ON THE GHOST CENTRE OF LIE SUPERALGEBRAS

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Abstract. We define a notion of *ghost centre* of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which is a sum of invariants with respect to the usual adjoint action (centre) and invariants with respect to a twisted adjoint action ("anticentre"). We calculate the anticentre in the case when the top external degree of $\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. We describe the Harish-Chandra projection of the ghost centre for basic classical Lie superalgebras and show that for these cases the ghost centre coincides with the centralizer of the even part of the enveloping algebra.

The ghost centre of a Lie superalgebra plays a role of the usual centre of a Lie algebra in some problems of representation theory. For instance, for $\mathfrak{g} = \mathrm{osp}(1, 2l)$ the annihilator of a Verma module is generated by the intersection with the ghost centre.

1. Introduction

1.1. Let $\mathfrak{g}_0$ be a complex finite dimensional Lie algebra and $\mathcal{Z}(\mathfrak{g}_0)$ be the centre of its universal enveloping algebra. Then $\mathcal{Z}(\mathfrak{g}_0)$ acts on a simple $\mathfrak{g}$-module by an infinitesimal character and consequently, such characters separate representations. Moreover, in the case when $\mathfrak{g}_0$ is semisimple, the annihilator of a Verma module is generated by the kernel of the corresponding infinitesimal character.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a complex finite dimensional Lie superalgebra and $\mathcal{Z}(\mathfrak{g})$ be the (super)centre of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. All $\mathfrak{g}$-modules considered below are assumed to be $\mathbb{Z}_2$-graded and "$\mathfrak{g}$-simple module" means simple as graded module. The centre $\mathcal{Z}(\mathfrak{g})$ acts on a simple $\mathfrak{g}$-module by an infinitesimal character, but, even in the "nice" case $\mathfrak{g} = \mathrm{osp}(1, 2l)$, the annihilator of a Verma module is not always generated by the kernel of the corresponding infinitesimal character. In [GL] we described, for the case $\mathfrak{g} = \mathrm{osp}(1, 2l)$, a polynomial subalgebra $\widetilde{\mathcal{Z}}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ which acts on a simple module by "supercharacter". The annihilator of a Verma module is generated by the kernel of the corresponding supercharacter.

In this paper we introduce a notion of *ghost centre* $\widetilde{\mathcal{Z}}(\mathfrak{g})$ (see 2.1.2). This is a subalgebra of $\mathcal{U}(\mathfrak{g})$ which contains both $\mathcal{Z}(\mathfrak{g})$ and a centre of $\mathcal{U}(\mathfrak{g})$ considered as associative algebra. The algebra $\widetilde{\mathcal{Z}}(\mathfrak{g})$ acts on a simple module by "supercharacter" and separates graded representations (see 2.2).

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By definition, $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is a sum of $\mathcal{Z}(\mathfrak{g})$ and so-called *anticentre* $\mathcal{A}(\mathfrak{g})$. The last one is the set of invariants of $\mathcal{U}(\mathfrak{g})$ with respect to a 'nonstandard adjoint action' $\mathrm{ad}'\,\mathfrak{g}$ introduced in [ABF]. The product of two elements from the anticentre lies in the centre and the product of an element from the centre and an element from the anticentre belongs to the anticentre.

As well as $\mathcal{Z}(\mathfrak{g})$ itself, $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is not easy to describe and, in general, it is not noetherian algebra. However, in the case when the top external degree $\Lambda^{top}\mathfrak{g}_1$ of $\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module, $\mathcal{A}(\mathfrak{g})$ itself as well as its image in the symmetric algebra can be easily described—see Theorem 3.3. The above condition on $\Lambda^{top}\mathfrak{g}_1$ holds for the simple finite-dimensional Lie superalgebras apart from the $W(n)$ type.

The existence of non-zero anticentral elements implies two "negative" results. The first one is that the direct generalization of the Gelfand-Kirillov conjecture does not hold for the Lie superalgebras with even dimensional $\mathfrak{g}_1$—see 3.5.2. The second one is that Separation theorem does not hold for the classical basic Lie superalgebras apart from the simple Lie algebras and the superalgebras $\mathrm{osp}(1,2l)$—see 4.5.

1.2. In the case $\mathfrak{g} = \mathrm{osp}(1,2l)$ Arnaudon, Bauer, Frappat ([ABF]) and Musson ([Mu]) constructed a remarkable even element T in the enveloping algebra $\mathcal{U}(\mathfrak{g})$. This element is $\mathrm{ad}'\,\mathfrak{g}$-invariant and its Harish-Chandra projection is the product of hyperplanes corresponding to the positive odd roots. The element T has been called 'Casimir's ghost' in [ABF], since its square belongs to the centre.

In 3.3 we construct such element $T \in \mathcal{A}(\mathfrak{g})$ for any $\mathfrak{g}$ such that $\Lambda^{top}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. The image of T in the symmetric algebra belongs to $\Lambda^{top}\mathfrak{g}_1$. In Section 4 we show that in the case when $\mathfrak{g}$ is a basic classical Lie superalgebra, the Harish-Chandra projection of T is also the product of hyperplanes corresponding to the positive odd roots.

In [S2] A. Sergeev described the set of 'anti-invariant polynomials' which are the invariants of the dual algebra $\mathcal{U}(\mathfrak{g})^*$ with respect to the nonstandard adjoint action $\mathrm{ad}'\,\mathfrak{g}$.

1.3. **Content of the paper.** In Section 2 we define our main objects: the anticentre $\mathcal{A}(\mathfrak{g})$ and the ghost centre $\widetilde{\mathcal{Z}}(\mathfrak{g})$. We describe the action of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ on the modules of finite length in the case when $\mathfrak{g}$ is finite dimensional.

In Section 3 we show that $\mathcal{A}(\mathfrak{g})$ is equal to zero if $\dim\mathfrak{g}_1$ is infinite. Moreover all elements of $\mathcal{A}(\mathfrak{g})$ are either even (if $\dim\mathfrak{g}_1$ is even) or odd (otherwise). We describe the structure of $\mathcal{A}(\mathfrak{g})$ when $\Lambda^{top}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. Namely, we prove that as a vector space $\mathcal{A}(\mathfrak{g})$ is isomorphic to $\mathcal{Z}(\mathfrak{g}_0)$. The central step of the proof is Theorem 3.2.3 which states that for any $\mathfrak{g}_0$-module L, the induced $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ and the coinduced $\mathfrak{g}$-module $\mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ are isomorphic up to grading. This theorem allows us to define T as a unique up to a scalar $\mathrm{ad}'\,\mathfrak{g}$-invariant element inside the $\mathrm{ad}'\,\mathfrak{g}$-module generated by 1.

In Section 4 we consider the case when $\mathfrak{g}$ is a complex classical basic Lie superalgebra. In this case, the Harish-Chandra projection of $\mathcal{Z}(\mathfrak{g})$ is described by Kac and Sergeev (see [S1]). In Corollary 4.2.4, we describe the Harish-Chandra projection of $\mathcal{A}(\mathfrak{g})$.

We say that an element $u\in\mathcal{U}(\mathfrak{g})$ acts on a module M by a *superconstant* if it acts by the multiplication by a scalar on each graded component $M_{\bar{i}}$ $(i=0,1)$. In the case when $\mathfrak{g}$ is finite dimensional and $\dim\mathfrak{g}_1$ is even, any element of $\tilde{\mathcal{Z}}(\mathfrak{g})$ acts on a simple module M by a superconstant (see 2.2). In Corollary 4.4.4 we show that if $\mathfrak{g}$ is a basic classical Lie superalgebra then any element of $\mathcal{U}(\mathfrak{g})$ acting by a superconstant on each simple finite dimensional module belongs to $\tilde{\mathcal{Z}}(\mathfrak{g})$. Moreover $\tilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the centre (and the centralizer) of the even part $\mathcal{U}(\mathfrak{g})_{\bar{0}}$ of the universal enveloping algebra. For the case $\mathfrak{g}=\mathrm{osp}(1,2l)$ the last result was proven in [GL].

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2. Ghost centre

In this paper the ground field is $\mathbb{C}$. Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a finite dimensional Lie superalgebra such that $\mathfrak{g}_1\neq 0$. Everywhere except 2.1, $\mathfrak{g}$ is assumed to be finite dimensional. All $\mathfrak{g}$-modules are assumed to be $\mathbb{Z}_2$-graded. We denote by Π the parity change functor: $\Pi(M):=M\otimes\Pi(\mathbb{C})$ where $\Pi(\mathbb{C})$ is the trivial odd representation. Denote by $\mathcal{U}(\mathfrak{g})$ the enveloping superalgebra of $\mathfrak{g}$ and by $\mathcal{Z}(\mathfrak{g})$ the (super)centre of $\mathcal{U}(\mathfrak{g})$.

