CHARACTERS OF HIGHEST WEIGHT MODULES OVER AFFINE LIE ALGEBRAS ARE MEROMORPHIC FUNCTIONS

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Abstract. We show that the characters of all highest weight modules over an affine Lie algebra with the highest weight away from the critical hyperplane are meromorphic functions in the positive half of the Cartan subalgebra, their singularities being at most simple poles at zeros of real roots. We obtain some information about these singularities. In particular, we show that this character is a holomorphic function iff the module is integrable.

0. Introduction

0.0.1. Let $\overline{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, and let $\mathfrak{g} = \overline{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the associated non-twisted affine Kac-Moody algebra [K]. Recall that the commutation relations on $\mathfrak{g}$ are:

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a|b)K, \quad [D, at^m] = mat^m, \quad [K, \mathfrak{g}] = 0,$$

where $a, b \in \overline{\mathfrak{g}}$, and $(-, -)$ is a non-degenerate invariant symmetric bilinear form on $\overline{\mathfrak{g}}$.

Choosing a Cartan subalgebra $\overline{\mathfrak{h}}$ of $\overline{\mathfrak{g}}$, and a triangular decomposition $\overline{\mathfrak{g}} = \overline{\mathfrak{n}}_- \oplus \overline{\mathfrak{h}} \oplus \overline{\mathfrak{n}}_+$, we have the associated Cartan subalgebra $\mathfrak{h} = \overline{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D$ and the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm = \overline{\mathfrak{n}}_\pm + \overline{\mathfrak{g}}[t^{\pm 1}]t^{\pm 1}$. Let $\Delta \supset \Delta_+ \supset \Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ be the multiset of all roots, the multiset of positive roots and the set of simple roots of $\mathfrak{g}$, respectively.

0.1. Let $M(\lambda)$ be the Verma module over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$. Any quotient $V(\lambda)$ of $M(\lambda)$, called a highest weight module, has weight space decomposition $V(\lambda) = \oplus_{\mu \in \mathfrak{h}^*} V_\mu(\lambda)$, where $\dim V_\mu(\lambda) < \infty$, hence we can define its character

$$\mathrm{ch}_{V(\lambda)}(h) = \sum_\mu \dim V_\mu(\lambda) e^{\mu(h)}, \; h \in \mathfrak{h}.$$

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0.1.1. The character of the Verma module is

$$\mathrm{ch}_{M(\lambda)}(h)=\frac{e^{\lambda(h)}}{R(h)},\quad \text{where } R(h):=\prod_{\alpha\in\Delta_+}(1-e^{-\alpha(h)}). \tag{1}$$

Since $\mathrm{mult}\,\alpha\leq n=\mathrm{rank}\,\mathfrak{g}$, it is easy to deduce that the series $\mathrm{ch}_{M(\lambda)}(h)$ converges to a holomorphic function in the domain

$$Y_>:=\{h\in\mathfrak{h}|\ \mathrm{Re}\,\alpha_i(h)>0,\ i=0,1,\dots,n\}.$$

Since $\mathrm{ch}_{V(\lambda)}(h)$ is majorized by $\mathrm{ch}_{M(\lambda)}(h)$, we deduce that $\mathrm{ch}_{V(\lambda)}(h)$ converges to a holomorphic function in $Y_>$ as well.

0.2. Let $\delta\in\mathfrak{h}^*$, defined by $\delta_{\overline{\mathfrak{h}}+\mathbb{C}K}=0,\ \delta(D)=1$, be the minimal positive imaginary root of $\mathfrak{g}$. Consider the interior of the complexified Tits cone

$$Y:=\{h\in\mathfrak{h}|\ \mathrm{Re}\,\delta(h)>0\}.$$

This is an open domain in $\mathfrak{h}$, which is W-invariant, where W is the Weyl group of $\mathfrak{g}$, and $\overline{Y}=\cup_{w\in W}w(\overline{Y}_>)$ [K].

It is easy to show that the product $R(h)$ converges to a holomorphic function in Y, hence $\mathrm{ch}_{M(\lambda)}(h)$ analytically extends from $Y_>$ to a meromorphic function on the whole domain Y. The problem, addressed in this paper is whether the same holds for $\mathrm{ch}_{V(\lambda)}(h)$.

0.3. Let $\rho\in\mathfrak{h}^*$ be such that $(\rho,\alpha_i)=\frac{1}{2}(\alpha_i,\alpha_i)$ for $i=0,1,\dots,n$. We prove the following result.

0.3.1. **Theorem.** *Let $L(\lambda)$ be the irreducible highest weight module over $\mathfrak{g}$ with highest weight λ, such that $(\lambda+\rho)(K)\neq 0$. Then $\mathrm{ch}_{L(\lambda)}(h)$ extends from $Y_>$ to a meromorphic function in Y, such that its numerator $N_\lambda(h):=R(h)\,\mathrm{ch}_{L(\lambda)}(h)$ is holomorphic in Y. The function $\mathrm{ch}_{L(\lambda)}(h)$ is holomorphic in Y iff $L(\lambda)$ is an integrable $\mathfrak{g}$-module.*

0.3.2. The proof in the case $(\lambda+\rho)(K)\not\in\mathbb{Q}_{\geq 0}$ is very easy (for arbitrary $V(\lambda)$). Indeed, it follows easily from [KK] (see [KT00]) that, in this case, N_λ is a finite linear combination (over $\mathbb{Z}$) of exponential functions e^μ, hence is a holomorphic function on the whole space $\mathfrak{h}$. (The difficult problem of computing the coefficients in N_λ in this case is solved in [KT00], but it is not needed here).

0.3.3. In the case $(\lambda+\rho)(K)\in\mathbb{Q}_{>0}$ there exists $x\in W(\lambda)$ such that $\lambda=x.\lambda'$, where $\lambda'\in Y_>$, $W(\lambda)$ denote the subgroup of W, generated by reflections s_α in $\alpha\in\Delta_+$, such that $(\alpha,\alpha)\neq 0$ and $2(\lambda,\alpha)/(\alpha,\alpha)\in\mathbb{Z}$, and $x.\lambda=x(\lambda+\rho)-\rho$ is the "shifted action" of W. It follows easily from [KK] (see [KT00]) that we then have:

$$N_\lambda=\sum_{y\in W(\lambda):\ y\geq x} c_{x,y}e^{y.\lambda},\quad c_{x,y}\in\mathbb{Z}, \tag{2}$$

which is an infinite sum if the group $W(\lambda)$ is infinite. In order to prove that $N_\lambda(h)$ converges to a holomorphic function in Y, we use the explicit formula for the $c_{x,y}$, given by the Kazhdan-Lusztig conjecture for $\mathfrak{g}$, proved in [KT00]:

$$c_{x,y}=(-1)^{l(y)-l(x)}Q_{x,y}(1), \tag{3}$$

where $Q_{x,y}(q)$ are the inverse Kazhdan-Lusztig polynomials for the Coxeter group $W(\lambda)$. (The length function $l(x)$ and the Bruhat order $\geq$ in (2), (3) are meant in the group $W(\lambda)$).

