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Fundamental Study

On the solvability of domino snake problems

Yael Etzion-Petruschka and David Harel*

Department of Applied Mathematics and Computer Science, The Weizmann Institute of Science, Rehovot 76100, Israel

Dale Myers

Department of Mathematics, University of Hawaii, Honolulu, HI 96822, USA

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Abstract

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In this paper we present an extensive treatment of *tile connectability problems*, sometimes called *domino snake problems*. The interest in such problems stems from their relationship to classical tiling problems, which have been established as an important, simple and useful tool for obtaining basic lower bound results in complexity and computability theory. We concentrate on the following two contrasting results: The general snake problem is undecidable in a half-plane (due to Ebbinghaus), but is decidable in the whole plane. This surprising decidability result was announced without proof by Myers in 1979. We provide here the first full proof, and show that the problem is actually PSPACE-complete.

We also prove many results concerning the difficulty of variants of these general snake problems and their extension to infinite snakes. In addition, we establish a resemblance between snake problems and classical tiling problems, considering the corresponding *bounded*, *unbounded* and *recurring* cases.

Correspondence to: D. Harel, Department of Applied Mathematics and Computer Science, The Weizmann Institute of Science, Rehovot 76100, Israel. Email: harel@wisdom.weizmann.ac.il.

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Contents

1. Introduction

1.1. Background

In the early 1960s Hao Wang introduced *domino tiling problems* [17]. Since then, these problems have been extensively investigated and have appeared repeatedly in the literature. A *domino*, or *a tile*, is a unit-sized square, fixed in orientation, with colored edges. The *type* of a tile is the quadruple containing the colors associated with its right, upper, left and lower edges, respectively. A finite set of tile types is called a *tiling system*. In general, a *tiling problem* is a decision problem that asks, given a tiling system $T=\{d_0, \ldots, d_m\}$, whether or not it is possible to tile some portion P of the integer grid $G=\mathscr{Z}\times\mathscr{Z}$ with dominoes (supply unlimited) taken from among the types in T. The rules of tiling are that each grid point of P is to be associated with a single domino type from T, and that adjacent edges are to be monochromatic. Constraints on the placement of certain dominoes, colors or combinations thereof may also be added to the rules.

The problems introduced by Wang are characterized by the fact that the portion of G to be tiled is *unbounded*; it may be G, the entire grid itself, a half-grid, a quadrant, and the like. These problems are Π_1^0-complete (i.e. when considering levels of undecidability, they reside in the co-r.e. level, at the base of the arithmetic hierarchy [14]). Other tiling problems, concerning tiling of *bounded* portions of the grid G, were considered by Lewis [11] and shown by him to be decidable, and complete in various complexity classes such as NP and PSPACE. *Recurring* tiling problems, characterized by the requirement that a designated domino or color occur infinitely often in the tiling, were introduced by Harel [7] and shown therein to be Σ_1^1-complete (i.e. they reside at the base of the highly undecidable analytic hierarchy [14]).

Undecidability proofs for the unbounded cases are usually based on reductions from the halting problem for Turing machines. Hardness proofs of the bounded cases are also based on such reductions, where the machines are nondeterministic and are bounded in time or space. The reductions are established by setting up a correspondence between the machine's computation and the tiled portion P of the plane. Each tiled row of P corresponds to a legal configuration of the machine, and adjacent rows correspond to legal transitions of the machine. Exceptions are the unbounded constraint-free versions of tiling problems (e.g. "can T tile G?") which require a more complicated correspondence [1, 13]. Undecidability of the recurring versions is similarly proved using reductions from Turing machines. Here the machines are nondeterministic and the problem considered is the existence of an infinite computation that reenters a "signaling situation" infinitely often [7].

Since tiling systems are strong enough to encode the computations of a Turing machine in a relatively straightforward way, and since the geometric and combinatorial structure of a tile is very simple, reductions from an instance of a domino tiling problem to instances of other problems are relatively easy to construct and comprehend. Thus, tiling problems have turned out to be quite powerful for proving undecidability and lower bounds on the complexity of various logical systems (see [6] for a survey). Domino tiling problems have also been used as alternative basic problems for reductions in the theory of NP-completeness [15, 16].

Less known is the family of *tile connectability problems*, or *domino snake problems*. In general, such a problem asks, given a tiling system T and two points $p, q \in \mathscr{Z} \times \mathscr{Z}$, whether the points can be connected within some portion P of the plane by a "domino snake" built of the types in T. A domino snake is a sequence of tiles on the plane in which successive tiles are adjacent along an edge and touching edges are monochromatic.

Connectability problems were investigated by Myers [12] and by Ebbinghaus [2, 3]. In 1979, Myers announced that the *unlimited connectability problem* (i.e. whether two given points can be connected by a domino snake within the whole plane) is decidable [12]. In contrast, Ebbinghaus [2] proved in 1982 that the problem becomes undecidable if instead of the whole plane, a half-plane or a quadrant is considered.[1] This difference in the solvability of the unbounded cases according to the portion of the plane has no analogue in the classical tiling problems, where one has undecidability in all cases.

A resemblance between snake problems and tiling problems was found by Ebbinghaus in the context of *bounded* snake problems, in which the allowed portion of the plane is bounded [3]. In analogy to the bounded tiling problems, which are complete for NP and PSPACE (for a square and a rectangle, respectively), Ebbinghaus showed that the corresponding bounded snake problems are also complete

[1] Ebbinghaus's result was actually obtained as a consequence of his undecidability proof of the *strict connectability problem*, where the snake must begin with a certain domino type placed at point p.

for these classes. An extension to infinite snakes also appears in Ebbinghaus's later paper [3].

Other applications of snake problems have recently appeared in connection with domino games [5] and the uncertainty principle for physical systems [10].

1.2. *Overview of the paper*

The two main goals of this paper are (i) to provide a full proof of the decidability result, and (ii) to investigate additional analogies and differences between domino snake problems and classical tiling problems. We survey existing results and address a sequence of additional variants of snake problems. Our results are divided into those for finite snakes (Section 2), and infinite snakes (Section 3).

Sections 2.1–2.3 deal with the decidable cases of finite snake problems. In Section 2.2, we present the bounded versions of snake problems, adding to the cases of a square and a rectangle, which are NP- and PSPACE-complete, respectively, a *fixed-width* rectangle version, which admits a polynomial-time algorithm. In Section 2.3, we prove the surprising result, to the effect that the general snake problem in the whole plane is decidable. This result was announced in 1979 [12], but a proof has not yet been published. Indeed proving it turns out to be quite a delicate task, and our proof in Section 2.3 contains some rather technically involved combinatorial/geometrical arguments. The decidability of another case, where the portion of the plane is limited to an infinite strip, is also proved at the same time.

Sections 2.4 and 2.5 deal with undecidable cases of finite snake problems. We consider versions of the strict connectability problem and a sequence of snake problems in various portions of the plane, showing them all to be undecidable.

The decidability of the unlimited connectability problem and the fact that the problem becomes undecidable for a half-grid or a quadrant, raise the question of the precise borderline between decidable and undecidable unbounded snake problems. Since all the reasonable variants we consider in the paper turn out to be undecidable, including the case in which only a single point of the grid is removed, we have come to believe that decidability in the whole plane is essentially a remarkable exception.

Section 3 deals with infinite snakes. In Section 3.2, we consider the problem of the existence of an infinite snake within a strip of fixed width. We prove this problem to be decidable by extending the proof techniques of Section 2 for finite snake problems. Sections 3.3 and 3.4 deal with strict versions and recurring versions of infinite snake problems in various portions of the plane. In analogy to the classical tiling problems, recurring versions of infinite snake problems are shown to be highly undecidable.

Figures 13 and 14 in Section 4 summarize the results.

1.3. *Preliminaries*

We regard a point $q \in \mathscr{Z} \times \mathscr{Z}$ as a unit square in the plane with center q. Given a tiling system T and a portion of the plane, $P \subseteq \mathscr{Z} \times \mathscr{Z}$, a *T-tiling* of P is a function,

$\tau: P \to T$, assigning to each grid point $q \in P$ a tile type $\tau(q) \in T$, such that adjacent edges are monochromatic. If T is clear from the context, we sometimes speak simply of a *tiling* instead of a T-tiling.