2.1. For a homogeneous $u\in\mathcal{U}(\mathfrak{g})$ denote by $d(u)$ its $\mathbb{Z}_2$-degree. For a $\mathcal{U}(\mathfrak{g})$-bimodule M one defines the adjoint action of $\mathfrak{g}$ on M by setting $(\mathrm{ad}\,g)m:=gm-(-1)^{d(g)d(m)}mg$ where $m\in M, g\in\mathfrak{g}$ are homogeneous elements and $d(m)$ denotes the $\mathbb{Z}_2$-degree of m. Define a twisted adjoint action ad' of $\mathfrak{g}$ on M as the adjoint action of $\mathfrak{g}$ on the bimodule $\Pi(M)$. One has

$$(\mathrm{ad}'\,g)(u)=gm-(-1)^{d(g)(d(m)+1)}mg.$$

Assume that M has a superalgebra structure such that $g(m_1m_2)=g(m_1)m_2$ and $(m_1m_2)g=m_1(m_2)g$ for all $g\in\mathfrak{g},\ m_1,m_2\in M$. Then for any homogeneous $m_1,m_2\in M$ and $g\in\mathfrak{g}$ one has

$$\begin{aligned}(\mathrm{ad}'\,g)(m_1m_2)&=((\mathrm{ad}\,g)m_1)m_2+(-1)^{d(g)d(m_1)}m_1((\mathrm{ad}'\,g)m_2)\\&=((\mathrm{ad}'\,g)m_1)m_2+(-1)^{d(g)(d(m_1)+1)}m_1((\mathrm{ad}\,g)m_2).\end{aligned}$$

Moreover if m is $\mathrm{ad}'\,\mathfrak{g}$-invariant then

$$(\mathrm{ad}'\,g)(m_1m)=((\mathrm{ad}\,g)m_1)m,\quad (\mathrm{ad}\,g)(m_1m)=((\mathrm{ad}'\,g)m_1)m. \tag{1}$$

2.1.1. **Example.** Let N be a $\mathcal{U}(\mathfrak{g})$-module and $\mathrm{End}(N)$ be the ring of its $\mathbb{C}$-linear endomorphisms. Then $\mathrm{End}(N)$ admits a natural structure of graded $\mathcal{U}(\mathfrak{g})$-bimodule. Let θ be the endomorphism of N which is equal to id (resp., $-\mathrm{id}$) on the even (resp., odd) component of N. Then θ is an even $\mathrm{ad}'\,\mathfrak{g}$-invariant homomorphism which commutes with the even elements of $\mathrm{End}(N)$ and anticommutes with the odd elements of $\mathrm{End}(N)$. The formulas (1) imply that $\mathrm{End}(N)$ considered as $\mathrm{ad}\,\mathfrak{g}$-module is isomorphic to $\mathrm{End}(N)$ considered as $\mathrm{ad}'\,\mathfrak{g}$-module. The similar assertion fails for $\mathcal{U}(\mathfrak{g})$ (the structure of $\mathcal{U}(\mathfrak{g})$ as $\mathrm{ad}'\,\mathfrak{g}$-module is given in Lemma 3.1.2).

2.1.2. Let us call *anticentre* $\mathcal{A}(\mathfrak{g})$ the set of elements of $\mathcal{U}(\mathfrak{g})$ which are invariant with respect to ad'. Remark that any even element of the anticentre anticommutes with odd elements of $\mathcal{U}(\mathfrak{g})$ and commutes with even ones and any odd element of the anticentre commutes with all elements of $\mathcal{U}(\mathfrak{g})$. Clearly the anticentre is a module over the centre and the product of any two elements of the anticentre belongs to the centre. For example, for $\mathfrak{g} = \mathrm{osp}(1, 2l)$ $\mathcal{A}(\mathfrak{g})$ is a free rank one module over $\mathcal{Z}(\mathfrak{g})$ (see [GL], 4.4.1). This is not true for a general Lie superalgebra.

Let us call *ghost centre* $\widetilde{\mathcal{Z}}(\mathfrak{g})$ the sum of $\mathcal{A}(\mathfrak{g})$ and $\mathcal{Z}(\mathfrak{g})$. It is clear that $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is a subalgebra of $\mathcal{U}(\mathfrak{g})$ which contains the centre of $\mathcal{U}(\mathfrak{g})$ considered as associative algebra.

In order to describe the action of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ on simple modules, note the following version of Schur's lemma for Lie superalgebras

2.1.3. **Lemma.** *Let $\mathfrak{g}$ be a finite or countable dimensional Lie superalgebra and $M = M_{\overline{0}} \oplus M_{\overline{1}}$ be a simple $\mathfrak{g}$-module. Then either $\mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}} = k\,\mathrm{id}$ or $\mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}} = k\,\mathrm{id} \oplus k\sigma$ where the odd element σ provides a $\mathfrak{g}$-isomorphism $M \xrightarrow{\sim} \Pi(M)$ and $\sigma^2 = \mathrm{id}$.*

Proof. Assume that $\phi \in \mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}}$ is even. Both homogeneous components $M_{\overline{0}}$ and $M_{\overline{1}}$ are simple $\mathcal{U}(\mathfrak{g})_{\overline{0}}$-modules. Since $\mathcal{U}(\mathfrak{g})_{\overline{0}}$ is a complex countable dimensional associative algebra, the restriction of ϕ on $M_{\overline{0}}$ (resp., on $M_{\overline{1}}$) is some constant c_0 (resp., c_1)— see [BZ]. Since M is simple, $c_0 = c_1$ and so $\phi = c_0\,\mathrm{id}$.

Assume that $\phi \in \mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}}$ is odd. Then ϕ^2 is even and so $\phi^2 = c\,\mathrm{id}$ for some $c \in \mathbb{C}$. If $c = 0$ then $\mathrm{Ker}\,\phi \neq 0$ and so $\phi = 0$. Otherwise ϕ is invertible and provides a $\mathfrak{g}$-isomorphism $M \xrightarrow{\sim} \Pi(M)$. Set $\sigma = \phi/\sqrt{c}$. Let ψ be another odd $\mathrm{ad}\,\mathfrak{g}$-invariant endomorphism such that $\psi^2 = \mathrm{id}$. Then $(\psi \pm \sigma)$ are also odd $\mathrm{ad}\,\mathfrak{g}$-invariant endomorphisms. Therefore $(\psi + \sigma)$ (resp., $(\psi - \sigma)$) is either isomorphism or zero. Since $(\psi + \sigma)(\psi - \sigma) = 0$, it implies that $\psi = \pm\sigma$. This proves the lemma. ☐

2.1.4. Using Example 2.1.1, we conclude that $\mathrm{End}(M)^{\mathrm{ad}'\,\mathfrak{g}} = \mathrm{End}(M)^{\mathrm{ad}\,\mathfrak{g}}\theta$. This implies the following lemma describing the action of $\widetilde{\mathcal{Z}}(\mathfrak{g})$ on simple modules.

Lemma. *Let $\mathfrak{g}$ be finite or countable dimensional Lie superalgebra, $M = M_{\overline{0}} \oplus M_{\overline{1}}$ be a simple $\mathfrak{g}$-module and z be an element of $\widetilde{\mathcal{Z}}(\mathfrak{g})$. Then the action of z on M is proportional* *to*

$$\begin{array}{ll} \mathrm{id}, & \text{if } z \in \mathcal{Z}(\mathfrak{g}) \text{ and } z \text{ is even}, \\ 0, & \text{if } z \in \mathcal{Z}(\mathfrak{g}) \text{ and } z \text{ is odd}, \\ \theta, & \text{if } z \in \mathcal{A}(\mathfrak{g}) \text{ and } z \text{ is even}, \\ \sigma\theta, & \text{if } z \in \mathcal{A}(\mathfrak{g}) \text{ and } z \text{ is odd}. \end{array}$$

2.2. **Case** $\dim \mathfrak{g}_1$ **is even.** In this case all elements of $\mathcal{A}(\mathfrak{g})$ are even (see 3.1.3). Denote by $\widetilde{\mathbb{C}}$ the algebra spanned by id and θ. Then $\widetilde{\mathbb{C}} = \mathbb{C}[\theta]/(\theta^2 - 1)$. Denote by π the algebra involution of $\widetilde{\mathbb{C}}$ sending θ to $-\theta$.

Definition. An algebra homomorphism $\chi : \tilde{\mathcal{Z}}(\mathfrak{g}) \to \widetilde{\mathbb{C}}$ is called *supercharacter* if $\chi(\mathcal{Z}(\mathfrak{g})) = \mathbb{C}$ and $\chi(\mathcal{A}(\mathfrak{g})) \subseteq \mathbb{C}\theta$.

By Lemma 2.1.4, $\tilde{\mathcal{Z}}(\mathfrak{g})$ acts on a simple modules M by a supercharacter χ_M. Moreover $\chi_{\Pi(M)} = \pi\chi_M$.