0.4. Using the recurrent definition of the polynomials $Q_{x,y}$, we prove the following estimate (which holds for polynomial growth Weyl groups):

$$|Q_{x,y}(1)|\leq (Cl(y)^{n+1})^{l(y)-l(x)}, \tag{4}$$

where C is a constant, independent of x and y. This estimate suffices to prove the theorem. We also show along the same lines that the theorem holds for arbitrary highest weight module $V(\lambda)$ with $(\lambda+\rho)(K)\neq 0$, and arbitrary affine Kac-Moody Lie algebras. The fact that $\mathfrak{g}$ is affine is used for the obvious, but crucial, observation that the affine Weyl group has polynomial growth (since it contains a finitely generated abelian subgroup of finite index).

The theorem probably holds also on the critical hyperplane $\{\lambda|\ \lambda(K)=-\rho(K)\}$, but it is unclear how to prove it since already the argument of [KK] collapses on this hyperplane.

1. Preliminaries

1.1. Let A be a symmetrizable Cartan matrix, let $\mathfrak{g}(A)=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}$ be the corresponding Kac-Moody algebra over $\mathbb{C}$ [K], Π be the set of simple roots, Δ_+ be the *multiset* of positive roots, and W be the Weyl group. Let Δ^{re} be the set of real roots and Δ^{im} be the multiset of imaginary roots.

1.1.1. Set $Q^+=\{\sum_{\alpha\in\Pi}m_\alpha\alpha|\ m_\alpha\in\mathbb{Z}_{\geq 0}\}$, and define the standard partial ordering on $\mathfrak{h}^*$: $\alpha\geq\beta$ for $\alpha-\beta\in Q^+$. For $\nu\in Q^+$ denote by $\operatorname{ht}\nu$ the *height* of ν: if $\nu=\sum_{\alpha\in\Pi}m_\alpha\alpha$ then $\operatorname{ht}\nu=\sum_{\alpha\in\Pi}m_\alpha$.

1.1.2. Denote by $(-,-)$ a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. It restricts to a non-degenerate bilinear form on $\mathfrak{h}$, and the latter induces one on $\mathfrak{h}^*$, also denoted by $(-,-)$. For $\alpha\in\Delta^{re}$, let $\alpha^\vee\in\mathfrak{h}$ be such that $\langle\alpha^\vee,\mu\rangle=2(\alpha,\mu)/(\alpha,\alpha)$; then the reflection $s_\alpha\in W$ is defined by $\mu\mapsto\mu-\langle\alpha^\vee,\mu\rangle\alpha$, and W is generated by these reflections.

1.2. Recall that W is a Coxeter group with the canonical system of generators $\{s_\alpha|\alpha\in\Pi\}$ (see [K], 3.13). Denote the unit element in W by e, the Bruhat ordering by $\geq$ (e is the minimal element) and the length function $l: W\to\mathbb{Z}_{\geq 0}$ by $l(w)$.

Fix $\rho\in\mathfrak{h}^*$ satisfying $\langle\alpha^\vee,\rho\rangle=1$ for all $\alpha\in\Pi$ and define the shifted action of W on $\mathfrak{h}^*$ by $w.\lambda:=w(\lambda+\rho)-\rho$.

1.2.1. For $w\in W$ set

$$S(w):=\Delta_+\cap w^{-1}\Delta_-.$$

For $\alpha\in\Pi$, $w\in W$ one has

$$S(s_\alpha w)=\left\{\begin{array}{ll} S(w)\cup\{w^{-1}\alpha\} & \text{if } w^{-1}\alpha\in\Delta_+,\\ S(w)\setminus\{-w^{-1}\alpha\} & \text{if } w^{-1}\alpha\in\Delta_-.\end{array}\right.$$

Moreover, $l(s_\alpha w)>l(w)$ iff $w^{-1}\alpha\in\Delta_+$. This gives $l(w)=|S(w)|=|w\Delta_+\cap\Delta_-|$ and $l(ws_\alpha)>l(w)$ iff $w\alpha\in\Delta_+$.

We will use the following lemma.

1.2.2. **Lemma.** *(see [K], Ex. 3.12) Let $w=s_{i_1}\dots s_{i_l}$ be a reduced expression of $w\in W$, where s_i denote the reflection with respect to the simple root α_i. Set $w^{(0)}=e$, $w^{(j)}=s_{i_1}\dots s_{i_j}$ and $\beta_j:=w^{(j)}\alpha_{i_{j+1}}$ for $j=0,\dots,l-1$. Then β_j are pairwise distinct, $\{\beta_j\}_{j=0}^{l-1}=S(w^{-1})$ and for any $\lambda\in\mathfrak{h}^*$*

$$\lambda-w\lambda=\sum_{j=0}^{l-1}\langle\alpha^\vee_{i_{j+1}},\lambda\rangle\beta_j.$$

Proof. The fact that $\{\beta_j\}_{j=0}^{l-1}=S(w^{-1})$ follows from the above description of $S(s_\alpha w)$; since $|S(w^{-1})|=l(w^{-1})=l$ the elements β_j are pairwise distinct.

One has

$$\lambda-w\lambda=w^{(l-1)}(\lambda-s_{i_l}\lambda)+(\lambda-w^{(l-1)}\lambda)=\langle\alpha^\vee_{i_l},\lambda\rangle w^{(l-1)}\alpha_{i_l}+(\lambda-w^{(l-1)}\lambda)$$

so the assertion follows by induction on $l=l(w)$. □

1.3. In this text $\mathfrak{g}$ will denote an (arbitrary) affine Lie algebra, unless otherwise stated. Let δ be the minimal positive imaginary root. We fix the form $(-,-)$ to be positive semidefinite on $\mathfrak{h}^*_{\mathbb{R}}=\sum_{\alpha\in\Pi}\mathbb{R}\alpha$; its kernel is $\mathbb{R}\delta$.

1.3.1. Set $E:=\mathbb{R}\Delta^{re}$, $E_{cl}=E/\mathbb{R}\delta$, and let $cl: E\to E_{cl}$ denote the projection. The form $(-,-)$ on $\mathfrak{h}^*$ induces a symmetric bilinear form on E_{cl} which is positive definite. Set $\Delta_{cl}=cl(\Delta^{re})$; this is a finite root system in E_{cl} (not necessarily reduced).