We are interested in special portions of the plane, called *snake skeletons*. A snake skeleton is an ordered sequence $(q_0, \ldots, q_n) \in (\mathscr{Z} \times \mathscr{Z})^n$, such that for each $0 \leqslant i \leqslant n-1$, q_i and q_{i+1} are adjacent. Given a tiling system T, a *T-snake* is a snake skeleton S together with a function, $\sigma: S \to T$, assigning to each skeleton point $q_i \in S$ a tile type $\sigma(q_i) \in T$, such that for each $0 \leqslant i \leqslant n-1$ the adjacent edges of $\sigma(q_i)$ and $\sigma(q_{i+1})$ are monochromatic. Note that if $q_i = q_j$ for $q_i, q_j \in S$, then $\sigma(q_i) = \sigma(q_j)$. A *$T$-snake connecting p and q*, where $p, q \in \mathscr{Z} \times \mathscr{Z}$, is a T-snake with the additional requirement that $q_0 = p$ and $q_n = q$. Given a T-snake σ, we use S_σ to denote its skeleton. Here again, if T is clear from the context, we speak of *snakes* instead of T-snakes.

Remark 1.1. The definition of a T-snake is liberal when it comes to distant parts of the snake that happen to "touch": when two tiles that are not consecutive in the skeleton sequence have adjacent edges in the plane, a T-snake does not require these edges to be monochromatic. Hence, a T-snake is not quite a tiling of the skeleton in the usual sense of tiling portions of the plane. The more constrained version in which every two adjacent tile-edges have to be monochromatic is called a *strong snake* in [4]. These were the snakes considered in [12]. It turns out that there are simple reductions between the two kinds of snakes, which the reader is invited to devise. Thus, it is possible to show that all the results of the paper hold for strong snakes too.

Problem 1.2 [*The general snake (connectability) problem*]. Given a tiling system T and two points p and q in some portion P of the plane, is there a T-snake σ connecting p and q whose skeleton S_σ lies entirely within P (i.e. $S_\sigma \subseteq P$)?

Note: We shall often say informally that σ lies within P, instead of saying that its skeleton S_σ does.

2. Finite snakes

2.1. Directed snakes

We start with definitions of *directed* and *fully directed* tiling systems over T. Given T, define the directed version $\hat{T}$ to be a new tiling system of size $4 \times |T|$. For each $t \in T$, $\hat{T}$ contains four tile types $t_\rightarrow, t_\leftarrow, t_\uparrow, t_\downarrow$, which are copies of t with the corresponding arrow in their centers (see Fig. 1(a)). The fully directed version $\hat{T}_\mathrm{f}$ is defined to be a new tiling system of size $12 \times |T|$. Here, for each $t \in T$, $\hat{T}_\mathrm{f}$ contains 12 tile types, which are copies of t with 12 kinds of arrows in their centers. The arrows are directed according to the 12 possible combinations of an ordered pair of different tile edges (see Fig. 1(b)). (Obviously, these versions can be obtained from the usual ones by using extra colors.)

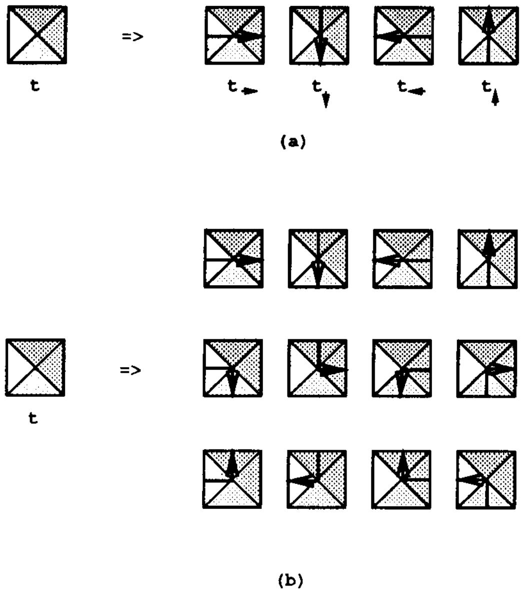


Fig. 1. Directed and fully directed tiling systems over T.

A *directed snake over T*, or simply a *$\hat{T}$-snake*, is an ordinary T-snake σ when the arrows are disregarded, and, in addition, for every pair of consecutive points q_i and q_{i+1} in the skeleton S_σ, the arrow in $\sigma(q_i)$ is directed towards the edge adjacent to q_{i+1} (see Fig. 2(a)). The direction of the arrow in $\sigma(q_n)$ is arbitrary. A *fully directed snake over T*, or a *$\hat{T}_{\mathrm{f}}$-snake*, is a T-snake σ when the arrows are disregarded, and, in addition, for every three consecutive points q_{i-1}, q_i and q_{i+1} in the skeleton S_σ, the arrow in $\sigma(q_i)$ is directed from the edge adjacent to q_{i-1} towards the edge adjacent to q_{i+1} (see Fig. 2(b)). For q_0 the source of the arrow in $\sigma(q_0)$ is arbitrary, and for q_n the target of the arrow in $\sigma(q_n)$ is arbitrary. Notice that the skeleton S of a (fully) directed snake cannot include loops (i.e. for all $q_i, q_j \in S$, $q_i = q_j$ iff $i = j$).

The following claim is immediate.

Claim 2.1. *Given a tiling system T and two points p, q in some portion P of the plane, there is a T-snake connecting p and q and lying entirely within P iff there is a $\hat{T}$-snake directed from p to q and lying entirely within P iff there is a $\hat{T}_{\mathrm{f}}$-snake fully directed from p to q and lying entirely within P.*

Hence, when considering specific snake problems, we can assume that tiling systems and snakes are (fully) directed.

2.2. *Bounded connectability problems*

For any pair of natural numbers (n, m), let P_{nm} denote the rectangle

$$\{(x, y) \mid 0 \leqslant x \leqslant n,\ 0 \leqslant y \leqslant m\}.$$

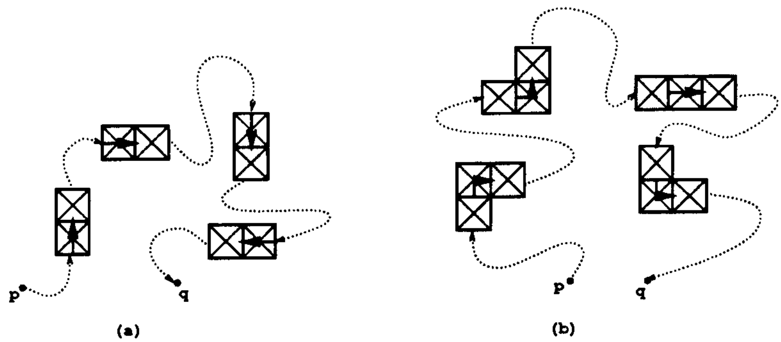


Fig. 2. Directed and fully directed snakes over T.

Problem 2.2 (*Connectability in a square*). Given a tiling system T and $n \in \mathcal{N}$ (in unary), is there a T-snake connecting the points $(0,0)$ and $(n,0)$ and lying entirely within the square P_{nn}?

Problem 2.3 (*Connectability in a rectangle*). Given a tiling system T and $n \in \mathcal{N}$ (in unary), is there some $m \in \mathcal{N}$ and a T-snake connecting the points $(0,0)$ and $(n,0)$ and lying entirely within the rectangle P_{nm}?

Problem 2.4 (*Connectability in a fixed-width rectangle*). Let k be a fixed natural number that is not part of the input. Given a tiling system T and $n \in \mathcal{N}$ (in unary), is there a T-snake connecting the points $(0,0)$ and $(k,0)$ and lying entirely within the fixed-width rectangle P_{kn}?

These problems are the "snake versions" of classical bounded tiling problems, i.e. the *square tiling* problem, the *rectangle tiling* problem and the *fixed-width tiling* problem (see e.g. [8, 11]). The complexity of Problems 2.2 and 2.3 was investigated in [3]. In analogy with the square tiling problem and the rectangle tiling problem which are NP- and PSPACE-complete, respectively, we have the following theorem.

