Q is of the form (*) then

$$sf(\langle Q\rangle, w) = \forall x_1 \ldots \forall x_n(w^Q)$$

where w^Q is w with $f_i^Q(\beta(y_i))$ substituted for every free occurrence of y_i in w.

$$sf(\neg P, w) = \neg sf(P, w), \qquad sf(P_1P_2, w) = sf(P_1, sf(P_2, w)).$$

Given a model I for L we say that I *satisfies* PM (written $I \models PM$) iff $I \models sop(P)\, sf(P, M)$, in the usual second-order sense.

A prefix P will be called a Σ_i^1 prefix and denoted by $P^{\langle i\rangle}$, if $sop(P)$ is a Σ_i^1 quantifier-prefix in the usual sense (see [3]); similarly, a Π_i^1 prefix will be denoted by $P^{[i]}$.

3. Results

In order to simplify the exposition of the following, we use the following notational convenience. For sets of formulae S and T of $\boldsymbol{Q}$ and L respectively, we write $S \equiv T$ to express the fact that for any $PM \in S$ there exists $w \in T$ such that $\models w \equiv sop(P)\, sf(P,M)$, and vice versa.

The following theorem establishes a tight link between alternating second order quantifiers in L, and forming compositions of alternating poq's in $\boldsymbol{Q}$.

Theorem. *For* $i \geqq 0$,

$$\text{(a)}\quad (\langle Q\rangle P^{[i]})\, M \equiv \Sigma_{i+1}^1, \qquad \text{(b)}\quad (\neg\langle Q\rangle P^{[i]})\, M \equiv \Pi_{i+1}^1.$$

Proof. Surely, given a prefix P' of the form $\langle Q\rangle P^{[i]}$, by definition $sop(P')$ is a Σ_{i+1}^1 prefix and $sf(P', M)$ has no second order quantifiers. Negation gives this direction for (b).

We concentrate on the $\Leftarrow$ direction. For $i = 0$ (a) simplifies to $\langle Q\rangle M \equiv \Sigma_1^1$, which is shown in Walkoe [2] and Enderton [1], and negation gives (b).

Assume (a) and (b) hold for $i - 1$ where $i > 0$. Without loss of generality we can assume that a Σ_{i+1}^1 formula is given in prenex form, w: $\exists f_1 \ldots \exists f_k \alpha R$, with matrix R and Π_i^1 prefix α. (The dual case where we are given a Π_{i+1}^1 formula is treated by carrying out the construction of this proof for its negation and then dualizing the prefix and negating the matrix.) Now use the inductive hypothesis to come up with $(\neg\langle Q'\rangle P'^{[i-1]})\ M'$ equivalent to αR. Denoting by $P^{[i]}$ the prefix $\neg\langle Q'\rangle P'^{[i-1]}$, we use a generalization of WALKOE's technique to construct $\langle Q\rangle$ and M such that $(\langle Q\rangle P^{[i]})\ M$ is equivalent to w:

Let there be n_j appearances of f_j in M', for $1 \leqq j \leqq k$, and let the arity of f_j be m_j. Define Q to be the poq

$$(u^1_{1,1}, u^1_{1,2}, \ldots, u^1_{1,m_1}, u^1_{2,1}, \ldots, u^1_{n_1,m_1}, u^2_{1,1}, \ldots, u^k_{n_k,m_k}; v^1_1, \ldots, v^1_{n_1}, v^2_1, \ldots, v^k_{n_k}; \beta)$$

with $\beta(v^j_s) = (u^j_{s,1}, \ldots, u^j_{s,m_j})$, where all the various v's and u's stand for new variables not appearing in $P^{[i]}M$. $\langle Q\rangle$ can be comprehended more easily by visualizing it as reading "for every $u^1_{1,1}, \ldots, u^1_{1,m_1}$ there exists v^1_1, and also, independently, for every $u^1_{2,1}, \ldots$ etc.".

$$\left\langle\begin{array}{l}
\forall u^1_{1,1} \ldots \forall u^1_{1,m_1} \exists v^1_1 \\
\forall u^1_{2,1} \ldots \forall u^1_{2,m_1} \exists v^1_2 \\
\quad\ldots \qquad\quad \ldots \\
\forall u^1_{n_1,1} \ldots \forall u^1_{n_1,m_1} \exists v^1_{n_1} \\
\quad\ldots \qquad\quad \ldots \\
\quad\ldots \qquad\quad \ldots \\
\forall u^k_{1,1} \ldots \forall u^k_{1,m_k} \exists v^k_1 \\
\quad\ldots \qquad\quad \ldots \\
\forall u^k_{n_k,1} \ldots \forall u^k_{n_k,m_k} \exists v^k_{n_k}
\end{array}\right\rangle$$

We now transform M' into a matrix M of the form $T \to (S \wedge M'')$ by the following process: T is taken to be the formula

$$\bigwedge_{j=1}^{k} \Big(\bigwedge_{h=1}^{n_j-1} \Big(\Big(\bigwedge_{p=1}^{m_j} u^j_{h,p} = u^j_{h+1,p}\Big) \to v^j_h = v^j_{h+1}\Big)\Big)$$

which essentially states that all the "lines" of $\langle Q\rangle$ which correspond to some f_j define the same function.

We now consider the appearances of the f_j's in M', working "from within". These $q = n_1 + \ldots + n_k$ appearances can be ordered by dependency, starting with those in which some f_j is applied to f-free terms. Define M''_0 as M' and S_0 as *true*. Assume the r'th appearance in the above order is $f_j(t_1, \ldots, t_{m_j})$, which in M''_{r-1} has already been modified to $f_j(t'_1, \ldots, t'_{m_j})$. Then M''_r is defined to be M''_{r-1} with the appropriate v^j_h substituted for this appearance, and S_r is

$$S_{r-1} \wedge \bigwedge_{s=1}^{m_j} (u^i_{h,s} = t'_s).$$

Take M'' to be M''_q, and S to be S_q. This process completes the construction of $(\langle Q\rangle P^{[i]})\ M$.