1.4. **Groups of polynomial growth.** Let G be a finitely generated group and B be its finite system of generators. For $g\in G$ denote by $l_B(g)$ the "length" of g, i.e. the smallest natural number k such that $g=e_1e_2\dots e_k$, where $e_i\in B\cup B^{-1}$. For $r\in\mathbb{R}_{>0}$ set

$$c_{G,B}(r):=|\{g\in G|\ l(g)\leq r\}|.$$

If B' is another finite system of generators then for some $a,b\in\mathbb{Z}_{>0}$ one has

$$c_{G,B}(ar)\leq c_{G,B'}(r)\leq c_{G,B}(br)\quad\text{for all } r.$$

A finitely generated group is called a group of a *polynomial growth* if there exist $C>0, m\geq 0$ such that $c_{G,B}(r)\leq cr^m$ for any $r\geq 1$. For instance, a finitely generated abelian group is of polynomial growth (if the rank is m then $c_{G,B}(r)\leq (r+m)^m<Cr^m$ for $r\geq 1$).

Let G be a group and G_1 be its finitely generated subgroup of finite index. Then G is also finitely generated. Let B (resp., B_1) be a finite system of generators of G (resp., G_1). Then there exist $a,b,C>0$ such that

$$c_{G_1,B_1}(ar)\leq c_{G,B}(r)\leq Cc_{G_1,B_1}(br)\quad\text{for all } r.$$

As a result, G is of polynomial growth iff G_1 is of polynomial growth.

2. Coxeter subgroups of W

2.1. Following [KT00], [MP] we call a subset Δ_1 of Δ^{re} a *subsystem* of Δ^{re} if $s_\alpha\beta\in\Delta_1$ for any $\alpha,\beta\in\Delta_1$. For a subsystem Δ_1 of Δ^{re} we set

$$\begin{array}{ll}\Delta_{1,\pm}:=\Delta_\pm\cap\Delta_1, & \Pi_1:=\{\alpha\in\Delta_{1,+}|\ s_\alpha(\Delta_{1,+}\setminus\{\alpha\})\subset\Delta_{1,+}\},\\ W_1:=\langle s_\alpha|\alpha\in\Pi_1\rangle, & S_1:=\{s_\alpha|\alpha\in\Pi_1\}.\end{array}$$

Notice that $\Delta_{1,+}=-\Delta_{1,-}$ since for $\alpha\in\Delta_1$ the element $-\alpha=s_\alpha\alpha$ lies in Δ_1.

2.2. **Lemma.** *(i) For $\alpha,\beta\in\Pi_1$ one has $\langle\alpha^\vee,\beta\rangle\in\mathbb{Z}_{\leq 0}$ if $\alpha\neq\beta$.*

(ii) $|\Pi_1|\leq|\Delta_{cl}|$ (see 1.3.1 for notation).

Proof. Take $\alpha\neq\beta$ and assume that $(\alpha,\beta)>0$. Clearly, $\langle\alpha^\vee,\beta\rangle\in\mathbb{Z}$. Since $(\alpha,\beta)>0$ and $s_\alpha\beta\in\Delta_+$ we get $\beta\geq\alpha$; similarly, $\alpha\geq\beta$, which is a contradiction. Hence $(\alpha,\beta)<0$ so $\langle\alpha^\vee,\beta\rangle\in\mathbb{Z}_{\leq 0}$ as required for (i). For (ii) notice that $(\alpha,\beta)<0$ implies that $\alpha-\beta$ is not proportional to δ so $cl(\alpha)\neq cl(\beta)$ for distinct $\alpha,\beta\in\Pi_1$. □

2.3. In the light of Lemma 2.2 the matrix $A:=(\langle\alpha^\vee,\beta\rangle)_{\alpha,\beta\in\Pi_1}$ is a generalized Cartan matrix. Let X be the Dynkin diagram constructed for Π_1 and A: the elements of Π_1 are nodes in X and the edges (and the arrows) are determined in the usual way by A. Note that the elements of Π_1 are not necessarily linearly independent.

Let $X_i, i=1,\dots,k$, be the connected components of X. We write $\alpha\in X_i$ if α is a node of X_i. The following result goes back to Dynkin (see also [B]).

2.3.1. **Proposition.** *(i) For each $i=1,\dots,k$ the roots from X_i are linearly independent.*

(ii) Any linear dependence of the elements of Π_1 is of the form $\sum_{i=1}^k \sum_{\alpha\in X_i} m_\alpha\alpha=0$, where $\sum_{\alpha\in X_i} m_\alpha\alpha\in\mathbb{C}\delta$ for each i.

(iii) The Dynkin diagram for Π_1 is a disjoint union of Dynkin diagrams of affine and finite types.

Proof. Any linear dependence $\sum_{\alpha\in\Pi_1} m_\alpha\alpha=0$ with $m_\alpha\in\mathbb{R}$ can be rewritten in the form $\sum_{\alpha\in S} m_\alpha\alpha=\sum_{\beta\in S'} k_\beta\beta$, where S,S' are non-empty subsets of Π_1 with empty intersection $S\cap S'=\emptyset$ and all coefficients are positive: $m_\alpha,k_\beta>0$. Put $\gamma:=\sum_{\alpha\in S} m_\alpha\alpha$. Then

$$(\gamma,\gamma)=\sum_{\alpha\in S,\beta\in S'} m_\alpha k_\beta(\alpha,\beta)\leq 0.$$

Since $(-,-)$ is semidefinite on $\mathfrak{h}^*_{\mathbb{R}}$ and the isotropic vectors are proportional to δ, we get that $\gamma\in\mathbb{R}\delta$ and that $(\alpha,\beta)=0$ for all $\alpha\in S,\beta\in S'$.

Let us show that for any i one has either $X_i\cap S=\emptyset$ or $X_i\subset S$. Indeed, assume that $X_i\cap S\neq\emptyset$ and $X_i\not\subset S$. Take $\alpha'\in X_i\setminus S$ which is connected to a node in $X_i\cap S$. Then

$$(\gamma,\alpha')=\sum_{\alpha\in S\cap X_i} m_\alpha(\alpha,\alpha')<0$$

since $(\alpha,\alpha')\leq 0$ for all α and $(\alpha,\alpha')<0$ if α' is connected to α. However $\gamma\in\mathbb{R}\delta$ so $(\gamma,\alpha')=0$, which is a contradiction.

By above, for any i one has either $X_i\cap S=\emptyset$ or $X_i\subset S$. The similar fact holds for S'. Since both S,S' are non-empty, their union does not lie in X_i: $S\cup S'\not\subset X_i$. Hence the roots of X_i are linearly independent as required for (i). For (ii) write $\gamma=\sum_{i=1}^k\gamma_i$, where $\gamma_i:=\sum_{\alpha\in X_i} m_\alpha\alpha$ and notice that

$$0=(\gamma,\gamma)=\sum_{i=1}^k(\gamma_i,\gamma_i).$$

Hence $(\gamma_i,\gamma_i)=0$ so $\gamma_i\in\mathbb{R}\delta$ for any i. This gives (ii).