Theorem 2.5 (Ebbinghaus [3]). *Problem* 2.2 *is* NP-*complete*; *Problem* 2.3 *is* PSPACE-*complete*.

As in the tiling analogues, simulations of Turing machine computations are used in all the proofs; the coding methods, however, vary. While combinations of colors were used for coding in the classical tiling problems, geometric shapes are used here.

To complete the picture, we have considered the connectability problem in a fixed-width rectangle (Problem 2.4). This problem admits a polynomial-time algorithm that is based on a reduction to a polynomial-time procedure for checking the existence of a *tiling* of a fixed-width rectangle. Hence, we first include a polynomial-time algorithm (whose existence was mentioned in [8]) for the latter case.

Problem 2.6 (*Tiling a fixed-width rectangle*). Let k be a fixed natural number that is not part of the input. Given a tiling system T and $n \in \mathcal{N}$ (in unary), is there a T-tiling of the fixed-width rectangle P_{kn}?

Theorem 2.7. *Problem* 2.6 *admits a polynomial-time algorithm.*

Proof. Consider a "slice" of P_{kn}, i.e. a segment of width k and height 1. There are at most $|T|^k$ possible ways to legally tile such a slice. Now, construct a directed graph $G=(V,E)$, whose set of vertices V corresponds to the set of all legally tiled slices. There is an edge from v to u, iff the tiled slice of u can be legally attached above the tiled slice of v. Note that the question of the existence of a legal tiling of P_{kn} is exactly the question of the existence of a directed path of length n in G. This question can be solved using the following polynomial-time algorithm: First, we check if the graph contains a cycle (applying, for example, a DFS procedure). If there is a cycle, then G contains a directed path of any length, in particular a path of length n. Otherwise, G is a directed acyclic graph, and the existence of a path of length n can be easily checked (applying, for example, a BFS-like procedure to each vertex $v \in V$). □

Theorem 2.8. *Problem* 2.4 *admits a polynomial-time algorithm.*

Proof. We reduce the problem to a certain kind of tiling problem for a fixed-width rectangle.

First, assume that snakes are fully directed and let us work with the fully directed version $\hat{T}_{\mathrm{f}}$ over the given tiling system T. Now, consider a tiling of the rectangle using the types of $\hat{T}_{\mathrm{f}}$ and an additional blank type (i.e. the type of a white tile containing no arrows). Rules of tiling are that two edges may be adjacent if and only if one of the following holds:

(1) One of the edges includes the head of an arrow, the other includes the tail of an arrow, and they are monochromatic.

(2) Neither of the edges includes the head or tail of an arrow (and there is no restriction on the coloring).

Adding boundary conditions that force the bottom-left and bottom-right edges of the rectangle to include, respectively, a starting arrow and a terminating arrow of a directed snake (and other boundary edges are arrow-less), ensures that a legal tiling of the rectangle P_{kn} exists if and only if there is a T-snake connecting $(0,0)$ and $(k,0)$ within P_{kn}. □

2.3. *The unlimited case*

Problem 2.9 (*Unlimited connectability*). Given a tiling system T and two points $p, q \in \mathcal{Z} \times \mathcal{Z}$, can p and q be connected by a T-snake?

For ease of exposition, we first address a simpler case of the problem, where the portion of the plane is limited to an infinite strip of fixed width. This case can also be treated as a separate result, since it is not implied by, nor does it directly imply, decidability of the general unlimited case.

2.3.1. Connectability in a strip

Let S_k denote a strip of width k on the grid $\mathscr{Z} \times \mathscr{Z}$. Without loss of generality, assume $S_k = \{(x, y) \mid 1 \leqslant x \leqslant k\}$.

Problem 2.10. Given two points $p, q \in S_k$ and a tiling system T, is there a T-snake connecting p and q and lying entirely within S_k?

Theorem 2.11. *Problem* 2.10 *is decidable.*

Proof. First, note that if such a snake exists, there is one of minimal length, call it σ_{m}. The basic idea of the proof is to use the properties of the minimal snake to bound its length by a recursive function of the size of the input. One can then run through all possible snakes up to that length to decide if any snake exists.

Throughout this proof, we work with the directed version $\hat{T}$ of the given tiling system and assume that the minimal-length snake σ_{m} is directed from p to q.

Consider the "slices" of S_k, i.e. all segments of width k and height 1. We identify a slice by its y coordinate, hence, for a fixed y_0, the slice $\{(x, y_0) \mid 1 \leqslant x \leqslant k\}$ is referred to as y_0. A slice is termed *relevant* if its intersection with the skeleton $S_{\sigma_{\mathrm{m}}}$ is not empty. Note that the number of possible ways that a relevant slice can contain tiles from the minimal snake σ_{m} is bounded by $(1 + |\hat{T}|)^k$.

Claim 2.12. *Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$. All the relevant slices of S_k are contained in the rectangle*

$$\{(x, y) \in S_k \mid \min(y_p, y_q) - (1 + |\hat{T}|)^k \leqslant y \leqslant \max(y_p, y_q) + (1 + |\hat{T}|)^k\}.$$

Proof. Assume S_k is tiled with the minimal snake σ_{m}, and assume the claim is false. There must be two relevant slices that are identically tiled and are both placed either above or below p and q. Without loss of generality, we can take $y_2 > y_1 > \max(y_p, y_q)$ as two identical slices. We use the arrows inside the tiles to simulate "travelling" along the skeleton of the snake, starting at p. However, whenever we have to enter a point above y_1, we continue from the corresponding point above y_2, and whenever we have to enter a point in y_2 from above, we continue from the corresponding point in y_1. Since the original "tour" was a legal snake leading from p to q within S_k, the new, truncated, one is also legal. Hence, "shifting" the slice y_2 and everything above it down to y_1, and eliminating the portion between y_1 and y_2, yields a shorter snake that connects p and q (see Fig. 3). This contradicts the minimality of the original snake, and completes the proof of the claim. □

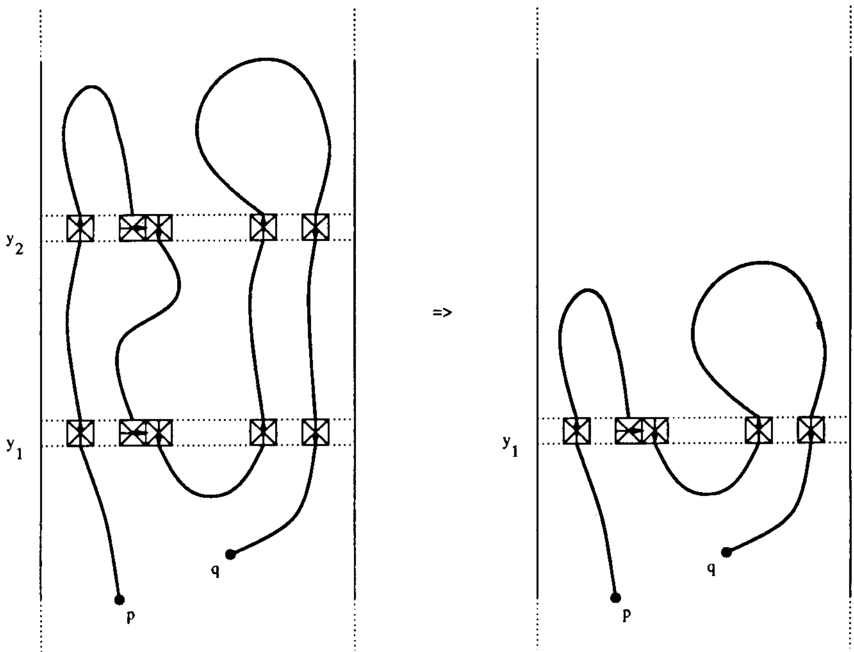


Fig. 3. "Shifting" the slice y_2 and everything above it to y_1, eliminating the portion between y_1 and y_2 altogether, yields a shorter snake that connects p and q. Note that there can be cases that are more complicated than the one illustrated here. For example, the cutting and pasting can introduce isolated "loops". However, the result will always include a shorter snake that connects p and q.