2.2.1. The standard consequence of Schur's lemma is the following statement. Any finite length module M has a unique decomposition into a direct sum of submodules M_i such that, for any fixed i, all simple subquotients of M_i have the same infinitesimal character and these characters are pairwise distinct for different i. Similarly, one can deduce from Lemma 2.1.4, that any finite length module M has a unique decomposition into a direct sum of submodules M_j such that, for any fixed j, all simple subquotients of M_j have the same supercharacter and these supercharacters are pairwise distinct for different j. This new decomposition is a refinement of the previous one. For example, let L be a simple module such that $\mathcal{A}(\mathfrak{g})$ does not lie in $\operatorname{Ann} L$. Then L and $\Pi(L)$ have different supercharacters. This, for instance, implies that though they have the same infinitesimal character, there are no non-trivial extensions of L by $\Pi(L)$.

2.3. **Case** $\dim \mathfrak{g}_1$ **is odd.** In this case all elements of $\mathcal{A}(\mathfrak{g})$ are odd (see 3.1.3). Retain notation of 2.2. The algebra spanned by id and $\sigma\theta$ (see Lemma 2.1.3) is isomorphic to $\widetilde{\mathbb{C}}$. However if L is a simple module such that $aL \neq 0$ for some $a \in \mathcal{A}(\mathfrak{g})$, then the product of θ and the image of a in $\operatorname{End} L$ provides an isomorphism $s : L \xrightarrow{\sim} \Pi(L)$. One can choose a such that $s^2 = \mathrm{id}$. There are two possible choices of such s which differ by sign. As a consequence, in this case, it is more natural to define a *odd supercharacter* as a pair of homomorphism $(\chi, \pi\chi)$ where χ satisfies the conditions given in Definition 2.2 and π is the involution of $\widetilde{\mathbb{C}}$ sending $\sigma\theta$ to $-\sigma\theta$. Observe that if $L \not\cong \Pi(L)$ then $\chi = \pi\chi$.

As in 2.2.1, odd supercharacters allows us to construct a decomposition of any module of finite length, but, probably, it always coincides with the decomposition coming from the infinitesimal characters.

2.3.1. **Example.** Let $\mathfrak{g}_1$ be generated by x and $\mathfrak{g}_0$ be generated by $[x, x]$. Then $\mathcal{U}(\mathfrak{g}) = \mathbb{C}[x]$, $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}[x^2]$ and $\mathcal{A}(\mathfrak{g}) = \mathbb{C}[x^2]x$ is a cyclic $\mathcal{Z}(\mathfrak{g})$-module generated by x. The list of the simple representations of $\mathfrak{g}$ is the following:

a) Two trivial representations (one is even and one is odd). The corresponding odd supercharacter sends $\mathcal{A}(\mathfrak{g})$ to zero.

b) Two-dimensional representations $L(\lambda)$ ($\lambda \in \mathbb{C} \setminus \{0\}$) spanned by v and xv where $x^2 v = \lambda v$. The corresponding odd supercharacter sends x to $\pm\sqrt{\lambda}\sigma\theta$. The representations $L(\lambda)$ and $\Pi(L(\lambda))$ are isomorphic.

3. Anticentre $\mathcal{A}(\mathfrak{g})$

Retain notation of Section 2. Assume that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra such that $\mathfrak{g}_1$ is finite dimensional and $\Lambda^{top}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. In this section we construct a linear injective map from the centre $\mathcal{Z}(\mathfrak{g}_0)$ to the anticentre $\mathcal{A}(\mathfrak{g})$—see Theorem 3.3. This allows us to describe the image of $\mathcal{A}(\mathfrak{g})$ in the symmetric algebra $\mathcal{S}(\mathfrak{g})$.

3.1. Denote by $\mathcal{F}$ the canonical filtration of $\mathcal{U}(\mathfrak{g})$ given by $\mathcal{F}^k := \mathfrak{g}^k$. Recall that this is an $\operatorname{ad}\mathfrak{g}$-invariant filtration and that the associated graded algebra $\operatorname{gr}_{\mathcal{F}}\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})$ is supercommutative. For $u \in \mathcal{U}(\mathfrak{g})$ denote its image in $\mathcal{S}(\mathfrak{g})$ by $\operatorname{gr} u$. Remark that $(\operatorname{ad}' x)(u) = 2xu - (\operatorname{ad} x)(u)$ for $x \in \mathfrak{g}_1$ and $u \in \mathcal{U}(\mathfrak{g})$. Therefore

$$\operatorname{gr}((\operatorname{ad}' x)(u)) = 2(\operatorname{gr} x)(\operatorname{gr} u), \quad \forall u \in \mathcal{U}(\mathfrak{g}), x \in \mathfrak{g}_1 \text{ s.t. } \operatorname{gr}(xu) = (\operatorname{gr} x)(\operatorname{gr} u). \tag{2}$$

3.1.1. Let L be an even vector space endowed by a structure of $\mathfrak{g}_0$-module. Denote by $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ the supervector space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} L$ (here $\mathcal{U}(\mathfrak{g})$ is considered as a right $\mathcal{U}(\mathfrak{g}_0)$-module) equipped with the natural left $\mathcal{U}(\mathfrak{g})$-module structure.

Let L be a submodule of $\mathcal{U}(\mathfrak{g}_0)$ with respect to $\operatorname{ad}\mathfrak{g}_0$-action. Denote by $(\operatorname{ad}'\mathfrak{g})(L)$ the $\operatorname{ad}'\mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})$ generated by L. Note that there is a natural surjective map from $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ to $(\operatorname{ad}'\mathfrak{g})(L)$ given by $u \otimes m \mapsto (\operatorname{ad}' u)m$ for $u \in \mathcal{U}(\mathfrak{g}), m \in L$.

3.1.2. **Lemma.** *Let L be a submodule of $\mathcal{U}(\mathfrak{g}_0)$ with respect to $\operatorname{ad}\mathfrak{g}_0$-action. The natural map $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L \to (\operatorname{ad}'\mathfrak{g})(L)$ is an isomorphism. Moreover $\mathcal{U}(\mathfrak{g}) = (\operatorname{ad}'\mathcal{U}(\mathfrak{g}))\mathcal{U}(\mathfrak{g}_0)$ and thus as $\operatorname{ad}'\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g})$ is isomorphic to $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}\mathcal{U}(\mathfrak{g}_0)$.*

Proof. Let $\{x_i\}_{i\in I}$ be an ordered basis of $\mathfrak{g}_1$. For any finite subset $J \subseteq I$ set $x_J := \prod_{i\in J} x_i$, where the product is taken with respect to the fixed order. Then the elements $\{\operatorname{gr} x_J\}_{J\subseteq I}$ form a basis of $\Lambda\mathfrak{g}_1 \subset \mathcal{S}(\mathfrak{g})$. Choose a basis $\{u_j\}_{j\in S}$ in L such that $\{\operatorname{gr} u_j\}_{j\in S}$ are linearly independent in $\operatorname{gr}_{\mathcal{F}}\mathcal{U}(\mathfrak{g}_0)$. Using (2) one concludes that $\operatorname{gr}(\operatorname{ad}' x_J)u_j = 2^{|J|}(\operatorname{gr} x_J)(\operatorname{gr} u_j)$ for all finite subsets $J \subseteq I, j \in S$. Therefore the elements $\{(\operatorname{ad}' x_J)u_j\}_{J\subseteq I, j\in S}$ are linearly independent. This proves the first assertion.

For the second assertion, note that $\operatorname{gr}\mathcal{U}(\mathfrak{g})$ is spanned by the elements of the form $(\operatorname{gr} x_J)(\operatorname{gr} u)$ with $u \in \mathcal{U}(\mathfrak{g}_0)$. Now $(\operatorname{gr} x_J)(\operatorname{gr} u) = \operatorname{gr}((\operatorname{ad}' x_J)u)/2^{|J|}$ and so $\operatorname{gr}(\operatorname{ad}'\mathcal{U}(\mathfrak{g}))\mathcal{U}(\mathfrak{g}_0) = \operatorname{gr}\mathcal{U}(\mathfrak{g})$. Therefore $\mathcal{U}(\mathfrak{g}) = (\operatorname{ad}'\mathcal{U}(\mathfrak{g}))\mathcal{U}(\mathfrak{g}_0)$ as required. ∎

The isomorphism $\mathcal{U}(\mathfrak{g}) \cong \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}\mathcal{U}(\mathfrak{g}_0)$ is proven in [S2], 3.2.