For (iii) fix i and let A_i be the corresponding submatrix of A: $A_i:=(\langle\alpha^\vee,\beta\rangle)_{\alpha,\beta\in X_i}$. Since the nodes of X_i are linearly independent, X_i is a standard Dynkin diagram. The matrix A_i is symmetrizable and the corresponding bilinear form on $\sum_{\alpha\in X_i}\mathbb{R}\alpha$ is the restriction of $(-,-)$. Therefore this form is either positive definite or positive semi-definite. Hence X_i is of either finite type or affine type. $\square$

2.3.2. **Corollary.** *(i) The group W_1, generated by S_1, is a direct product of Coxeter groups of finite or affine types corresponding to the connected components of X.*

(ii) The group W_1 is the Weyl group corresponding to the Dynkin diagram X: it is a Coxeter group with the canonical generator system S_1, and its length function $l_1 : W_1 \to \mathbb{Z}_{\geq 0}$ is given by $l_1(w) = |w\Delta_{1,+} \cap \Delta_{1,-}|$.

Proof. Let W'_i be the Coxeter group corresponding to the Dynkin diagram X_i and $\Delta(X_i) \subset \Delta_1$ be the root system of X_i. Since the Dynkin diagram of Π_1 is the disjoint union of X_is, there exists a homomorphism $\phi : W'_1 \times W'_2 \times \ldots \times W'_k \to W_1$. Take $w'_1 w'_2 \ldots w'_k \in \operatorname{Ker}\phi$, where $w'_i \in W'_i$. Assume that $w'_i \neq e$. Then there exists $\alpha \in X_i$ such that $l_i(s_\alpha (w'_i)^{-1}) < l_i((w'_i)^{-1})$, where l_i stands for the length function $l_i : W'_i \to \mathbb{Z}_{\geq 0}$. By 1.2 $w'_i\alpha \in \Delta_- \cap \Delta(X_i)$. The elements $w'_i\alpha, \alpha$ lie in $\Delta(X_i)$ so they are stable with respect to the action of W'_j for $j \neq i$. Thus $(w'_1 w'_2 \ldots w'_k)\alpha = w'_i\alpha \in \Delta_-$ so $(w'_1 w'_2 \ldots w'_k)\alpha \neq \alpha$ which contradicts to the assumption $w'_1 w'_2 \ldots w'_k \in \operatorname{Ker}\phi$. Hence $\operatorname{Ker}\phi = e$ and this proves (i); (ii) follows from (i). ☐

2.4. *Remark.* By [KT98], 2.2.7, for any root system Δ one has $\Delta_{1,+} \subset \sum_{\alpha \in \Pi_1} \mathbb{Z}_{\geq 0}\alpha$, $\Delta_1 = W_1 \Pi_1$, and so $W_1 = \langle s_\alpha | \alpha \in \Delta_1 \rangle$.

3. Kazhdan-Lusztig polynomials

We recall the construction of Kazhdan-Lusztig polynomials introduced in [KL79]. In this section W is a Coxeter group, isomorphic to the Weyl group of a Kac-Moody algebra.

3.1. **Polynomials $P_{x,y}(q)$.** The Kazhdan-Lusztig polynomials $P_{x,y}(q) \in \mathbb{Z}[q]$ $(x, y \in W)$ can be described recursively in the following way (see [KL79], 2.2 c):

(a) $P_{x,y} = 0$ if $x \not\leq y$; $P_{x,x} = 1$;

(b) if $x < y$ and s is a simple reflection such that $sy < y$ then

$$P_{x,y} = q^{1-c} P_{sx,sy} + q^c P_{x,sy} - \sum_{z:\ x \leq z < sy,\ sz < z} q^{\frac{l(y)-l(z)}{2}} \mu(z, sy) P_{x,z},$$

where $c = 1$ if $sx < x$, $c = 0$ if $sx > x$ and $\mu(z, w)$ is the coefficient of $\frac{l(w)-l(z)-1}{2}$th power of q in $P_{z,w}$ (defined to be 0 if $l(ys) - l(z)$ is even). Notice that the degree of $P_{x,y}$ is not greater than $\frac{l(y)-l(x)-1}{2}$.

Due to non-negativity of coefficients of $P_{x,y}(q)$ [KL79], [KL80],[H], the above recursive description of $P_{x,y}$ gives $P_{x,y}(1) \leq P_{sx,sy}(1) + P_{x,sy}(1)$ if $sy < y$, hence

$$P_{x,y}(1) \leq l(y)^{l(y)-l(x)}. \tag{5}$$

3.2. **Polynomials $Q_{x,y}(q)$.** The inverse Kazhdan-Lusztig polynomials $Q_{x,y}(q)\in\mathbb{Z}[q]$ $(x,y\in W)$ are defined by the property

$$\sum_w (-1)^{l(w)-l(x)}Q_{x,w}P_{w,y}=\delta_{x,y}.$$

One has $Q_{x,y}=0$ for $y\not\geq x$, $Q_{x,x}=1$, and for $y>x$ the degree of $Q_{x,y}$ is not greater than $\frac{l(y)-l(x)-1}{2}$.

3.2.1. For any fixed $z\in W$ the matrix $(P_{w,y})_{w,y\leq z}$ is a square matrix of finite size; by above, this matrix is inverse to the matrix $((-1)^{l(w)-l(x)}Q_{x,w})_{x,w\leq z}$. Therefore

$$\sum_{w:x\leq w\leq y}(-1)^{l(y)-l(w)}P_{x,w}Q_{w,y}=\delta_{x,y}.$$

Using that $P_{x,x}=1$, we express $Q_{x,y}$ via the rest of summands to obtain

$$Q_{x,y}=\sum_{w:x<w\leq y}(-1)^{l(w)-l(x)+1}P_{x,w}Q_{w,y}\quad\text{for}\quad y>x. \tag{6}$$