Proof of Theorem 2.11 (*conclusion*). Theorem 2.11 now follows, since the length of the minimal snake connecting p and q is bounded by a simple recursive function of the size of the input (i.e. the distance between p and q, the cardinality of T and the width k of the strip). □

Corollary 2.13. *Problem* 2.10 *is* PSPACE-*complete.*

Proof. The proof of Theorem 2.11 actually implies that there exists a snake connecting p and q within S_k if and only if there is such a snake within a finite part of the strip whose height is exponential in the size of the input. Using the same trick used in the proof of Theorem 2.8 to reduce the question of the existence of a snake to the question of the existence of a tiling (where the rules of tiling are changed accordingly), one can solve the problem using a simple recursive PSPACE procedure. Hardness is achieved using the reduction from a space bounded Turing machine presented by [3] in the proof that the connectability problem in a rectangle is PSPACE-hard. □

Theorem 2.11 can be strengthened by considering strips having curved borderlines. Call such a strip a *corridor*. Formally, a corridor of width k, denoted C_k, is a portion of the plane, where for any fixed y_0, there is some integer x' and a single slice $\{(x, y_0) \mid x'+1 \leqslant x \leqslant x'+k\} \subseteq C_k$.

Problem 2.14. Given two points $p, q \in \mathscr{Z} \times \mathscr{Z}$, a tiling system T and an integer k (in unary), is there a T-snake connecting p and q such that its skeleton lies within some corridor of width k?

Theorem 2.15. *Problem* 2.14 *is* PSPACE-*complete.*

Proof. The proof is essentially the same as that of the strip case in Theorem 2.11 and Corollary 2.13. □

Theorem 2.15 holds even for corridors with a slanted base line (as opposed to the horizontal base line used implicitly in the previous definition of C_k). A corridor of width k with respect to the base line l is a portion of the plane which contains for each line l' parallel to l, a single slice induced by a segment of length k. Generally, a *slice* is the set of all unit squares whose interior intersects some given line or line segment. Note that the "shifting mechanism" used in the proofs of Theorems 2.11 and 2.15 for horizontal slices can also be used for "slanted slices" having the same shape and size.

2.3.2. The main theorem

Theorem 2.16. *Problem* 2.9 *is* PSPACE-*complete.*

Proof. At the heart of the proof is the same principle as in the proof of Theorem 2.11. Again, we use the properties of the minimal-length snake connecting p and q to bound its length.

Throughout the proof we denote by S the skeleton of a minimal snake from p to q. Let S_1^R be a duplicate of S in which the initial point is q. Denote its terminal point by q_1. Inductively, let S_i^R be a duplicate of S_{i-1}^R with initial point q_{i-1}, and terminal point q_i. Similarly, define S_1^L to be a duplicate of S with terminal point p, and denote by p_1 its initial point. Again, inductively, S_i^L is a duplicate of S_{i-1}^L with terminal point p_{i-1}, and initial point p_i. See Fig. 4.

Lemma 2.17. *S has no self-intersections.*

Proof. This is trivial. Simply delete the "loop" at an intersection point to obtain a shorter snake, thus contradicting minimality. □

Lemma 2.18. *S has no intersections*[2] *with S_1^R or S_1^L.*

[2] Here, and in the sequel, snake intersections are assumed not to include the endpoints.

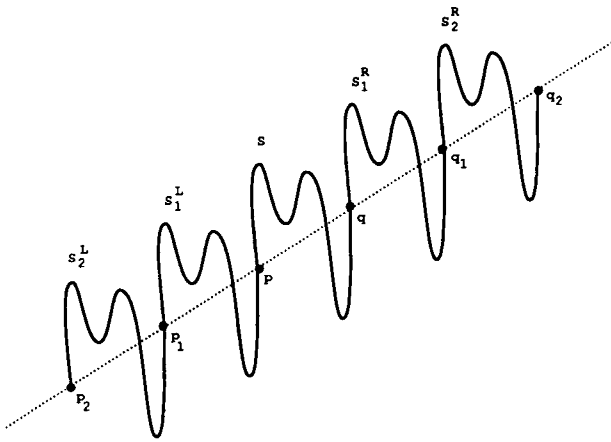


Fig. 4. Duplicates of S from its right and left.

Proof. We prove the lemma for S_1^R. Suppose q' is an intersection point of S and S_1^R. Since S_1^R is a duplicate of S, there is a point p' on S at the same relative position as q' on S_1^R. Formally, if $p=(x_p, y_p)$, $q=(x_q, y_q)$ and $q'=(x_q+\Delta x, y_q+\Delta y)$, then $p'=(x_p+\Delta x, y_p+\Delta y)$. Since $p', q' \in S$, we can shift the entire segment of the snake between p' and q' by $-\Delta x$ in the x-coordinate and $-\Delta y$ in the y-coordinate, obtaining a shorter snake connecting p and q (see Fig. 5). Again, the minimality of S is contradicted. □

Lemma 2.19. *Let Ω be the set of snakes consisting of S and all the S_i^L and S_i^R. Then no two members of Ω intersect.*

Proof. It suffices to show, by induction on n, that for any $n>0$, no sequence of n successive members of Ω contains an intersection point. For $n=1$ and $n=2$, the result follows from Lemmas 2.17 and 2.18, respectively. Suppose that no sequence of $n \geqslant 2$ successive members of Ω has an intersection point. Consider a sequence of $n+1$ successive members of Ω. By the inductive hypothesis, the only possible intersection is between the first and last duplicates of S in the sequence. Denote them by S_1 and S_{n+1}. Choose an arbitrary duplicate between them, say S_j, $1<j<n+1$. Consider the infinite line l through p and q. Regard the half-plane on one side of l as being positive and the other half as being negative. Now, let a and b be extremal points on S_j as far as distance to l is concerned, where a is of maximal distance relative to the positive half-plane, and b is of maximal distance relative to the negative half-plane. (Clearly, a and b might be located on l itself.) See Fig. 6 for example. Denote by S^* the segment of S_j that connects a and b. By the inductive hypothesis, $S_1 \cap S^* = \emptyset$ and $S_{n+1} \cap S^* = \emptyset$. Also, neither S_1 nor S_{n+1} can 'take a detour around' S^* because they are all duplicates of the same snake, so that their extremal points are at the same distance from l as are a and b. Hence, S^* is a borderline that separates S_1 from S_{n+1}, which, therefore, cannot intersect. □

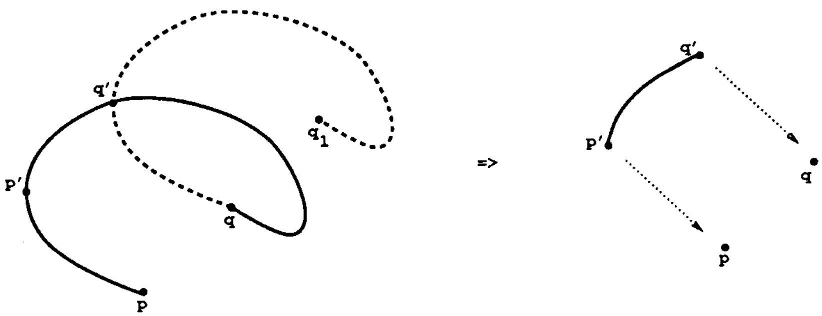


Fig. 5. We can shift the entire segment of the snake between p' and q' to obtain a shorter snake connecting p and q.

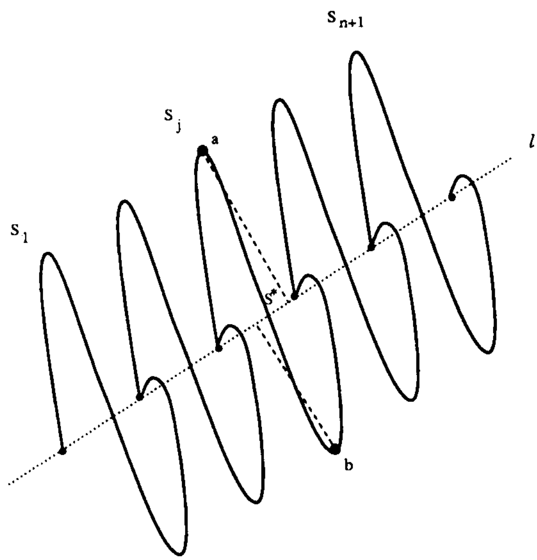


Fig. 6. As far as distance to l is concerned, a and b are the extremal points on S_j. S^* is the segment of S_j that connects a and b and creates a borderline that separates S_1 and S_{n+1}.