We now argue that $\vDash w \equiv sop(P)\ sf(P, M)$ with P: $\langle Q\rangle P^{[i]}$ and M: $T \to (S \wedge M'')$. By definition, $sop(P) = sop(\langle Q\rangle)\ sop(P^{[i]}) = \exists g_1 \ldots \exists g_q\ sop(P^{[i]})$ for some new function symbols g_j, and

$$\begin{aligned} sf(P, M) &= sf(\langle Q\rangle, sf(P^{[i]}, T \to (S \wedge M''))) \\ &= \forall u^1_{1,1} \ldots \forall u^k_{n_k,m_k} (\neg sf(P^{\langle i\rangle}, T \to (S \wedge M''))^Q), \end{aligned}$$

where $P^{\langle i\rangle}$ is $P^{[i]}$ with the leading negation dropped. For the sake of the following remarks we abbreviate $\exists f_1 \ldots \exists f_k$ to $\exists f$, $\exists g_1 \ldots \exists g_q$ to $\exists g$ and $\forall u^1_{1,1} \ldots \forall u^k_{n_k, m_k}$ to $\forall u$. Surely $\forall u(\neg sf(P^{\langle i\rangle}, T \to (S \wedge M''))^Q)$ is, by virtue of T not containing any variable appearing in $P^{\langle i\rangle}$, logically equivalent to $\forall u \neg(T^Q \to sf(P^{\langle i\rangle}, (S \wedge M''))^Q)$ or $\forall u(T^Q) \wedge \forall u(\neg sf(P^{\langle i\rangle}, S \wedge M'')^Q)$. Careful application of the definitions involved establishes the additional fact that $\forall u(\neg sf(P^{\langle i\rangle}, S \wedge M'')^Q)$ is in fact logically equivalent to $\neg sf(P^{\langle i\rangle}, (M')^g_f)$ where $(M')^g_f$ is M' with the corresponding new function symbols $g_1, \ldots, g_q$ replacing the q appearances of the symbols $f_1, \ldots, f_k$.

Using the inductive hypothesis, we have to show that the following two formulae are equivalent:

$$w_1 \colon \exists f\, sop(P^{[i]}) \neg\, sf(P^{\langle i\rangle}, M') \text{ and } w_2 \colon \exists g\, sop(P^{[i]}) (\forall u(R^Q) \wedge \neg sf(P^{\langle i\rangle}, (M')^g_f)).$$

Indeed, $I \models w_1$ asserts the existence of an assignment of k functions to the symbols $f_1, \ldots, f_k$ satisfying $sop(P^{[i]})\, sf(P^{[i]}, M')$. To obtain $I \models w_2$, simply assign to $g_1, \ldots, g_{n_1}$ the function assigned to f_1; to $g_{n_1+1}, \ldots, g_{n_1+n_2}$ the function assigned to f_2; etc. Trivially $\forall u(R^Q)$ is satisfied, and hence $I \models w_2$. Conversely, if $I \models w_2$, $\forall u(R^Q)$ forces the assignment to $g_1, \ldots, g_q$ to be such that $g_1, \ldots, g_{n_1}$ are assigned the same function; $g_{n_1+1}, \ldots, g_{n_1+n_2}$ are assigned the same function; etc. This assignment of k functions to the g's, when transformed appropriately to the f_j's yields $I \models w_1$. $\square$

As an example of the technique of the proof of the theorem, take w to be $\exists f_1 \exists f_2 \alpha R$, and M' to be of the form $M'(f_1(g(x), f_2(y)), f_2(f_1(f_2(z), x)))$, involving these two terms and possibly other f_i-free terms. Using new variable symbols v_j and u_j, we take $\langle Q\rangle$ to be $\langle u_1, \ldots, u_7; v_1, \ldots, v_4; \beta\rangle$, with $\beta(v_1) = \{u_1, u_2\}$, $\beta(v_2) = \{u_3, u_4\}$ and $\beta(v_j) = \{u_{j+2}\}$ for $3 \leqq j \leqq 5$, more vividly displayed as

$$\left\langle \begin{matrix} \forall u_1 \forall u_2 \exists v_1 \\ \forall u_3 \forall u_4 \exists v_2 \\ \forall u_5 \exists v_3 \\ \forall u_6 \exists v_4 \\ \forall u_7 \exists v_5, \end{matrix} \right\rangle$$

and M as

$$((u_1 = u_3 \wedge u_2 = u_4) \to v_1 = v_2 .\wedge.\, u_5 = u_6 \to v_3 = v_4 .\wedge.\, u_6 = u_7 \to v_4 = v_5) \to$$
$$((y = u_5 \wedge z = u_6 \wedge g(x) = u_1 \wedge x = u_4 \wedge v_3 = u_2 \wedge v_4 = u_3 \wedge v_2 = u_7) \wedge R(v_1, v_5)).$$

Corollary. $\boldsymbol{Q} \equiv$ L.

Proof. The previous theorem establishes the equivalence in expressive power, of L and a subset of the wff's of $\boldsymbol{Q}$. Conversely, by the definition of $I \models PM$, every wff of $\boldsymbol{Q}$ is equivalent to a formula of L. $\square$

References

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[3] Rogers, H., Theory of Recursive Functions and Effective Computability. McGraw-Hill, New York 1967.

(Eingegangen am 8. November 1977)