3.2.2. **Lemma.** *Assume that W has polynomial growth, i.e. for some constants $C>0,\ n\geq 0$ one has*

$$|\{w|\ l(w)\leq k\}|<Ck^n,\quad \textit{for all positive integers}\ \ k.$$

Put $N:=\max(n+1;2)$. Then for all $x,y\in W$ one has

$$|Q_{x,y}(1)|\leq (Cl(y)^N)^{l(y)-l(x)}.$$

Proof. The proof is by induction on $l:=l(y)-l(x)$. For $l\leq 2$ one has $Q_{x,y}=0$ or 1 so the assertion holds. Assume that $l\geq 3$. From (6) one sees that

$$|Q_{x,y}(1)|\leq Cl(y)^n\max_{w:x<w\leq y}|P_{x,w}(1)Q_{w,y}(1)|.$$

Set $a:=l(x),b:=l(y),t:=l(w)$ and note that $t\in[a+1,b]$. By (5) one has $P_{x,w}(1)\leq t^{t-a}$. The induction hypothesis gives

$$|Q_{x,y}(1)|\leq C^{b-a}b^n\max_{t\in[a+1,b]}\big(b^{N(b-t)}t^{t-a}\big).$$

Set $g(t):=b^{N(b-t)}t^{t-a}$. Then

$$g'(t)=g(t)(-N\ln b+\ln t+1-a/t).$$

For $t\in[a+1,b]$ one has $-N\ln b+\ln t+1-a/t<-(N-1)\ln b+1<0$ since $b=l(y)\geq 3$ and $N>1$. Hence $g'(t)<0$ for $t\in[a+1,b]$ and therefore

$$\max_{t\in[a+1,b]}g(t)=g(a+1)=b^{N(b-a-1)}(a+1)\leq b^{N(b-a-1)+1}.$$

Thus

$$|Q_{x,y}(1)|\leq C^{b-a}b^{n+N(b-a-1)+1}=C^{b-a}b^{N(b-a)}.\quad\square$$

4. Characters of irreducible modules with non-critical highest weights

In this section we recall the results of M. Kashiwara and T. Tanisaki [KT00]. Let $\mathfrak{g}$ be an affine Lie algebra and $\lambda\in\mathfrak{h}^*$ be a non-critical weight, i.e. $(\delta,\lambda+\rho)\neq 0$.

4.1. **Notations.** For $\lambda\in\mathfrak{h}^*$ set denote by $\Delta(\lambda)$ the set of real roots satisfying $\langle\alpha^\vee,\lambda{+}\rho\rangle\in\mathbb{Z}$ and by $\Delta_0(\lambda)$ the set of real roots satisfying $\langle\alpha^\vee,\lambda+\rho\rangle=0$. Clearly, both $\Delta(\lambda)$ and $\Delta_0(\lambda)$ are root subsystems in a sense of 2.1. Denote by $W(\lambda)$ (resp., $W_0(\lambda)$) the subgroup of W generated by $\{s_\alpha|\alpha\in\Delta(\lambda)\}$ (resp., by $\{s_\alpha|\alpha\in\Delta_0(\lambda)\}$).

Since $(\delta,\lambda+\rho)\neq 0$ one has $cl(\alpha)\neq cl(\beta)$ for $\alpha,\beta\in\Delta_{+,0}(\lambda)$ (see 1.3.1 for notation). Therefore $\Delta_{+,0}(\lambda)$ is finite, so $W_0(\lambda)$ is a finite Coxeter group.

4.1.1. By Remark 2.4, $W(\lambda)$ is a Coxeter group with the canonical generator system $\{s_\alpha|\alpha\in\Pi(\lambda)\}$, where $\Pi(\lambda)$ for $\Delta(\lambda)$ is defined in 2.1.

Denote the Bruhat ordering of $W(\lambda)$ by $\geq_\lambda$ and and the length function by $l_\lambda: W(\lambda)\to\mathbb{Z}_{\geq 0}$. For $x,y\in W(\lambda)$ denote by $P^\lambda_{x,y}(q)\in\mathbb{Z}[q]$ (resp., by $Q^\lambda_{x,y}(q)\in\mathbb{Z}[q]$) the corresponding Kazhdan-Lusztig polynomial (resp., inverse Kazhdan-Lusztig polynomial).

4.2. Let $\mathcal{C}$ be the set of non-critical weights, i.e.

$$\mathcal{C}:=\{\lambda\in\mathfrak{h}^*|\ (\delta,\lambda+\rho)\neq 0\}.$$

Let

$$\begin{array}{l}\mathcal{C}^+:=\{\lambda\in\mathcal{C}|\ \langle\alpha^\vee,\lambda+\rho\rangle\geq 0 \text{ for any } \alpha\in\Delta(\lambda)\cap\Delta_+\},\\ \mathcal{C}^-:=\{\lambda\in\mathcal{C}|\ \langle\alpha^\vee,\lambda+\rho\rangle\leq 0 \text{ for any } \alpha\in\Delta(\lambda)\cap\Delta_+\}.\end{array}$$

4.3. Observe that $\Delta(\lambda)=\emptyset$ forces $\Delta(w.\lambda)=\emptyset$ for any $w\in W$. Therefore if $\Delta(\lambda)=\emptyset$, then $M(w.\lambda)=L(w.\lambda)$ for any $w\in W$.

Let $\lambda\in\mathcal{C}$ be such that $\Delta(\lambda)\neq\emptyset$. In the light of Lemma 2.10 of [KT00] there are the following cases:

(a) If $(\delta,\lambda+\rho)\not\in\mathbb{Q}$, then $|(W(\lambda).\lambda)\cap\mathcal{C}^+|=|(W(\lambda).\lambda)\cap\mathcal{C}^-|=1$.

(b) If $(\delta,\lambda+\rho)\in\mathbb{Q}_{>0}$, then $|(W(\lambda).\lambda)\cap\mathcal{C}^+|=1$ and $|(W(\lambda).\lambda)\cap\mathcal{C}^-|=0$.

(c) If $(\delta,\lambda+\rho)\in\mathbb{Q}_{<0}$, then $|(W(\lambda).\lambda)\cap\mathcal{C}^-|=1$ and $|(W(\lambda).\lambda)\cap\mathcal{C}^+|=0$.

In particular, if $M(\lambda')$ is not simple then there exists $\lambda\in\mathcal{C}^+\cup\mathcal{C}^-$ and $w\in W(\lambda)$ such that $\lambda'=w.\lambda$.

4.3.1. **Lemma.** *Let $\lambda\in\mathcal{C}^+\cup\mathcal{C}^-$.*

(i) The group $W_0(\lambda)$ coincides with the stabilizer of λ in $W(\lambda)$ and is generated by the set $\{s_\alpha|\alpha\in\Pi_0(\lambda)\}$, where $\Pi_0(\lambda)=\Pi(\lambda)\cap\{\alpha|\langle\alpha^\vee,\lambda+\rho\rangle=0\}$.

(ii) For any $w\in W(\lambda)$ the coset $wW_0(\lambda)$ contains a unique minimal element and a unique maximal element.