3.1.3. **Corollary.** *If $\mathfrak{g}_1$ has infinite dimension then $\mathcal{A}(\mathfrak{g}) = 0$. If $\dim \mathfrak{g}_1$ is even, all elements of $\mathcal{A}(\mathfrak{g})$ are even and if $\dim \mathfrak{g}_1$ is odd, all elements of $\mathcal{A}(\mathfrak{g})$ are odd.*

Proof. Retain notation of Lemma 3.1.2. Any element of $\mathcal{U}(\mathfrak{g})$ can be written in a form $u = \sum_J (ad' x_J) u_J$ where $u_J \in \mathcal{U}(\mathfrak{g}_0)$. Take $u \neq 0$ and set $m = \max\{|J| \mid u_J \neq 0\}$. Assume that $m < \dim \mathfrak{g}_1$. Take J such that $|J| = m$ and $u_J \neq 0$; take $i \in I \setminus J$. Modulo $\sum_{|J'|<m+1} (\mathrm{ad}'\, x_{J'}) \mathcal{U}(\mathfrak{g}_0)$ one has

$$(ad' x_i) u = \sum_{|J'|=m} (ad' x_i x_{J'}) u_{J'} \neq 0.$$

Thus if $u \in \mathcal{A}(\mathfrak{g})$ then $m = \dim \mathfrak{g}_1$. Since $\mathcal{A}(\mathfrak{g})$ is a $\mathbb{Z}_2$-graded subspace of $\mathcal{U}(\mathfrak{g})$, the assertion follows. □

3.2. Ind **and** Coind**.** Let L be an even vector space endowed by a structure of left $\mathfrak{g}_0$-module. Denote by $\mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ the supervector space $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), L)$ (here $\mathcal{U}(\mathfrak{g})$ is considered as a left $\mathcal{U}(\mathfrak{g}_0)$-module) equipped with the following left $\mathcal{U}(\mathfrak{g})$-module structure: $(uf)(u') := f(u'u)$ for any $f \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), L),\ u, u' \in \mathcal{U}(\mathfrak{g})$. The aim of this subsection is to prove that $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ and $\mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ (resp., $\Pi(\mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L)$) are isomorphic if $\dim \mathfrak{g}_1$ is even (resp., odd).

3.2.1. Retain notation of Lemma 3.1.2. For $k \in \mathbb{N}$ set

$$\mathcal{F}_o^k := \sum_{J \subseteq I, |J| \leq k} \mathcal{U}(\mathfrak{g}_0) x_J.$$

One has $x_J x_{J'} = \pm x_{J \cup J'}$ modulo $\mathcal{F}_o^{|J|+|J'|-1}$. This implies that $\mathcal{F}_o^p \mathcal{F}_o^q \subseteq \mathcal{F}_o^{p+q}$ and thus $\mathcal{F}_o$ is a filtration of $\mathcal{U}(\mathfrak{g})$. In particular, $\mathcal{F}_o^k$ are $\mathcal{U}(\mathfrak{g}_0)$-bimodules and the filtration does not depend from the choice of $\{x_i\}_{i \in I}$.

Consider $\mathcal{U}(\mathfrak{g})$ as a left $\mathcal{U}(\mathfrak{g}_0)$-module. Denote by ι a $\mathfrak{g}_0$-homomorphism from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g}_0)$ such that $\ker \iota = \mathcal{F}_o^{|I|-1}$ and $\iota(x_I) = 1$. Recall that $\ker \iota$ does not depend from the choice of basis in $\mathfrak{g}_1$. Note that $u_0 x_I = x_I u_0$ modulo $\mathcal{F}_o^{|I|-1}$ for any $u_0 \in \mathcal{U}(\mathfrak{g}_0)$, since $\Lambda^{top} \mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. Thus $\iota : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}_0)$ is a homomorphism of $\mathcal{U}(\mathfrak{g}_0)$-bimodules.

Define a map $(.|.)$ from $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} \mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g}_0)$ by setting $(u|u') = \iota(uu')$. For any subsets J, J' of I set $\delta_{J,J'} = 1$ if $J = J'$ and $\delta_{J,J'} = 0$ otherwise.

3.2.2. **Lemma.** *For any $J \subseteq I$ there exist $u_J, v_J \in \mathcal{U}(\mathfrak{g})$ such that $(u_J | x_{J'}) = (x_{J'} | v_J) = \delta_{J,J'}$.*

Proof. We prove the existence of v_J by induction on $r = |I \setminus J|$. For $r = 0$, $J = I$ and $v_I = 1$ satisfies the conditions.

Fix $J \subseteq I$. For any $J' \subseteq I$ such that $|J'| \leq |J|$, one has $x_{J'} x_{I\setminus J} = \pm x_{I\setminus J\cup J'}$ modulo $\ker\iota = \mathcal{F}_o^{|I|-1}$. Thus $(x_{J'}|x_{I\setminus J}) = 0$ for $J \neq J'$ and $(x_J|x_{I\setminus J}) = \pm 1$. Set

$$v := x_{I\setminus J} - \sum_{|J'|>|J|} v_{J'}(x_{J'}|x_{I\setminus J}).$$

Then $(x_{J'}|v) = 0$ for any $J' \subseteq I, J' \neq J$ and $(x_J|v) = \pm 1$. This proves the assertion.

The existence of u_J can be shown similarly. □

3.2.3. **Theorem.** *Assume that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra such that $\Lambda^{top}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. Then for any $\mathfrak{g}_0$-module L the linear map Ψ defined by*

$$\Psi(u' \otimes m)(u) := (u|u')m, \ \forall m \in L, \ u, u' \in \mathcal{U}(\mathfrak{g})$$

provides an isomorphism $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L \xrightarrow{\sim} \Pi^{\dim \mathfrak{g}_1}(\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L)$.

Proof. For any $u_0 \in \mathcal{U}(\mathfrak{g}_0)$ one has $(u|u'u_0)m = \iota(uu'u_0)m = \iota(uu')u_0 m = (u|u')u_0 m$ and thus ϕ is well-defined on $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} L$. Moreover

$$\Psi(u' \otimes m)(u_0 u) = (u_0 u|u')m = u_0(u|u')m = u_0 \Psi(u' \otimes m)(u)$$

and so $\Psi(u' \otimes m)$ is a $\mathcal{U}(\mathfrak{g}_0)$-linear map.

For any $s \in \mathcal{U}$ one has

$$\Psi(su' \otimes m)(u) = (u|su')m = (us|u')m = \Psi(u' \otimes m)(us) = (s\Psi(u' \otimes m))(u)$$

and so Ψ is a homomorphism of left $\mathcal{U}(\mathfrak{g})$-modules.

Since $\Psi(1 \otimes m)(x_J) = \delta_{I,J} m$, the element $\Psi(1 \otimes m)$ is even iff $x_I \in \mathcal{U}(\mathfrak{g})$ is even. Thus Ψ is a map from $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ to $\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ if $\dim \mathfrak{g}_1$ is even and from $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ to $\Pi(\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L)$ if $\dim \mathfrak{g}_1$ is odd.

Any element of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ can be written in the form $\sum_{J\subseteq I} x_J \otimes m_J$ where $m_J \in L$. Fix $J' \subseteq I$ and choose $u_{J'} \in \mathcal{U}(\mathfrak{g})$ as in Lemma 3.2.2. Then $\Psi(\sum_{J\subseteq I} x_J \otimes m_J)(u_{J'}) = m_{J'}$. This implies that $\ker \Psi = 0$.

Fix $J \subseteq I$ and choose $v_J \in \mathcal{U}(\mathfrak{g})$ as in Lemma 3.2.2. Then for any $m \in L$ one has $\Psi(v_J \otimes m)(x_{J'}) = \delta_{J,J'} m$. This implies the surjectivity of Ψ and completes the proof. □

3.3. Retain notation of Lemma 3.2.2.

Theorem. *Assume that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra such that $\Lambda^{top}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module. Then the map $\phi : z \mapsto (\operatorname{ad}' v_\emptyset)z$ provides a linear isomorphism $\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{A}(\mathfrak{g})$. Moreover one has $\operatorname{gr}\phi(z) = x \operatorname{gr} z$ where x is an element of $\Lambda^{top}(\mathfrak{g}_1)$.*

Proof. The proof follows from Lemma 3.1.2 and Theorem 3.2.3. We give a full detail below.

Denote by $\widetilde{\epsilon}$ a trivial even representation of $\mathfrak{g}$ and let e be a non-zero vector of $\widetilde{\epsilon}$. There is a canonical bijection Φ from $\mathrm{Hom}_{\mathfrak{g}_0}(\widetilde{\epsilon}, \mathcal{U}(\mathfrak{g}_0))$ onto $\mathrm{Hom}_{\mathfrak{g}}(\widetilde{\epsilon}, \mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathcal{U}(\mathfrak{g}_0))$ given by

$$\Phi(\psi)(e)(u) = \psi(ue) \quad \forall \psi \in \mathrm{Hom}_{\mathfrak{g}_0}(\widetilde{\epsilon}, \mathcal{U}(\mathfrak{g}_0)), u \in \mathcal{U}(\mathfrak{g}).$$

Combining the map Φ with the natural bijection $\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}_0}(\widetilde{\epsilon}, \mathcal{U}(\mathfrak{g}_0))$, we obtain the bijection

$$\Phi' : \mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(\Pi^{\dim \mathfrak{g}_1}(\widetilde{\epsilon}), \mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathcal{U}(\mathfrak{g}_0))$$

given by

$$(\Phi'(z)(e))(x_J) = \delta_{J,\emptyset} z.$$

In view of Theorem 3.2.3, Φ' induces the bijection

$$\Phi'' : \mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(\Pi^{\dim \mathfrak{g}_1}(\widetilde{\epsilon}), \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathcal{U}(\mathfrak{g}_0))$$

given by

$$\Phi''(z)(e) = v_\emptyset \otimes z.$$

Finally, Lemma 3.1.2 implies that the map sending z to $(\mathrm{ad}' v_\emptyset)z$ provides a linear isomorphism $\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{A}(\mathfrak{g})$ and moreover $\mathcal{A}(\mathfrak{g})$ lies in $\mathcal{U}(\mathfrak{g})_{\overline{0}}$ if $\dim \mathfrak{g}_1$ is even and in $\mathcal{U}(\mathfrak{g})_{\overline{1}}$ if $\dim \mathfrak{g}_1$ is odd.