Proof of Theorem 2.16 (*continued*). Let us be given a line segment L in the infinite plane. The set of integer grid points of $\mathscr{Z} \times \mathscr{Z}$ with the property that L strictly intersects their unit squares (touching edges is not enough) is called a *slice* of the grid. The *size* of a finite slice is the number of grid-points it contains. The distance between two points p and q in the grid is taken to be the size of the maximal slice from among the set of all slices that are induced by line segments connecting the unit square of p with the unit square of q. Use $d(p, q)$ to denote such a distance. In the following, we will talk about infinite slices of the grid that are induced by infinite lines parallel to the line l that passes through p and q. We call these *l-slices.*

Lemma 2.20. *Let V be an l-slice and let $P = V \cap S$. P inherits an ordering from the order of points on S from p to q. Let A and B be two successive points in P, then the size of V's segment between A and B is bounded by $d(p, q)$.*

Proof. Denote by A' and B' the translates of A and B on S_1^R. It is easy to verify that A' and B' are also contained in the slice V (see Fig. 7(a)). Whereas S_1^R is the first duplicate of S, the distance between any point on S and its translate on S_1^R is equal to the distance between p, the starting point of S, and q, the starting point of S_1^R, which is p's translate on S_1^R. So we have $d(A, A') = d(B, B') = d(p, q)$. Hence, if the size of V's segment between A and B were more than $d(p, q)$, A' would have been located between A and B, and B' would be beyond B. Now, since A and B are successive points in P, V contains no additional points of S between A and B. Hence, the segment of S connecting A to B lies entirely on one side of V, and the segment of S_1^R connecting A' and B' must lie entirely on that same side too. But if the order of the points in V is A, A', B and B', then the two segments must intersect (see Fig. 7(b)). This contradicts Lemma 2.18. □

Lemma 2.21. *Let V be an l-slice. Then $|V \cap S| \leqslant d(p, q)$.*

Proof. Let $b_0 \in V \cap S$. We may consider all the translates of b_0 in the sets S_i^R. Denote them by $b_1, b_2, \ldots$. Similarly, denote the translates of b_0 in the S_i^L by $b_{-1}, b_{-2}, \ldots$.

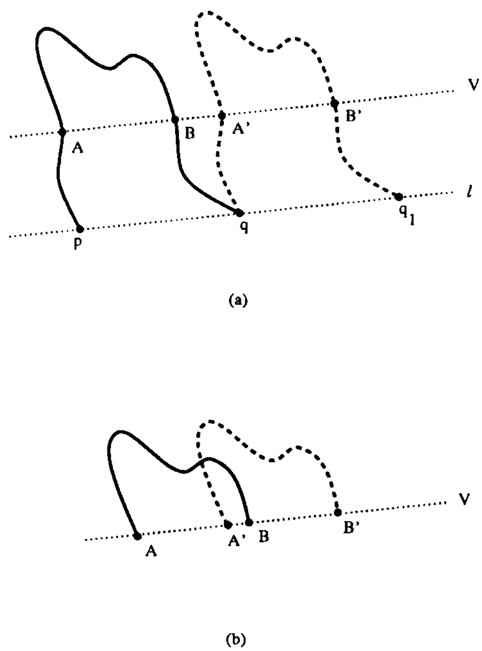


Fig. 7. (a) The translates of A and B on S_1^R. (b) Since the segment of S_1^R connecting A' and B' must be located at the same side of the slice as the correspondent segment of S, they must intersect.

Clearly, all the b_i are in V. Partition the slice V into equal sized blocks by the b_i, with block i being the portion of V between b_i and b_{i+1}. Now if S crosses the same relative position in blocks i and j ($i<j$), then it must intersect S^{R}_{j-i} (see Fig. 8), contradicting Lemma 2.19. Thus, S cannot cross V at similarly positioned points within any two blocks. Since the size of a block is at the most $d(p,q)$, the claim follows. □

Proof of Theorem 2.16 (*conclusion*). From Lemmas 2.20 and 2.21 we conclude that the snake S is actually confined to a corridor of polynomial width. Hence, Theorem 2.15 implies immediately that Problem 2.9 is decidable, and is actually in PSPACE. That Problem 2.9 is PSPACE-hard is achieved by applying simple changes to the reduction from a space bounded Turing machine that Ebbinghaus [3] presented in his proof of the PSPACE-hardness of the connectability problem in a rectangle. □

2.4. *The strict case*

The basic version of the strict connectability problem (Problem 2.22) was formulated by Ebbinghaus [2] and proved by him to be undecidable.

Problem 2.22 (*Strict connectability*). Given a tiling system T, a tile type $\tau_0 \in T$, and two points $p, q \in \mathscr{Z} \times \mathscr{Z}$, is there a T-snake σ connecting p and q, such that $\sigma(p) = \tau_0$?

Theorem 2.23 (Ebbinghaus [2]). *Problem 2.22 is complete for r.e.*

The proof involves a reduction from the halting problem for two-register machines. An alternative proof, based on a reduction from the Post correspondence problem, appears in [4].

The following two strict versions of the general snake problem were also found to be complete for r.e., as an immediate result of Ebbinghaus' construction.

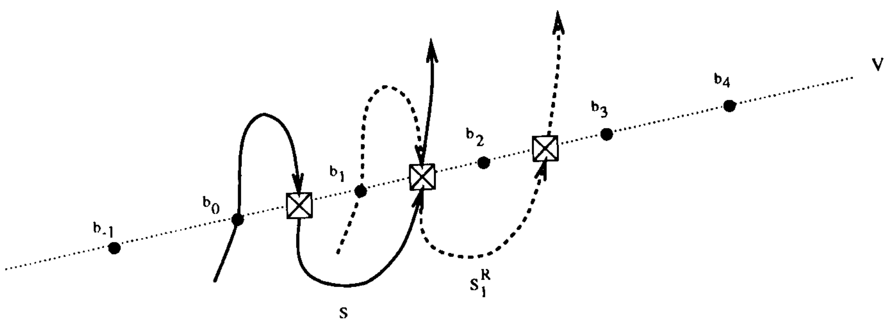


Fig. 8. S crosses the same relative position in blocks $i=0$ and $j=1$. Hence, it must intersect $S^{\mathrm{R}}_{j-i}=S^{\mathrm{R}}_1$.

Problem 2.24. Given a tiling system T, a tile type $\tau_0 \in T$ and two points $p, q \in \mathscr{Z} \times \mathscr{Z}$, is there a T-snake σ connecting p and q, such that for some $l \in S_\sigma, \sigma(l) = \tau_0$?

Problem 2.25. Given a tiling system T, a tile type $\tau_0 \in T$ and three points $p, q, l \in \mathscr{Z} \times \mathscr{Z}$, is there a T-snake σ connecting p and q, such that $l \in S_\sigma$ and $\sigma(l) = \tau_0$?

2.5. Connectability in portions of the plane

We now limit the portion of the plane in various ways. We have already considered the bounded cases of a square and a rectangle, and the semi-bounded case of a strip. In these cases, the snake problem was proved to be decidable. The completely unlimited case is also decidable. The problems presented now deal with limited, but unbounded, portions of the plane. That is, they are not totally bounded in either of their dimensions. As we shall show, all these "intermediate" cases give rise to undecidable connectability problems.

Problem 2.26. Given a tiling system T and two points p and q , is there a T-snake σ connecting p and q and lying entirely within the upper half-plane, $\mathscr{Z} \times \mathscr{N}$?

Problem 2.27. Given a tiling system T and two points p and q, is there a T-snake σ connecting p and q and lying entirely within the positive quadrant, $\mathscr{N} \times \mathscr{N}$?