Proof. (i) Fix $\lambda \in \mathcal{C}^+$. Take $w \in W(\lambda)$ and let $w = s_{i_1} \dots s_{i_l}$ be a reduced expression of $w \in W$, where each s_{i_j} is a simple reflection with respect to $\alpha_{i_j} \in \Pi(\lambda)$. By Lemma 1.2.2,

$$\lambda - w.\lambda = (\lambda + \rho) - w(\lambda + \rho) = \sum_{j=0}^{l-1} \langle \alpha^\vee_{i_{j+1}}, \lambda + \rho \rangle \beta_j$$

for some $\beta_j \in \Delta_+ \cap \Delta(\lambda)$. The condition $\lambda \in \mathcal{C}^+$ ensures that the coefficient of β_j is non-negative for any j. If $\lambda - w.\lambda = 0$ this implies $\langle \alpha^\vee_{i_{j+1}}, \lambda + \rho \rangle = 0$ for any j. Therefore the stabilizer of λ in $W(\lambda)$ is generated by simple reflections of $W(\lambda)$ and $W_0(\lambda)$ coincides with this stabilizer. The proof for $\lambda \in \mathcal{C}^-$ is completely similar.

(ii) Let $x \in wW_0(\lambda)$ be a minimal element. Let us show that for any $y \in W_0(\lambda)$ one has $l(xy) = l(x) + l(y)$. The proof is by induction on $l(y)$. Assume that $l(xy) = l(x) + l(y)$ and that $\alpha \in \Pi_0(\lambda)$ is such that $ys_\alpha > y$. By 1.2.1, $ys_\alpha > y$ gives $y(\alpha) \in \Delta_{0,+}(\lambda)$; from (i) one sees that $y(\alpha)$ is a non-negative linear combination of elements of $\Pi_0(\lambda)$. The minimality of x implies $x(\beta) \in \Delta_+$ for any $\beta \in \Pi_0(\lambda)$. Therefore $xy(\alpha) \in \Delta_+$ so $l(xys_\alpha) = l(xy) + 1$ by 1.2.1. Hence $l(xy) = l(x) + l(y)$ for any $y \in W_0(\lambda)$, and (ii) follows from the finiteness of $W_0(\lambda)$. □

4.3.2. Take $\lambda' \in \mathcal{C}$ such that $M(\lambda')$ is not simple and choose $\lambda \in \mathcal{C}^+ \cup \mathcal{C}^-$ such that $\lambda' = w.\lambda$ for some $w \in W(\lambda)$. By Lemma 4.3.1, $wW_0(\lambda) = \{x \in W(\lambda) |\ x.\lambda = \lambda'\}$ and this set contains a unique minimal element and a unique maximal element, which we denote respectively by $w_s(\lambda'; \lambda)$ and $w_l(\lambda'; \lambda)$.

4.3.3. **Theorem.** ([KT00], *Thm. 1.1). Take $\lambda' \in \mathcal{C}$ such that $M(\lambda')$ is not simple. Take $\lambda \in \mathcal{C}^+ \cup \mathcal{C}^-$ such that $\lambda' = w.\lambda$ for some $w \in W(\lambda)$ and define $w_s(\lambda'; \lambda)$, $w_l(\lambda'; \lambda)$ as in 4.3.2.*

(i) If $\lambda \in \mathcal{C}^+$ set $x := w_l(\lambda'; \lambda)$ and observe that $\lambda' = x.\lambda$. One has

$$\mathrm{ch}_{L(x.\lambda)} = \sum_{y \in W(\lambda): y \geq_\lambda x} (-1)^{l_\lambda(y) - l_\lambda(x)} Q^\lambda_{x,y}(1)\, \mathrm{ch}_{M(y.\lambda)}\,.$$

(ii) If $\lambda \in \mathcal{C}^-$ set $z := w_s(\lambda'; \lambda)$ and observe that $\lambda' = z.\lambda$. One has

$$\mathrm{ch}_{L(z.\lambda)} = \sum_{y \in W(\lambda): y \leq_\lambda z} (-1)^{l_\lambda(z) - l_\lambda(y)} P^\lambda_{y,z}(1)\, \mathrm{ch}_{M(y.\lambda)}\,.$$

5. Proof of Theorem 5.1

5.1. **Theorem.** *Let $V(\Lambda)$ be a highest weight module with highest weight Λ over an affine Lie algebra, and assume that $(\Lambda + \rho, \delta) \neq 0$. Then the function $\mathrm{ch}_{V(\Lambda)}(h)$ is meromorphic in the domain $Y := \{h \in \mathfrak{h} |\ \mathrm{Re}\langle h, \delta \rangle > 0\}$.*

5.1.1. *Remark.* In fact, we will show that $R(h)\,\mathrm{ch}_{V(\Lambda)}(h)$ is holomorphic in Y, where R is the Weyl denominator, which is holomorphic in Y as well (see 5.3).

5.2. Write $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ and define a grading on Q^+ by setting $\deg \alpha_0 = 1$, $\deg \alpha_i = 1$ for $i = 1, \dots, n$. Consider the induced grading on Δ_+: $\Delta_+ = \cup_{j\geq 0}\Delta_{+,j}$. From the structure theory of affine Lie algebras ([K], Ch. VI) one knows that the multiset Δ contains at most $n := \dim \mathfrak{h} - 2$ copies of each imaginary root which is of the form $k\delta$, and that all real roots have multiplicity one and are of the form $k\delta + \alpha$ for $\alpha \in \Delta_{cl}$; in particular,

$$\forall j \quad |\Delta_{+,j}| \leq N, \quad \text{where } N := |\Delta_{cl}| + \dim \mathfrak{h} - 2. \tag{7}$$

5.3. Consider the infinite product

$$R(h) := \prod_{\alpha\in\Delta_+} (1 - e^{-\alpha(h)}),\ \ h \in Y.$$

It is well-known that $R(h)$ converges to a holomorphic function in the domain Y. Indeed, using the Weierstrass criterion, it is enough to show that

$$\sum_{\alpha\in\Delta_+} |e^{-\langle h,\alpha\rangle}| < \infty \text{ if } h \in Y.$$

From 5.2 one sees that

$$\sum_{\alpha\in\Delta_+} |e^{-\langle h,\alpha\rangle}| < \big(\sum_{\alpha\in\Delta_{cl}} |e^{-\langle h,\alpha\rangle}|\big)\big(\sum_{k=0}^{\infty} |e^{-k\langle h,\delta\rangle}|\big)^N.$$

The first sum is finite; for the second sum one has

$$\sum_{k=0}^{\infty} |e^{-k\langle h,\delta\rangle}| = \sum_{k=0}^{\infty}\big(e^{-\,\mathrm{Re}\langle h,\delta\rangle}\big)^k < \infty \ \text{ if } \ \mathrm{Re}\langle h,\delta\rangle > 0.$$

5.3.1. *Remark.* This argument implies the following: $\mathrm{ch}_{M(\lambda)}$ converges in the domain $Y_> := \{h \in \mathfrak{h}|\ \mathrm{Re}\,\alpha_i(h) > 0,\ i = 0, 1, \dots, n\}$ to a holomorphic function. The same holds for the character $\mathrm{ch}_{V(\lambda)}(h)$ since it is majorized by $\mathrm{ch}_{M(\lambda)}(\mathrm{Re}(h))$ (where $\mathrm{Re}(h) \in \mathfrak{h}_{\mathbb{R}}$ is such that $\alpha(\mathrm{Re}(h)) = \mathrm{Re}\,\alpha(h)$ for any $\alpha \in \Delta$); for $h \in Y_>$ one has $\mathrm{Re}(h) \in Y_>$. Since the summands in $\mathrm{ch}_{M(\lambda)}(\mathrm{Re}(h))$ are positive real numbers $\mathrm{ch}_{V(\lambda)}(h)$ is holomorphic in $Y_>$ by the Weierstrass criterion.