The proof of Lemma 3.2.2 shows that $v_\emptyset = \pm x_I + \sum_{J \neq \emptyset} x_{I \setminus J} d_J$ where d_J are some elements of $\mathcal{U}(\mathfrak{g}_0)$. Therefore

$$\phi(z) := (\mathrm{ad}' v_\emptyset)(z) = (\mathrm{ad}'(x_I + \sum_{\substack{J \subset I \\ \neq}} c_J x_J))z \tag{3}$$

where c_J are some scalars. By the formula (2), $\mathrm{gr}\, \phi(z) = x \,\mathrm{gr}\, z$ for $x := \mathrm{gr}\, x_I \in \Lambda^{top}(\mathfrak{g}_1)$. This completes the proof. □

3.4. The condition on $\Lambda^{top}(\mathfrak{g}_1)$ is essential for both Theorem 3.2.3 and Theorem 3.3.

3.4.1. **Example.** Let $\mathfrak{g}$ be the Lie superalgebra $W(1)$ spanned by the even element g and the odd element x subject to the relations $[x, x] = 0, [g, x] = x$. Then $\mathcal{Z}(\mathfrak{g}) = \widetilde{\mathcal{Z}}(\mathfrak{g}) = \mathbb{C}$ and $\mathcal{A}(\mathfrak{g}) = 0$.

3.4.2. For Theorem 3.2.3 the condition on Λ^{top} is not only sufficient but also necessary. In fact, assume that the isomorphism Ψ exists for a trivial $\mathfrak{g}_0$-module ϵ. Then one has the following isomorphisms of $\mathfrak{g}_0$-modules

$$\Lambda \mathfrak{g}_1 \xrightarrow{\sim} \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \epsilon \xrightarrow{\sim} \mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} \epsilon \xrightarrow{\sim} (\Lambda \mathfrak{g}_1)^*.$$

Thus $\Lambda \mathfrak{g}_1$ is a self-dual $\mathfrak{g}_0$-module. The last is equivalent to the condition $\Lambda^{top}(\mathfrak{g}_1) \xrightarrow{\sim} \epsilon$ if $\mathfrak{g}_1$ is assumed to be finite dimensional.

3.4.3. We describe below for which simple Lie superalgebras the condition $\Lambda^{top}(\mathfrak{g}_1) \xrightarrow{\sim} \epsilon$ holds.

A classification theorem of Kac (see [K1], 4.2.1) states that any complex simple finite dimensional Lie superalgebra is isomorphic either to one of the classical Lie superalgebra or to one of the Cartan Lie superalgebras $W(n), S(n), \widetilde{S}(n), H(n)$.

Evidently the condition holds if $\mathfrak{g}_0$ is semisimple or if $\mathfrak{g}_1 \cong \mathfrak{g}_1^*$ as $\mathfrak{g}_0$-module. In particular the condition holds for all classical Lie superalgebras. It is easy to check that the above condition holds also for the Cartan Lie superalgebras $S(n), \widetilde{S}(n), H(n)$ and does not hold for the Cartan Lie superalgebras $W(n)$ with $n \neq 2$.

3.4.4. Retain the assumption of Theorem 3.3.

Definition. Denote by T a non-zero $\operatorname{ad}'\mathfrak{g}$-invariant element belonging to $(\operatorname{ad}'\mathcal{U}(\mathfrak{g}))(1)$.

The element T is defined up to a non-zero scalar and it is even iff $\dim \mathfrak{g}_1$ is even. Observe that, up to a scalar, T is a unique element of the anticentre whose image in $\mathcal{S}(\mathfrak{g})$ belongs to $\Lambda^{top}(\mathfrak{g}_1)$.

3.5. **Remarks.**

3.5.1. Here we consider $\mathcal{U}(\mathfrak{g})$ as an associative algebra and denote its centre by Z. Evidently $Z \cap \mathcal{U}(\mathfrak{g})_{\overline{0}} = \mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_{\overline{0}}$ and $Z \cap \mathcal{U}(\mathfrak{g})_{\overline{1}} = \mathcal{A}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_{\overline{1}}$. Hence, in the case when $\mathfrak{g}$ is finite dimensional,

$$\begin{array}{lll} Z = \mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_{\overline{0}} & \text{if} \quad \dim \mathfrak{g}_1 & \text{is even,} \\ Z = (\mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_{\overline{0}}) \oplus \mathcal{A}(\mathfrak{g}) & \text{if} \quad \dim \mathfrak{g}_1 & \text{is odd.} \end{array}$$

3.5.2. In most of the cases $\mathcal{U}(\mathfrak{g})$ is not a domain— see [AL]. However, even if $\mathcal{U}(\mathfrak{g})$ is a domain (for example $\mathfrak{g} = \operatorname{osp}(1, 2l)$) the direct generalization of the Gelfand-Kirillov conjecture does not hold for Lie superalgebras.

In fact, let k be a field of characteristic zero and $A_n(k)$ be a Weyl algebra over k. Recall that the centre of a Weyl skew field $W_n(k)$ coincides with k and that $A_n(\overline{k}) = A_n(k) \otimes_k \overline{k}$. Therefore a Weyl skew field does not contain non-central elements whose squares are central. Take any non-zero $a \in \mathcal{A}(\mathfrak{g})$. If $\dim \mathfrak{g}_1$ is even then $a \notin Z$, but $a^2 \in Z$. This implies that a Weyl skew field and a skew field of fractions of $\mathcal{U}(\mathfrak{g})$ are not isomorphic if $\dim \mathfrak{g}_1$ is even and $\Lambda^{top}\mathfrak{g}_1$ is a trivial $\mathfrak{g}$-module.

4. The case of basic classical Lie superalgebras

In this section $\mathfrak{g}$ denotes a basic classical Lie superalgebra (see [K2] and 4.1 below) such that $\mathfrak{g}_1 \neq 0$. In this case the dimension of $\mathfrak{g}_1$ is even and so all elements of $\mathcal{A}(\mathfrak{g})$ even. In particular, they anticommute with the odd elements of $\mathcal{U}(\mathfrak{g})$ and commute with the even ones.

In this section we show that the restriction of the Harish-Chandra projection $\mathcal{P}$ on $\mathcal{A}(\mathfrak{g})$ is an injection and describe its image. We also prove that $\widetilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the centralizer of $\mathcal{U}(\mathfrak{g})_{\overline{0}}$ and with the set of the elements of $\mathcal{U}(\mathfrak{g})$ acting by superconstants on each simple finite dimensional module.

4.1. **Notation.** A finite dimensional simple Lie superalgebra $\mathfrak{g}$ is called basic classical if $\mathfrak{g}_0$ is reductive and $\mathfrak{g}$ admits a non-degenerate invariant bilinear form. The list of basic classical Lie superalgebras is the following as determined by Kac (see [K2]):

a) simple Lie agebras
b) $A(m,n),\ B(m,n),\ C(n),\ D(m,n),\ D(2,1,\alpha),\ F(4),\ G(3)$.

Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_0$ and a triangular decomposition $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$. For a $\mathcal{U}(\mathfrak{g})$-module M and an element $\mu\in\mathfrak{h}^*$ set $M|_\mu=\{m\in M|\ hm=\mu(h)m,\ \forall h\in\mathfrak{h}\}$. When we use the notation $\mathcal{U}(\mathfrak{g})|_\mu$, the action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ is assumed to be the adjoint action. For $\mu\in\mathfrak{h}^*$ we say that μ is *even* if $\mathcal{U}(\mathfrak{g})|_\mu$ is a non-zero subspace of the even part of $\mathcal{U}(\mathfrak{g})$. We say that μ is *odd* if $\mathcal{U}(\mathfrak{g})|_\mu$ is a non-zero subspace of the odd part of $\mathcal{U}(\mathfrak{g})$. Since $\mathfrak{g}$ is a basic classical Lie superalgebra, μ is either even or odd in the case when $\mathcal{U}(\mathfrak{g})|_\mu\neq 0$.

The Harish-Chandra projection $\mathcal{P}:\mathcal{U}(\mathfrak{g})\to\mathcal{S}(\mathfrak{h})$ is the projection with respect to the following triangular decomposition $\mathcal{U}(\mathfrak{g})=\mathcal{U}(\mathfrak{h})\oplus(\mathcal{U}(\mathfrak{g})\mathfrak{n}^++\mathfrak{n}^-\mathcal{U}(\mathfrak{g}))$ (we identify $\mathcal{S}(\mathfrak{h})$ and $\mathcal{U}(\mathfrak{h})$). An element a of $\mathcal{U}(\mathfrak{g})|_0$ acts on a primitive vector of weight μ ($\mu\in\mathfrak{h}^*$) by multiplication by the scalar $\mathcal{P}(a)(\mu)$. Thus the restriction of $\mathcal{P}$ on $\mathcal{U}(\mathfrak{g})|_0=\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ is an algebra homomorphism from $\mathcal{U}(\mathfrak{g})|_0$ to $\mathcal{S}(\mathfrak{h})$.