Problems 2.26 and 2.27 were proved to be complete for r.e. in [2]. The proofs rely on the construction for the strict case. A circular version of the general snake problem and a 3-dimensional version were also proved in [2] to be complete for r.e., relying on the same construction. Applying simple changes to this construction, we have been able to prove that the following additional problems are hard for r.e.

Problem 2.28. Given a tiling system T, two points p and q, and some increasing linear function $f(x) = ax + b$, $a > 0$, is there a T-snake σ connecting p and q and lying entirely within $\{(x, y) \mid (-b/a) \leqslant x,\ 0 \leqslant y \leqslant f(x)\}$?

Problem 2.29. Given a tiling system T, two points p and q, and some closed portion P of the plane, is there a T-snake σ connecting p and q and lying entirely within $(\mathscr{Z} \times \mathscr{Z}) - P$?

Problem 2.30. Given a tiling system T, two points p and q, and a set of k additional points $c_1, \ldots, c_k$, is there a T-snake σ connecting p and q such that $c_1, \ldots, c_k \in S_\sigma$?

To prove that Problem 2.28, for example, is hard for r.e., we use the fact that the snake in the original construction of [2] is already located under some specific linear function. Thus, the tiling system can be modified to allow the tiled snake to be "stretched" along the x-coordinate, yielding a new snake having the same properties as the original one, but located under the required linear function.

Applying simple changes to the construction of the alternative proof of Theorem 2.23 (details appear in [4]), we are also able to show that the following problem is complete for r.e.

Problem 2.31. Given a tiling system T, two points p and q, and some point $l \in \mathscr{Z} \times \mathscr{Z}$, is there a T-snake σ connecting p and q and lying entirely within $\mathscr{Z} \times \mathscr{Z} - \{l\}$?

This result is to be contrasted with Theorem 2.16. Removing a single point from the plane results in undecidability!

3. Infinite snakes

In this section we consider *infinite snake problems*. Formally, an *infinite snake skeleton* is an infinite ordered sequence $\{q_n \in \mathscr{Z} \times \mathscr{Z}\}_{n \in \mathscr{Z}}$ such that for each $j \in \mathscr{Z}$, q_j and q_{j+1} are adjacent. A *one-way infinite snake skeleton* is an infinite snake skeleton in which the n's come from $\mathscr{N}$ instead of $\mathscr{Z}$. Given a tiling system T, an *infinite T-snake* is an infinite snake skeleton S together with a function $\sigma : S \to T$, assigning to each skeleton point $q_i \in S$ a tile type $\sigma(q_i) \in T$, such that for each $i \in \mathscr{Z}$ the adjacent edges of $\sigma(q_i)$ and $\sigma(q_{i+1})$ are monochromatic. If $q_i = q_j$ for $q_i, q_j \in S$, then $\sigma(q_i) = \sigma(q_j)$. A *one-way infinite T-snake* is defined similarly, except that the skeleton S is one-way infinite. Given an infinite (one-way infinite) T-snake σ, we denote its skeleton by S_σ.

In general, an infinite snake problem asks, given a tiling system T and some portion P of the plane, whether there is an infinite (one-way infinite) T-snake whose skeleton lies entirely within P.

3.1. Directed snakes

Recall the directed and fully directed tiling system over T ($\hat{T}$ and $\hat{T}_{\mathrm{f}}$, respectively), that were used in Section 2.1 to define directed and fully directed versions of finite snakes. Similarly, we define directed and fully directed versions of infinite and one-way infinite snakes.

A *directed infinite snake over T*, or an *infinite $\hat{T}$-snake*, is an infinite T-snake σ when the arrows are disregarded, and, in addition, for every pair of consecutive points q_i and q_{i+1} in the skeleton S_σ, the arrow in $\sigma(q_i)$ is directed towards the edge adjacent to q_{i+1}. A *fully directed infinite snake over T*, or an *infinite $\hat{T}_{\mathrm{f}}$-snake*, is an infinite T-snake σ when the arrows are disregarded, and, in addition, for every three consecutive points q_{i-1}, q_i and q_{i+1} in the skeleton S_σ, the arrow in $\sigma(q_i)$ is directed from the edge adjacent to q_{i-1} towards the edge adjacent to q_{i+1}. Note that the infinite skeleton S of a (fully) directed snake cannot include loops (i.e. for all $q_i, q_j \in S$, $q_i = q_j$ iff $i = j$). The formal definitions of a *one-way infinite $\hat{T}$-snake* and a *one-way infinite $\hat{T}_{\mathrm{f}}$-snake* are similar. (The arrow in $\sigma(q_0)$ for the first point q_0 in the one-way infinite skeleton of a one-way infinite $\hat{T}_f$-snake σ is directed from an arbitrary edge towards the edge adjacent to q_1.)

The following two claims are immediate.

Claim 3.1. *Given a tiling system T and some portion P of the plane, there is an infinite T-snake lying entirely within P iff there is an infinite $\hat{T}$-snake lying entirely within P iff there is an infinite $\hat{T}_f$-snake lying entirely within P.*

Claim 3.2. *Given a tiling system T and some portion P of the plane, there is a one-way infinite T-snake lying entirely within P iff there is a one-way infinite $\hat{T}$-snake lying entirely within P iff there is a one-way infinite $\hat{T}_f$-snake lying entirely within P.*

Hence, when considering specific infinite snake problems, we can assume that tiling systems and snakes are (fully) directed.

3.2. Infinite snakes in a strip

Let S_k denote a strip of width k in the grid $\mathscr{Z} \times \mathscr{Z}$. Without loss of generality, assume $S_k = \{(x, y) \mid 1 \leqslant x \leqslant k\}$. Now, consider the following decision problem.

Problem 3.3. Given a tiling system T and a natural number k, is there an infinite T-snake whose skeleton lies entirely within S_k?

Theorem 3.4. *Problem* 3.3 *is decidable.*

Proof. At the heart of the proof, we show that the existence of an infinite T-snake within the strip S_k necessarily implies the existence of a periodic infinite T-snake within S_k (i.e. a snake built of repetitions of a certain shape and pattern). We provide a constructive method for finding such a periodic snake, if it exists. The proof relies on a combination of the techniques used in the proofs of Theorems 2.8 and 2.11 (i.e. the existence of a PTIME algorithm for the connectability problem in a fixed-width rectangle and the existence of a PSPACE algorithm for the connectability problem in a strip).

Throughout the proof, we consider full tilings (not snakes) of the strip S_k using the types of $\hat{T}_f$ and an additional blank type (i.e. a white tile containing no arrows). Rules of tiling are changed so that two edges may be adjacent if and only if one of the following holds:

(1) One of the edges includes the head of an arrow, the other includes the tail of an arrow, and they are monochromatic.

(2) Neither of the edges includes the head or tail of an arrow (and there is no restriction on the coloring).

Boundary conditions are added, so that the edges adjacent to the boundaries of the strip include neither the head nor the tail of an arrow. Note that under these new rules the only possible tilings of S_k are those that are totally blank or those whose nonblank tiles create patterns of legal (closed or infinite) fully directed snakes. See Fig. 9.

In the sequel, we use $S_{(k,n)}$ to denote a segment of the strip with width k and height n. Without loss of generality, assume that $S_{(k,n)}$ is the segment $\{(x,y) \mid 1 \leqslant x \leqslant k, 1 \leqslant y \leqslant n\} \subset S_k$. We also use the following terminology: types of $\hat{T}_f$ whose bottom edge includes the tail of an arrow are termed *entries* and types of $\hat{T}_f$ whose bottom edge includes the head of an arrow are termed *exits*. See Fig. 10(a). A tiled "slice" of S_k (i.e. a segment of width k and height 1) is said to have a *periodic pattern* if the absolute value of the difference between the number of entries in the slice and the number of exits in the slice equals 1. An example is presented in Fig. 10(b).

Lemma 3.5. *An infinite $\hat{T}_f$-snake within S_k, exists iff there is a legal tiling of $S_{(k,n)}$ for some $2 \leqslant n \leqslant (1+|\hat{T}_f|)^k + 1$, such that the bottom slice, $y=1$, and the top slice, $y=n$, are identically tiled with a periodic pattern.*

Proof. ($\Leftarrow$) This direction is proved using constructive arguments, based on the pigeonhole principle, as follows.