5.4. Since $R(h)$ is holomorphic in Y, $\mathrm{ch}_{M(\lambda)}(h) = R^{-1}(h)e^{\langle h,\lambda\rangle}$ is meromorphic in Y for any λ.

5.4.1. Consider the case when $(\Lambda+\rho,\delta)\notin\mathbb{Q}_{\geq 0}$. By 4.3 $\mathrm{ch}_{L(\Lambda)}$ is given by Theorem 4.3.3 (ii) so it is a finite sum of functions $\mathrm{ch}_{M(y.\lambda)}(h)$. Hence $\mathrm{ch}_{L(\Lambda)}(h)$ is meromorphic in Y.

The assumption $(\Lambda+\rho,\delta)\notin\mathbb{Q}_{\geq 0}$ implies that $\langle\alpha^\vee,\Lambda+\rho\rangle\notin\mathbb{Z}_{>0}$ for $\alpha^\vee\in\Delta_j$ with $j>>0$ (see 5.2 for notation). Then [KK], $M(\Lambda)$ has finite length. Therefore $\mathrm{ch}_{V(\Lambda)}$ is a finite linear combination of $\mathrm{ch}_{L(w.\Lambda)}(h)$. Clearly, $(w.\Lambda+\rho,\delta)=(\Lambda+\rho,\delta)$ and so, by above, $\mathrm{ch}_{L(w.\Lambda)}(h)$ is meromorphic in Y for any w. Hence $\mathrm{ch}_{V(\Lambda)}$ is meromorphic in Y.

5.4.2. Therefore it remains to consider the case when $(\Lambda+\rho,\delta)\in\mathbb{Q}_{>0}$.

5.5. Fix Λ such that $(\Lambda+\rho,\delta)\in\mathbb{Q}_{>0}$. Take $\lambda\in\mathcal{C}^+$ and $x\in W(\lambda)$ such that $\Lambda=x.\lambda$.

In this case $\mathrm{ch}_{L(\Lambda)}$ is given by Theorem 4.3.3 (i):

$$\mathrm{ch}_{L(\Lambda)}(h)=\mathrm{ch}_{L(x.\lambda)}(h)=R^{-1}(h)\sum_{z\in W(\lambda):\ z\geq_\lambda x}(-1)^{l_\lambda(z)-l_\lambda(x)}Q^\lambda_{z,y}(1)e^{\langle h,z.\lambda\rangle}$$

for some $\lambda\in\mathcal{C}^+,\ x\in W(\lambda)$ (see 4.1 for notation).

All simple subquotients of $M(\Lambda)=M(x.\lambda)$ are of the form $L(y.\lambda)$ for some $y\geq_\lambda x$. By Lemma 4.3.1, the condition $y\geq_\lambda x$ ensures that y is the maximal element of $yW_0(\lambda)$. Combining Theorem 4.3.3 and the definition of $Q_{x,y}$ one sees that the multiplicity of $L(y.\lambda)$ in $M(x.\lambda)$ is $P^\lambda_{x,y}(1)$. Since $V(\Lambda)=V(x.\lambda)$ is a quotient of $M(x.\lambda)$, and the multiplicity of $L(y.\lambda)$ in $M(x.\lambda)$ is $P^\lambda_{x,y}(1)$ [KT00], the multiplicity of $L(y.\lambda)$ in $V(x.\lambda)$ is not greater than $P^\lambda_{x,y}(1)$, that is

$$\mathrm{ch}_{V(x.\lambda)}(h)=\sum_{y\in W(\lambda):\ y\geq_\lambda x}a_y\,\mathrm{ch}_{L(y.\lambda)}(h),\quad \text{for some } a_y\in\mathbb{Z}_{\geq 0},\quad a_y\leq P^\lambda_{x,y}(1).$$

Hence

$$\begin{array}{l}|R(h)\,\mathrm{ch}_{V(x.\lambda)}(h)|\leq\sum_{z\in W(\lambda):\ z\geq_\lambda x}b_z e^{\mathrm{Re}\langle h,z.\lambda\rangle},\quad\text{where}\\ b_z=\sum_{y\in W(\lambda):\ x\leq_\lambda y\leq_\lambda z}|P^\lambda_{x,y}(1)Q^\lambda_{y,z}(1)|.\end{array}\tag{8}$$

From now on we fix $\lambda\in\mathcal{C}^+$ such that $(\lambda+\rho,\delta)\in\mathbb{Q}_{>0}$ and fix $x\in W(\lambda)$.

5.5.1. Retain notation of 1.3.1. For $a,b>0$ set

$$Y_{a,b}:=\{h\in\mathfrak{h}:\mathrm{Re}\langle h,\delta\rangle>a,\ |\langle h,\alpha\rangle|<b\ \text{ for any }\ \alpha\in\Delta_{cl}\}.$$

5.5.2. In view of (8), by the Weierstrass criterion, it remains to verify that the sum

$$\sum_{l=0}^{\infty}\Big(\sum_{z\in W(\lambda):\ z\geq_\lambda x,\ l(z)=l}\ \sum_{y\in W(\lambda):\ x\leq_\lambda y\leq_\lambda z}|P^\lambda_{x,y}(1)Q^\lambda_{x,y}(1)|\Big)e^{\mathrm{Re}\langle h,z.\lambda-\lambda\rangle}\tag{9}$$

converges to a uniformly bounded function of h in the domain $Y_{a,b}$, for any $a,b>0$.

5.6. **Proposition.** *There exists $C>0$ such that for all but finitely many $y\in W(\lambda)$ one has*

$$\mathrm{Re}\langle h,\lambda-y.\lambda\rangle > Cl_\lambda(y)^2 \quad \textit{for any}\ \ h\in Y_{a,b}.$$

We prove this proposition in 5.6.1-5.6.7 below.

5.6.1. Retain notation of 4.1 and set $\Delta^\pm(\lambda):=\Delta(\lambda)\cap\Delta^\pm$, $\Pi_0(\lambda):=\{\alpha\in\Pi(\lambda)|\ \langle\alpha^\vee,\lambda+\rho\rangle=0\}$. Recall that $W_0(\lambda)$ is a finite group generated by $\{s_\alpha|\alpha\in\Pi_0(\lambda)\}$; let L be the length of the longest elements of $W_0(\lambda)$.