Denote by Δ_0 the set of non-zero even roots of $\mathfrak{g}$. Denote by Δ_1 the set of odd roots of $\mathfrak{g}$. Set $\Delta=\Delta_0\cup\Delta_1$. Set

$$\overline{\Delta}_0:=\{\alpha\in\Delta_0|\ \alpha/2\not\in\Delta_1\},\quad \overline{\Delta}_1:=\{\beta\in\Delta_1|\ 2\beta\not\in\Delta_0\}.$$

Note that $\overline{\Delta}_1$ is the set of isotropic roots. Denote by Δ^+ the set of positive roots and define $\Delta^-,\Delta_0^{\pm},\Delta_1^{\pm},\overline{\Delta}_0^{\pm}$ as usual.

Denote by W the Weyl group of Δ_0. For any $\alpha\in\Delta_0$ let $s_\alpha\in W$ be the corresponding reflection. Let W' be the subgroup of W generated by the reflections $s_\alpha,\alpha\in\overline{\Delta}_0$. Note that $W=W'$ iff all odd roots are isotropic. Otherwise (if $\mathfrak{g}$ is of the type $B(m,n)$ or $G(3)$) W' is a subgroup of index two.

Set

$$\rho_0:=\tfrac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha,\quad \rho_1:=\tfrac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha,\quad \rho:=\rho_0-\rho_1.$$

Define the translated action of W on $\mathfrak{h}^*$ by the formula:

$$w.\lambda=w(\lambda+\rho)-\rho,\quad \forall\lambda\in\mathfrak{h}^*,w\in W.$$

Define the left translated action of W on $\mathcal{S}(\mathfrak{h})$ by setting $w.f(\lambda) = f(w^{-1}.\lambda)$ for any $\lambda \in \mathfrak{h}^*$.

Denote by $(-,-)$ a non-degenerate W-invariant bilinear form on $\mathfrak{h}^*$.

4.1.1. For $\lambda \in \mathfrak{h}^*$ denote a graded $\mathfrak{g}$-Verma module of the highest weight λ by $\widetilde{M}(\lambda)$ where the grading is fixed in such a way that a highest weight vector has degree zero. By [LM], an element of $\mathcal{U}(\mathfrak{g})$ annihilating the modules $\widetilde{M}(\lambda)$ for λ running through a Zariski dense subset of $\mathfrak{h}^*$, is equal to zero.

We use the following result which is a consequence of a theorem of Kac (see [Ja], 2.4)

4.1.2. **Lemma.** *Assume that a pair (n, α) belongs to the following set*

$$(\mathbb{N}^+ \times \overline{\Delta}_0^+) \cup ((1+2\mathbb{N}) \times (\Delta_1^+ \setminus \overline{\Delta}_1^+))$$

and $\lambda \in \mathfrak{h}^*$ is such that $(\lambda+\rho, \alpha) = n$. Then $\widetilde{M}(\lambda)$ contains a primitive vector of the weight $\lambda - n\alpha$. If $\alpha \in \overline{\Delta}_1^+$ and $(\lambda+\rho,\alpha) = 0$ then $\widetilde{M}(\lambda)$ contains a primitive vector of the weight $\lambda - \alpha$.

Remark. Note that the formula for Shapovalov determinants presented in [Ja], 2.4 contains misprints; the correct formula reads as follows

$$\begin{aligned}
\det S_\eta &= A \cdot B \cdot C,\\
A &= \prod_{n=1}^{\infty} \prod_{\gamma \in \overline{\Delta}_0^+} (h_\gamma + (\rho,\gamma) - n(\gamma,\gamma)/2)^{P(\eta - n\gamma)},\\
B &= \prod_{n=1}^{\infty} \prod_{\gamma \in \Delta_1^+ \setminus \overline{\Delta}_1^+} (h_\gamma + (\rho,\gamma) - (2n-1)(\gamma,\gamma)/2)^{P(\eta-(2n-1)\gamma)},\\
C &= \prod_{\gamma \in \overline{\Delta}_1^+} (h_\gamma + (\rho,\gamma))^{P_\gamma(\eta-\gamma)}.
\end{aligned}$$

This formula can be proven following the proof of [J], 6.1—6.11.

4.2. For $\beta \in \mathfrak{h}^*$ denote by h_β the element of $\mathfrak{h}$ such that $\mu(h_\beta) = (\mu, \beta)$ for any $\mu \in \mathfrak{h}^*$. Set

$$t := \prod_{\alpha \in \Delta_1^+} (h_\alpha + (\rho, \alpha)).$$

4.2.1. **Lemma.** *The restriction of the Harish-Chandra projection $\mathcal{P}$ provides a linear injective map $\mathcal{A}(\mathfrak{g}) \to t\mathcal{S}(\mathfrak{h})^{W.}$.*

Proof. Recall that $a \in \mathcal{U}(\mathfrak{g})|_0$ acts on a primitive vector of weight μ ($\mu \in \mathfrak{h}^*$) by multiplication by the scalar $\mathcal{P}(a)(\mu)$. Fix a non-zero $a \in \mathcal{A}(\mathfrak{g})$. Since $\mathcal{A}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})|_0$, a acts on the even component of $\widetilde{M}(\lambda)$ by multiplication by the scalar $\mathcal{P}(a)(\lambda)$ and on the odd component of $\widetilde{M}(\lambda)$ by multiplication by the opposite scalar. The intersection of the annihilators of all Verma modules is zero (see 4.1.1) and so $\mathcal{P}(a)$ is a non-zero polynomial in $\mathcal{S}(\mathfrak{h})$.

Choose a pair (n,α) and an element λ satisfying the assumption of Lemma 4.1.2. Note that $n\alpha$ is even iff α is even. This implies that $\mathcal{P}(a)(\lambda)=\mathcal{P}(a)(\lambda-n\alpha)$ for $\alpha\in\overline{\Delta}_0^+$ and $\mathcal{P}(a)(\lambda)=-\mathcal{P}(a)(\lambda-n\alpha)$ for $\alpha\in\Delta_1^+\setminus\overline{\Delta}_1^+$. Observe that for $\alpha\in\overline{\Delta}_0^+$ $\lambda-n\alpha=s_\alpha.\lambda$ and so $\mathcal{P}(a)(\lambda)=\mathcal{P}(a)(s_\alpha.\lambda)$. For fixed $\alpha\in\overline{\Delta}_0^+$ the set of λ such that $(\lambda+\rho,\alpha)\in\mathbb{N}^+$ is a Zariski dense subset of $\mathfrak{h}^*$. Thus $\mathcal{P}(a)\in\mathcal{S}(\mathfrak{h})$ is W'.-invariant.

Take $\alpha\in(\Delta_1^+\setminus\overline{\Delta}_1^+)$. Then $2\alpha\in\Delta_0^+$ and $\lambda-n\alpha=s_{2\alpha}.\lambda$; arguing as above we obtain that $s_{2\alpha}.\mathcal{P}(a)=-\mathcal{P}(a)$. In particular, $\mathcal{P}(a)$ is divisible by $(h_\alpha+(\rho,\alpha))$.

Now take $\alpha\in\overline{\Delta}_1^+$. Then α is isotropic. In particular, $(\lambda+\rho,\alpha)=0$ implies $(\lambda-\alpha+\rho,\alpha)=0$. Using Lemma 4.1.2, we conclude that $(\lambda+\rho,\alpha)=0$ implies that $\mathcal{P}(a)(\lambda)=(-1)^n\mathcal{P}(a)(\lambda-n\alpha)$ for any $n\in\mathbb{N}$. Therefore $\mathcal{P}(a)(\lambda)=0$ if $(\lambda+\rho,\alpha)=0$. Thus $\mathcal{P}(a)$ is divisible by $(h_\alpha+(\rho,\alpha))$.