Regard the tiled segment $S_{(k,n)}$ as a "tiled block". Concatenate a sequence of $2k$ copies of this block, such that each pair of consecutive blocks overlap in their

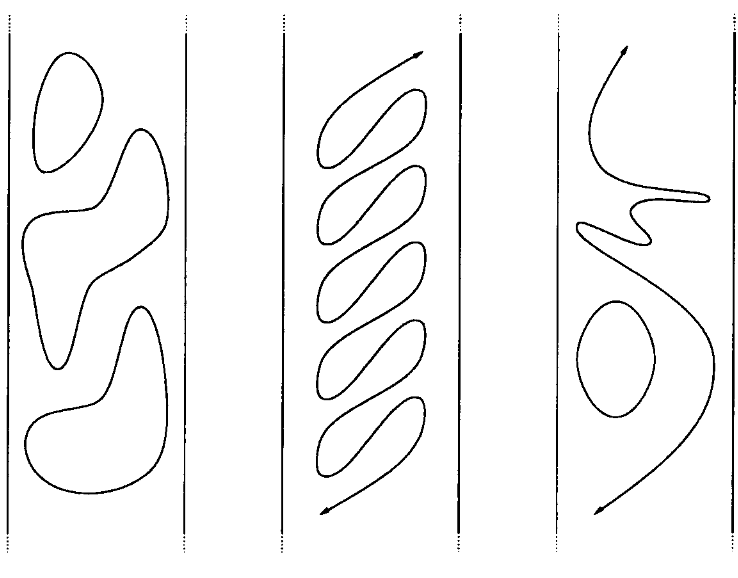
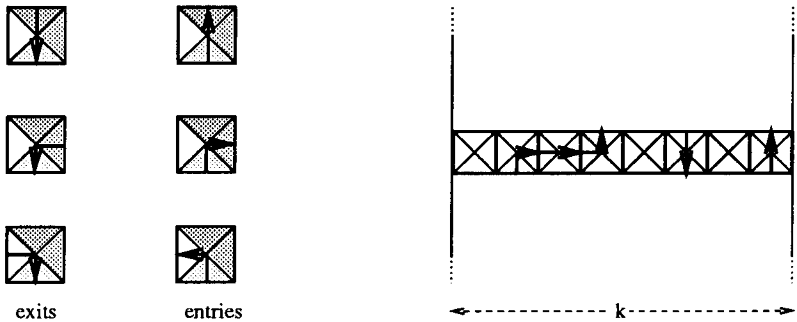


Fig. 9. Under the new rules, the only possible tilings of S_k that are not totally blank may be (a) ones whose nonblank tiles create legal closed snakes, (b) ones whose nonblank tiles create legal infinite snakes, (c) ones whose nonblank tiles create both closed and infinite snakes.



(a) (b)

Fig. 10. (a) Exits and entries; (b) a tiled slice with a periodic pattern.

identically tiled edge-slices. In this way, we obtain a tiling of $S_{(k,2k(n-1)+1)}$, which is a legal tiling, since the original block was legally tiled. See Fig. 11.

By the construction, all the slices $\{y=i(n-1)+1\}_{0\leqslant i\leqslant 2k}$ are identically tiled with a periodic pattern. Start from the leftmost tile that contains an entry or an exit in the middle slice $y=k(n-1)+1$. Use the arrows inside the tiles to simulate "traveling" within the tiled part of the strip. This "tour" terminates when we reach an untiled point or a point that has already been visited. Since only a finite part of the strip is tiled, the process of traveling must eventually terminate. Moreover, notice that the rules of tiling allow such a tour to proceed only along a legal fully directed snake. Thus, one of the two following possibilities must hold:

(1) The most recently visited point is tiled with a type having an arrow directed to the starting point (i.e. we have traveled along a legal closed snake).

(2) The most recently visited point is tiled with a type having an arrow directed to a point outside the tiled part of the strip (i.e. we have traveled along a legal snake ending in one of the edge-slices).

In the former case, mark all the points in the middle slice that were already visited during the tour, and begin a new tour from the leftmost unmarked point that is tiled with a type containing an entry or an exit. We claim that there must be at least one such unmarked point in the middle slice. The reason is that a closed traveling path crosses an equal number of entries and exits in each slice, but the middle slice is periodically tiled, and so has at least one additional entry or exit.

Since there is only a finite number of points in a slice, the latter case, where the tour terminates by reaching an untiled point, must eventually occur. In this case, the most recently traveled path must have crossed at least $k+1$ slices from among the set $\{y=i(n-1)+1\}_{0\leqslant i\leqslant 2k}$. Each such slice contains at most k entries and exits, so there must be a pair of two different slices that were visited at the same specific entry or exit. (Recall that all slices in the set $\{y=i(n-1)+1\}_{0\leqslant i\leqslant 2k}$ are identically tiled, thus having exactly the same entries and exits in their pattern.) Without loss of generality, assume

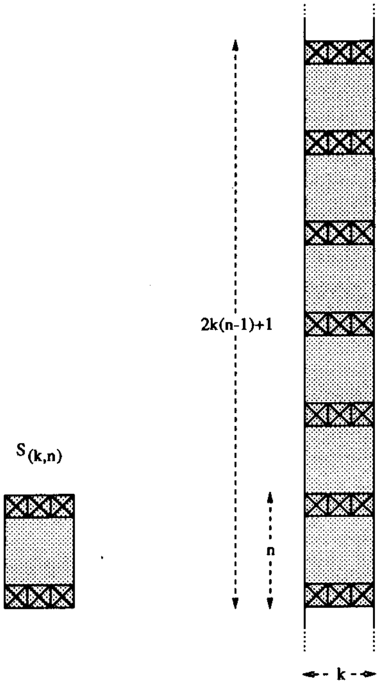


Fig. 11. Concatenation of $2k$ copies of the tiled block $S_{(k,n)}$, such that each pair of consecutive blocks overlaps in their identically tiled edge-slices, yields a legal tiling of $S_{(k,2k(n-1)+1)}$.

the existence of a pair of two different such slices that were visited at the same specific entry. Refer to this entry as the *snake entry*. The existence of an infinite fully directed snake is now derived from the fact that in a tiling of the infinite strip S_k, that is built of concatenations of infinitely many tiled blocks, one can simply use the directions of the arrows to travel, indefinitely, from each snake entry to the next one along a legal fully directed snake. (One can also travel indefinitely backwards, moving along from each snake entry to the previous one.)

($\Rightarrow$) The existence of an infinite $\hat{T}_f$-snake in the strip S_k immediately implies the existence of a one-way infinite $\hat{T}_f$-snake σ in the strip. Without loss of generality, assume that the first point of σ's skeleton, q_0, is lower than all the other points in S_σ. Consider a tiling of the "semi-infinite" part of the strip S_k above q_0, in which each point $p \in S_k \cap S_\sigma$ is tiled with the type $\sigma(p)$ and all other points in $S_k - S_\sigma$ are tiled with the blank type. Clearly, this is a legal tiling. It is also easy to check that all the slices of the tiled part of S_k have a periodic pattern. To complete the proof of this direction, notice that there are only $(1+|\hat{T}_f|)^k$ ways of tiling a slice of S_k with different patterns, so there must be two identically tiled slices within a segment of height $(1+|\hat{T}_f|)^k+1$. □

Proof of Theorem 3.4 (*conclusion*). To complete the proof of the theorem, one can check all possible tilings of $S_{(k,n)}$ for $2 \leqslant n \leqslant (1+|\hat{T}_f|)^k+1$, to decide if any infinite

$\hat{T}_f$-snake within S_k exists. By Claim 3.1, this is also a decision procedure for the existence of an infinite T-snake in S_k. □

An immediate corollary from the proof of Theorem 3.4 is that Problem 3.3 is in PSPACE if k is given in unary. In order to decide whether there exists an infinite snake in the given strip, we carry out a reduction to the rectangle tiling problem, which is PSPACE-complete [11]. Although we have not managed to prove a matching lower bound, we do conjecture that Problem 3.3 (with k in unary) is indeed PSPACE-complete. This conjecture gains some support from the corresponding result for the finite case (Corollary 2.13).