Hence $\mathcal{P}(a)$ is divisible by $(h_\alpha+(\rho,\alpha))$ for any $\alpha\in\Delta_1^+$. This implies that $\mathcal{P}(a)$ is divisible by t. Since t is W'.-invariant, $\mathcal{P}(a)/t$ is also W'.-invariant. For any $\alpha\in(\Delta_0^+\setminus\overline{\Delta}_0^+$, both $\mathcal{P}(a)$ and t are antiinvariant with respect to the action of s_α. Thus $\mathcal{P}(a)/t$ is invariant with respect to the action of s_α and so $\mathcal{P}(a)/t$ is W.-invariant. This completes the proof. □

4.2.2. Define a filtration on $\mathfrak{g}$ by setting $\mathcal{F}_u^0=0$, $\mathcal{F}_u^1=\mathfrak{g}_1$, $\mathcal{F}_u^2=\mathfrak{g}$ and extend it canonically to an increasing filtration on $\mathcal{U}(\mathfrak{g})$. Let $z\in\mathcal{Z}(\mathfrak{g}_0)$ have a degree r with respect to the canonical filtration. Then, by (3), $\phi(z)\in\mathcal{F}_u^{\dim\mathfrak{g}_1+2r}$ and so $\mathcal{P}(\phi(z))$ is a polynomial of degree less than or equal to $(\dim\mathfrak{g}_1+2r)/2=|\Delta_1^+|+r$. In particular, $\mathcal{P}(\phi(1))$ is a polynomial of degree less than or equal to $|\Delta_1^+|$ and so it is equal to t up to a non-zero scalar. Recall that the map ϕ depends on the choice of basis $\{x_i\}_{i\in I}$; choose a basis $\{x_i\}_{i\in I}$ such that $\mathcal{P}(\phi(1))=t$.

4.2.3. Fix $r\in\mathbb{N}$ and set $Z_r:=\mathcal{Z}(\mathfrak{g}_0)\cap\mathcal{F}^r$. Denote by S_r the space of W.-invariant polynomials of degree less than or equal to r. Take $z\in Z_r$. Combining Lemma 4.2.1 and 4.2.2, we conclude that $(\mathcal{P}(\phi(z_r))/t)\in S_r$. Recall that $\operatorname{gr}\mathcal{Z}(\mathfrak{g}_0)\xrightarrow{\sim}\mathcal{S}(\mathfrak{h})^W$ as graded algebras and so $\dim Z_r=\dim S_r$. Since ϕ is a linear isomorphism, it follows that $\mathcal{P}(Z_r)=tS_r$.

4.2.4. **Corollary.** *The restriction of the Harish-Chandra projection $\mathcal{P}$ provides a linear bijective map $\mathcal{A}(\mathfrak{g})\to t\mathcal{S}(\mathfrak{h})^{W.}$. In particular, $\mathcal{P}(T)=t$.*

4.2.5. **Lemma.** *Any non-zero element $z\in\mathcal{A}(\mathfrak{g})$ is a non-zero divisor in $\mathcal{U}(\mathfrak{g})$.*

Proof. Assume that $zu=0$. Recall that z acts by multiplication by $\mathcal{P}(z)(\lambda)$ (resp., $-\mathcal{P}(z)(\lambda)$) on the even (resp., odd) graded component of $\widetilde{M}(\lambda)$. Therefore u annihilates

$\widetilde{M}(\lambda)$ when λ is such that $\mathcal{P}(z)(\lambda) \neq 0$. Since $\mathcal{P}(z) \neq 0$, the set $\{\lambda | \ \mathcal{P}(z)(\lambda) \neq 0\}$ is a Zariski dense subset of $\mathfrak{h}^*$. By 4.1.1, it implies that $u = 0$. □

4.2.6. *Remark.* On the contrary to the central elements, $\operatorname{gr} z$ is a zero-divisor for any $z \in \mathcal{A}(\mathfrak{g})$. In fact, by (1) $(\operatorname{ad}\mathcal{U}(\mathfrak{g}))(z) = (\operatorname{ad}'\mathcal{U}(\mathfrak{g}))(1)z$ and thus $(\operatorname{ad}\mathcal{U}(\mathfrak{g}))z$ contains Tz. Therefore z and Tz have the same degree with respect to the canonical filtration and so $\operatorname{gr} T \operatorname{gr} z = 0$. In particular, T^2 is a central element whose degree is equal to $\dim \mathfrak{g}_1$.

4.2.7. **Corollary.**

$$\mathcal{Z}(\mathfrak{g}) \cap \mathcal{A}(\mathfrak{g}) = 0.$$

Proof. For any $z \in \mathcal{Z}(\mathfrak{g}) \cap \mathcal{A}(\mathfrak{g})$ and any odd element $u \in \mathfrak{g}_1$ one has $zu = 0$. Hence $z = 0$ by Lemma 4.2.5. □

4.3. **The structure of $\widetilde{\mathcal{Z}}(\mathfrak{g})$.** The algebra $\widetilde{\mathcal{Z}}(\mathfrak{g})$ has the following easy realization. Consider the algebra $\widetilde{\mathcal{S}(\mathfrak{h})} := \mathcal{S}(\mathfrak{h})[\xi]/(\xi^2 - 1)$. Define a map $\mathcal{P}' : \widetilde{\mathcal{Z}}(\mathfrak{g}) \to \widetilde{\mathcal{S}(\mathfrak{h})}$ by setting $\mathcal{P}'(z) = \mathcal{P}(z)$ for $z \in \mathcal{Z}(\mathfrak{g})$ and $\mathcal{P}'(z) = \mathcal{P}(z)\xi$ for $z \in \mathcal{A}(\mathfrak{g})$. Since $\widetilde{\mathcal{Z}}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})|_0$, the restriction of $\mathcal{P}$ on $\widetilde{\mathcal{Z}}(\mathfrak{g})$ is an algebra homomorphism. Taking into account Corollary 4.2.4, we conclude that $\mathcal{P}'$ provides an algebra isomorphism from $\widetilde{\mathcal{Z}}(\mathfrak{g})$ onto the subalgebra $(\mathcal{P}(\mathcal{Z}(\mathfrak{g})) \oplus t\mathcal{S}(\mathfrak{h})^{W.}\xi)$ of $\widetilde{\mathcal{S}(\mathfrak{h})}$.

4.3.1. Assume that $\mathfrak{g}$ is of the type $B(m,n)$ or $G(3)$. Then $W' \neq W$ and so t is not $W.$-invariant. Therefore $\mathcal{P}(\mathcal{A}(\mathfrak{g})) \cap \mathcal{P}(\mathcal{Z}(\mathfrak{g})) = \{0\}$. Then, using Corollary 4.2.4, we conclude that the restriction of the Harish-Chandra projection provides an algebra isomorphism $\widetilde{\mathcal{Z}}(\mathfrak{g}) \cong (\mathcal{P}(\mathcal{Z}(\mathfrak{g})) \oplus t\mathcal{S}(\mathfrak{h})^{W.})$.

In all other cases, $\mathcal{P}(\mathcal{A}(\mathfrak{g})) \subset \mathcal{P}(\mathcal{Z}(\mathfrak{g}))$.

4.3.2. In the case when $\mathfrak{g} = \operatorname{osp}(1,2l)$, $\mathcal{Z}(\mathfrak{g})$ is a polynomial algebra and $\mathcal{A}(\mathfrak{g})$ is a cyclic $\mathcal{Z}(\mathfrak{g})$ module generated by T. In other cases (when $\mathfrak{g}$ is basic classical Lie superalgebra) this does not hold. However, a similar result hold after a certain localization.

More precisely, if $\mathfrak{g} \neq \operatorname{osp}(1,2l)$ (that is $\mathfrak{g}$ is not of the type $B(0,l)$) then $\mathcal{P}(\mathcal{Z}(\mathfrak{g}))$ is strictly contained in $\mathcal{S}(\mathfrak{h})^{W.}$. However, since the product of two elements from the anticentre belongs to the centre, $\mathcal{P}(\mathcal{Z}(\mathfrak{g}))$ contains $t^2\mathcal{S}(\mathfrak{h})^{W.}$. Set $Q := T^2$, $q := t^2$. Then the localized algebras $\mathcal{Z}(\mathfrak{g})[Q^{-1}]$ and $\mathcal{S}(\mathfrak{h})^{W.}[q^{-1}]$ are isomorphic. Moreover $\widetilde{\mathcal{Z}}(\mathfrak{g})[Q^{-1}] = \mathcal{Z}(\mathfrak{g})[Q^{-1}] \oplus \mathcal{A}(\mathfrak{g})[Q^{-1}]$ and $\mathcal{A}(\mathfrak{g})[Q^{-1}]$ is a cyclic $\mathcal{Z}(\mathfrak{g})[Q^{-1}]$-module generated by T.

4.4. **The action of $\tilde{\mathcal{Z}}(\mathfrak{g})$ on the simple modules.** Let us say that an element $u \in \mathcal{U}(\mathfrak{g})$ acts on a $\mathcal{U}(\mathfrak{g})$-module M by a superconstant if it acts by a multiplication by a scalar on each graded component of M. By 2.2, each element of $\tilde{\mathcal{Z}}(\mathfrak{g})$ acts by a superconstant on any simple module. In this subsection we shall prove that actually $\tilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the set of elements of $\mathcal{U}(\mathfrak{g})$ which act by superconstants on each simple finite dimensional module. Moreover $\tilde{\mathcal{Z}}(\mathfrak{g})$ coincides with the centralizer of the even part $\mathcal{U}(\mathfrak{g})_{\bar{0}}$ in $\mathcal{U}(\mathfrak{g})$.

4.4.1. By definition, $\tilde{\mathcal{Z}}(\mathfrak{g})$ lies in the centralizer of $\mathcal{U}(\mathfrak{g})_{\bar{0}}$ in $\mathcal{U}(\mathfrak{g})$ and even in the centre of $\mathcal{U}(\mathfrak{g})_{\bar{0}}$ since all elements of $\tilde{\mathcal{Z}}(\mathfrak{g})$ are even.

Let A be a centralizer of $\mathcal{U}(\mathfrak{g})_{\bar{0}}$ in $\mathcal{U}(\mathfrak{g})$ and a be an element of A. Clearly, a acts by a superconstant on any Verma module. On the even component of $\widetilde{M}(\lambda)$ a acts by $\mathcal{P}(a)(\lambda)$. Let $f(a)$ be the function $\mathfrak{h}^* \to k$ such that a acts by $f(a)(\lambda)$ on the odd component of $\widetilde{M}(\lambda)$.

4.4.2. **Lemma.** *For any $a \in A$ the function $f(a) : \mathfrak{h}^* \to k$ is polynomial.*

Proof. Choose $y \in \mathfrak{n}_1^-$ and $x \in \mathfrak{n}_1^+$ such that $h := [y, x] \in \mathcal{S}(\mathfrak{h})$ and $h \neq 0$. For each $\mu \in \mathfrak{h}^*$ choose a highest weight vector $v_\mu \in \widetilde{M}(\mu)$. Then yv_μ is odd and so

$$xayv_\mu = f(a)(\mu)xyv_\mu = f(a)(\mu)h(\mu)v_\mu.$$

Since $xay \in \mathcal{U}(\mathfrak{g})|_0$ one has $xayv_\mu = \mathcal{P}(xay)(\mu)v_\mu$. Thus

$$f(a)(\mu)h(\mu) = \mathcal{P}(xay)(\mu). \tag{4}$$

This implies that $\mathcal{P}(xay)(\mu)$ vanishes on the whole hyperplane $\{\mu|\ h(\mu) = 0\}$. Therefore h divides $\mathcal{P}(xay)(\mu)$ and so $f(a) = \mathcal{P}(xay)/h$ is a polynomial. ☐

4.4.3. Lemma 4.4.2 implies that an element $a \in A$ acts by $\mathcal{P}(a)(\lambda)$ on the even component of $\widetilde{M}(\lambda)$ and by $f(\lambda)$ on the odd component of $\widetilde{M}(\lambda)$. Arguing as in 4.2.1, we obtain that $P' := \mathcal{P}(a) - f(a)$ belongs to $t\mathcal{S}(\mathfrak{h})^{W\cdot}$. Similarly, $P := \mathcal{P}(a) + f(a)$ belongs to $\mathcal{S}(\mathfrak{h})^{W\cdot}$ and moreover for any $\alpha \in \overline{\Delta_1}^+$ one has $P(\lambda - \alpha) = P(\lambda)$ if $(\lambda + \rho, \alpha) = 0$. By [S1] and Corollary 4.2.4, this implies that $P = \mathcal{P}(z)$ for some $z \in \mathcal{Z}(\mathfrak{g})$ and $P' = \mathcal{P}(z')$ for some $z' \in \mathcal{A}(\mathfrak{g})$. Then $a - (z + z')/2$ kills any Verma module and so $a = (z + z')/2$. This proves that $\tilde{\mathcal{Z}}(\mathfrak{g}) = A$.

The intersection of the annihilators of all simple highest weight modules is equal to zero (see 4.1.1). This implies that the set of elements of $\mathcal{U}(\mathfrak{g})$ acting by superconstants on each graded simple finite dimensional module coincides with A. Hence we obtain

4.4.4. **Corollary.** *If $\mathfrak{g}$ is a basic classical Lie superalgebra then the following algebras coincide*

i) The algebra of elements of $\mathcal{U}(\mathfrak{g})$ which act by superconstants on each graded simple finite dimensional module.

ii) The algebra $\widetilde{\mathcal{Z}}(\mathfrak{g})$.

iii) The centre of $\mathcal{U}(\mathfrak{g})_{\overline{0}}$.

iv) The centralizer of $\mathcal{U}(\mathfrak{g})_{\overline{0}}$.

4.5. **A remark concerning the separation theorem.** An important structure theorem of Kostant asserts that for any finite dimensional semisimple Lie algebra there exists an $\operatorname{ad}\mathfrak{g}$-submodule $\mathcal{H}$ of $\mathcal{U}(\mathfrak{g})$ such that the multiplication map induces the isomorphism $\mathcal{H}\otimes\mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$. In [Mu], I. Musson proved the similar assertion for $\mathfrak{g} = \operatorname{osp}(1,2l)$. These theorems are called the separation theorems. We shall show that Separation theorem does not hold for any basic classical Lie superalgebra apart from finite dimensional simple Lie algebras and $\mathfrak{g} = \operatorname{osp}(1,2l)$.

4.5.1. Denote by V (resp., $\widetilde{V}$) a trivial representation of $\mathfrak{g}_0$ (resp., $\mathfrak{g}$).

Lemma. *Assume that* $\mathfrak{g}_1$ *contains a non-zero element* x *such that* $[x,x]=0$. *Then the trivial submodule* $\widetilde{V}$ *is not a direct summand of* $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V$.

Proof. Denote a generator of $V\subset \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V$ by v and a generator of the trivial submodule of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V$ by v'. Retain notation of 3.3. Choose an ordered basis $\{x_i\}_{i\in I}$ of $\mathfrak{g}_1$ in such a way that $[x_1,x_1]=0$. Write $v'=\sum_{J\subset I} c_J x_J\otimes v$ where $c_J\in\mathbb{C}$. One has $x_1x_J=0$ when $1\in J$ and $x_1x_J = x_{\{1\}\cup J}$ otherwise. Since $x_1v'=0$, this implies that $c_\emptyset = 0$.

Recall that $\operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V,\widetilde{V})$ is one-dimensional and is spanned by an element f such that $f(V)\neq 0$ and $f(\mathcal{U}(\mathfrak{g})\mathfrak{g}V)=0$. Thus $f(v')=0$ and so the trivial submodule $\widetilde{V}$ is not a direct summand of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V$. □

4.5.2. Let $\mathfrak{g}$ be a basic classical Lie superalgebra which is neither simple Lie algebra nor $\operatorname{osp}(1,2l)$. Then $\mathfrak{g}_1$ contains a non-zero element x such that $[x,x]=0$. From (1) it follows that the multiplication by T gives a $\mathfrak{g}$-map from the $\operatorname{ad}'\mathfrak{g}$-module generated by 1 onto the $\operatorname{ad}\mathfrak{g}$-module generated by T. Since T is a non-zero divisor, this map is an isomorphism. In particular, $T^2\in(\operatorname{ad}\mathcal{U}(\mathfrak{g}))T$ and $(\operatorname{ad}\mathcal{U}(\mathfrak{g}))T\cong \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V$ where V is a trivial $\mathfrak{g}_0$-module. Using Lemma 4.5.1, we conclude that T^2 spans a trivial $\operatorname{ad}\mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})$ which is not a direct summand. On the other hand, 1 spans a trivial $\operatorname{ad}\mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})$ which is a direct summand, since $\mathcal{U}(\mathfrak{g})=\mathbb{C}\oplus\mathcal{U}(\mathfrak{g})\mathfrak{g}$ as $\operatorname{ad}\mathfrak{g}$-module.

Assume now that Separation theorem holds for $\mathcal{U}(\mathfrak{g})$. Then $\mathcal{Z}(\mathfrak{g})\cong(\mathcal{H})^{\mathfrak{g}}\otimes\mathcal{Z}(\mathfrak{g})$ as $\mathcal{Z}(\mathfrak{g})$-module. Therefore $(\mathcal{H})^{\mathfrak{g}}$ is one-dimensional. Thus all trivial $\operatorname{ad}\mathfrak{g}$-submodules of $\mathcal{U}(\mathfrak{g})$ are either direct summands or are not direct summands of $\mathcal{U}(\mathfrak{g})$. This gives a contradiction.

5. Questions

5.1. The centralizer of $\mathcal{U}(\mathfrak{g})_{\overline{0}}$ contains $\widetilde{\mathcal{Z}}(\mathfrak{g})$. Do they coincide provided that $\Lambda^{top}\mathfrak{g}_1$ is a trivial $\mathfrak{g}_0$-module? Note that the condition on $\Lambda^{top}\mathfrak{g}_1$ is essential— see Example 3.4.1.

5.2. Let C be the set of the elements of $\mathcal{U}(\mathfrak{g})$ which act by a superconstant on each simple module. Clearly, C is a subalgebra of $\mathcal{U}(\mathfrak{g})$. By 2.2, C contains $\widetilde{\mathcal{Z}}(\mathfrak{g})$ if $\dim \mathfrak{g}_1$ is even. Assume that $\dim \mathfrak{g}_1$ is even and that the intersection of all graded primitive ideals of $\mathcal{U}(\mathfrak{g})$ is zero. Does this imply that $C = \widetilde{\mathcal{Z}}(\mathfrak{g})$?

5.3. In the case when $\mathfrak{g}$ is a basic classical Lie superalgebra both answers are positive— see Corollary 4.4.4.

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