3.3. The strict case

The following two problems are straightforward extensions of the strict connectability problem (Problem 2.22) to infinite snakes.

Problem 3.6. Given a tiling system T and a specific tile type $\tau_0 \in T$, is there a one-way infinite T-snake σ, such that $\sigma(q_0)=\tau_0$, where q_0 is the first point of σ's skeleton?

Problem 3.7. Given a tiling system T and a specific tile type $\tau_0 \in T$, is there an infinite T-snake σ, that contains τ_0?

Problem 3.6 has already been considered in [3]. The following is a direct result of the methods used in the proofs for the bounded cases of snake problems [3] and for the strict connectability problem [2].

Theorem 3.8 (Ebbinghaus [3]). *Problem* 3.6 *is complete for co-r.e.*

Using methods introduced in [4] for a PCP-based undecidability proof for the strict connectability problem, we have also managed to provide a similar result for Problem 3.7.

Theorem 3.9. *Problem* 3.7 *is complete for co-r.e.*

A recurring theme in the present paper is concerned with comparing classical tiling problems and snake problems. It is, therefore, natural to consider snake versions of the *recurring tiling problems* of [7].

Problem 3.10. Given a tiling system T and a specific tile type $\tau_0 \in T$, is there a one-way infinite T-snake σ, in which τ_0 occurs infinitely often?

Problem 3.11. Given a tiling system T and a specific tile type $\tau_0 \in T$, is there an infinite T-snake σ, in which τ_0 occurs infinitely often?

In analogy to recurring tiling problems, which are Σ_1^1-complete [7], we have the following theorem.

Theorem 3.12. *Problems* 3.10 *and* 3.11 *are* Σ_1^1*-complete.*

Theorem 3.12 is proved using tile construction ideas from [2], but the general line of proof is analogous to the proof that recurring tiling is Σ_1^1-complete [7]. We omit the details.

We note that [3] also considers infinite snake problems with restrictions on the structure of the snake. It is shown therein that the following two problems are Σ_1^1-complete.

Problem 3.13. Given a tiling system T, is there a one-way infinite T-snake σ, whose skeleton S_σ does not ultimately become a straight line?

Problem 3.14. Given a tiling system T, is there a one-way infinite T-snake σ, whose skeleton S_σ is nonrecursive?

It is possible to show that problems 3.6, 3.10, 3.13 and 3.14 remain in their undecidability level even when a half-plane or a quadrant are considered, rather than the entire plane. Moreover, the undecidability level does not change even when the portion of the plane is limited to $\{(x, y) \mid (-b/a) \leqslant x, 0 \leqslant y \leqslant f(x)\}$, where $f(x) = ax + b$, $a > 0$. The proofs are obtained by very simple changes to the basic proofs.

4. Discussion

Figures 12 and 13 summarize the results proved or stated in the paper. These results, taken together, point to a clear analogy between snake problems and classical tiling problems. The complexity results for the corresponding fixed-width, bounded, unbounded and recurring cases of snake and tiling problems are essentially the same. Furthermore, the proof methods used for the lower bounds on snake problems are conceptually the same as those used for tiling problems; all are based on simulations of computations of Turing machines or register machines. Tiling problems have been very helpful in establishing lower bounds on the difficulty of other problems e.g. in logics of programs [6]. Thus, a potential direction of future work on snake problems would be to find similar applications. It should be noted that tiling problems are combinatorial in nature, while snake problems have unique geometric properties that might turn out to be useful for other applications. This direction is corroborated by recent applications of snake problems to domino games [5] and the uncertainty principle for physical systems [10].

The general snake problem in the whole plane appears to be unique among snake problems, since all other reasonable unbounded versions we have considered

Finite snake (connectability) problems		
Bounded	Problem 2.4 (connectability in a fixed-width rectangle)	PTIME
	Problem 2.2 (connectability in a square)	NP [3]
	Problem 2.3 (connectability in a rectangle)	PSPACE [3]
Semi-bounded	Problem 2.10 (connectability in a strip)	PSPACE
	Problem 2.14 (connectability in a corridor)	PSPACE
Unbounded	Problem 2.9 (unlimited connectability)	PSPACE
Strict	Problem 2.22 (strict connectability)	r.e. [2]
	Problem 2.24 (strict connectability – version I)	r.e.
	Problem 2.25 (strict connectability – version II)	r.e.
Portions of the plane	Problem 2.26 (connectability in a half-plane)	r.e. [2]
	Problem 2.27 (connectability in a quadrant)	r.e. [2]
	Problem 2.28 (connectability under a linear function)	r.e.
	Problem 2.29 (connectability in the plane outside a forbidden area)	r.e.
	Problem 2.31 (connectability in the plane except for a forbidden point)	r.e.
	Problem 2.30 (connectability through a given set of points)	r.e.

Fig. 12. Summary of results concerning finite snake problems.

(including strict cases and those with limited portions of the plane) were found to be undecidable. The decidability of the general snake problem in the whole plane should also be contrasted with the undecidability of its tiling counterpart. It should be noted, though, that the general tiling problem in the whole plane is also unique among tiling problems, since its undecidability is much harder to prove [1, 13].

Infinite snake problems		
Semi-bounded	Problem 3.3 (infinite snake in a strip)	in PSPACE (hardness conjectured)
Unbounded	Problem 4.1 (unlimited infinite snake)	open
Strict	Problem 3.6 (strict, one-way infinite snake)	co-r.e. [3]
	Problem 3.7 (strict, infinite snake)	co-r.e.
Recurring	Problem 3.10 (recurring, one-way infinite snake)	Σ_1^1
	Problem 3.11 (recurring, infinite snake)	Σ_1^1
Miscellaneous	Problem 3.13 (infinite skeleton is not ultimately a straight line)	Σ_1^1 [3]
	Problem 3.14 (infinite skeleton is not recursive)	Σ_1^1 [3]
Portions of the plane	Problems 3.6, 3.10, 3.13, 3.14 in a half-plane, a quadrant or under a linear function, remain in the same complexity classes.	

Fig. 13. Summary of results concerning infinite snake problems.

A partial explanation for the decidability of the unlimited connectability problem may be obtained by analyzing the central argument of the proof, stated in Lemma 2.18. This lemma states that the minimal-length snake connecting two points in the plane cannot intersect with its right and left duplicates; hence, the right and left translates of each of the other points in the minimal snake's skeleton cannot "participate" in the skeleton either. This is a rather strong statement, which leads to decidability for the unlimited case. However, it fails even if only elementary constraints on the portion of the plane, the structure of the snake, or the existence of certain types, are added.

One question left open here is that of determining the exact complexity of the strip case of infinite snake problems. As mentioned earlier, we conjecture that it is PSPACE-complete. Another question is whether or not the unlimited case of infinite snake problems is decidable.

Problem 4.1. Given a tiling system T, is there an infinite T-snake within the infinite grid $G=\mathscr{Z}\times\mathscr{Z}$?

We conjecture that Problem 4.1 is undecidable, but it seems that this would be difficult to prove. We have not been able to find a way to adjust the proof techniques of the other undecidability results for this purpose.

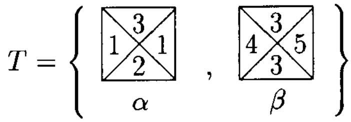


Fig. 14.

Finally, consider the connection, for a given tiling system T, between the existence of a T-tiling of the infinite grid G, and the T-connectability of *every* two points in G. Obviously, the former implies the latter. The converse, however, is false.

Proposition 4.2. *The existence of a T-tiling of G does not necessarily follow from the fact that there is a T-snake connecting any two points in G.*

Proof. Consider the tiling system given in Fig. 14. Clearly, a T-tiling of G does not exist (in fact, even a 2×2 square cannot be tiled by T). Yet, for any pair of points $p, q \in G$, where $p=(x_p, y_p)$, $q=(x_q, y_q)$ and $y_p \leqslant y_q$, type α can be repeatedly used to tile a horizontal T-snake connecting the point p with the point (x_q, y_p) and type β can then be repeatedly used to continue with a vertical T-snake connecting (x_q, y_p) to q. □

Additional observations of this kind appear in [4].

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