5.6.2. Fix $y\in W(\lambda)$ and write its reduced decomposition in the Coxeter group $W(\lambda)$ in the form $y=y_1s_{i_1}y_2s_{i_2}\ldots y_ms_{i_m}y_{m+1}$, where $y_1,\ldots,y_{m+1}\in W_0(\lambda)$ and each s_{i_j} is the reflection with respect to $\alpha_{i_j}\in\Pi(\lambda)\setminus\Pi_0(\lambda)$. By above, $l_\lambda(y_j)\leq L$, so

$$m>\frac{l_\lambda(y)}{L+1}-1. \tag{10}$$

5.6.3. Let X be the Dynkin diagram of $\Pi(\lambda)$ constructed as in 2.3. By Proposition 2.3.1, X is a disjoint union of Dynkin diagrams of finite and affine types; denote by Π' the set of simple roots for the diagram X and by Δ' the set of roots for X (the elements of Π' are linearly independent whereas the elements of $\Pi(\lambda)$ may be linearly dependent, see Proposition 2.3.1 (ii)). Extend the natural identification of Π' and $\Pi(\lambda)$ to the linear map $\iota:\mathbb{R}\Pi'\to\mathbb{R}\Pi(\lambda)$. Retain notation of 1.2.1. For $w\in W(\lambda)$ set $S'(w):=\Delta'_+\cap w^{-1}(\Delta'_-)$. By 1.2.1, the elements of $S'(w)$ are pairwise distinct.

From Lemma 1.2.2 we obtain

$$\lambda-y.\lambda=(\lambda+\rho)-y(\lambda+\rho)=\sum_{j=1}^m\langle\alpha_{i_j}^\vee,\lambda+\rho\rangle\iota(\beta'_j), \tag{11}$$

where $\{\beta'_j\}_{j=1}^m$ is a subset of $S'(y^{-1})\subset\Delta_+^{re}$. We claim that the elements $\iota(\beta'_j)$ are pairwise distinct. Indeed, take $\beta'_i,\beta'_j\in S'(y^{-1})$ such that $\beta'_i-\beta'_j\in\mathrm{Ker}\,\iota$. Each element of Δ' lies in the root system of a certain connected component of X. Let $\beta'_i\in\Delta(X_s)$, $\beta'_j\in\Delta(X_t)$. By Proposition 2.3.1 (i), $\mathrm{Ker}\,\iota\cap\Delta(X_s)=0$ so $\beta'_i-\beta'_j\not\in\mathrm{Ker}\,\iota$ if $s=t$. By Proposition 2.3.1 (ii) for $s\neq t$ the assumption $\beta'_i-\beta'_j\in\mathrm{Ker}\,\iota$ gives $\iota(\beta'_i)\in\mathbb{C}\delta$. This contradicts the fact that β'_i is a real root.

By above, $\beta_j:=\iota(\beta_j)$ $(j=1,\ldots,m)$ are pairwise distinct elements of Δ_+^{re}; set $R(y):=\{\beta_j\}_{j=1}^m$.

5.6.4. Fix $h\in Y_{a,b}$. Take $D\in\mathfrak{h}^*$ such that $\langle D,\Delta_{cl}\rangle=0$, $\langle D,\delta\rangle=1$. Write $h=a'D+h'$, where h' lies in the span of Δ and notice that $\mathrm{Re}\,a'>a$. One has $\lambda-y.\lambda=\langle D,\lambda-y.\lambda\rangle\delta+cl(\lambda-y.\lambda)$ and so

$$\langle h,\lambda-y.\lambda\rangle=a'\langle D,\lambda-y.\lambda\rangle+\langle h',\lambda-y.\lambda\rangle=a'\langle D,\lambda-y.\lambda\rangle+\langle h,cl(\lambda-y.\lambda)\rangle.$$

5.6.5. Recall that $\langle \alpha_{i_j}^\vee, \lambda+\rho\rangle \in \mathbb{Z}_{>0}$ for $j=1,\dots,m$ and that $\langle D,\gamma\rangle\in\mathbb{Z}_{\geq 0}$ for any $\gamma\in\Delta_+$. From (11) we get

$$\langle D, \lambda - y.\lambda\rangle \geq \sum_{j=1}^m \langle D,\beta_j\rangle.$$

Setting $b' := \max_{\alpha\in\Pi(\lambda)}\langle\alpha^\vee,\lambda+\rho\rangle$, we obtain from (11)

$$|\langle h, cl(\lambda - y.\lambda)\rangle| \leq \sum_{j=1}^m |\langle \alpha_{i_j}^\vee, \lambda+\rho\rangle||\langle h, cl(\beta_j)| < mbb' \ \text{ for } h\in Y_{a,b}.$$

5.6.6. Retain notation of 5.6.3. By 5.2, the cardinality of $X_k=\{\beta\in\Delta_+ |\ \langle D,\beta\rangle < k\}$ is not greater than Nk so

$$|\{\beta\in R(y)|\ \langle D,\beta\rangle\geq k\}| = |R(y)\setminus X_k| > m - Nk.$$

Set $m' := [m/N]$. Then

$$\begin{aligned}\textstyle\sum_{j=1}^m\langle D,\beta_j\rangle &= \textstyle\sum_{k=1}^\infty |\{\beta\in R(y)|\ \langle D,\beta\rangle\geq k\}| > \sum_{k=1}^{m'} |\{\beta\in S_\lambda(w)|\ \beta\rangle\geq k\}| \\ &> \textstyle\sum_{k=1}^{m'}(Nm'-Nk) = Nm'(m'-1)/2 > \frac{N}{2}(\frac{m}{N}-1)(\frac{m}{N}-2).\end{aligned}$$

Hence

$$\sum_{j=1}^m\langle D,\beta_j\rangle > \frac{m^2}{8N} \text{ if } m > 4N. \tag{12}$$

5.6.7. By (10), $m\to\infty$ if $l_\lambda(y)\to\infty$. Now 5.6.4,5.6.5 and (12) give

$$\mathrm{Re}\langle h,\lambda-y.\lambda\rangle \geq a\frac{m^2}{8N} - mbb' > a\frac{m^2}{16N} \text{ for } l_\lambda(y) >> 0.$$

Combining with (10) we obtain for $M := 2^7L^2N$

$$\mathrm{Re}\langle h,\lambda-y.\lambda\rangle \geq a\frac{l_\lambda(y)^2}{M} \text{ for } l_\lambda(y) >> 0.$$

This completes the proof of Proposition 5.6. □

5.7. Recall that the Coxeter group of affine type is a semidirect product of a finite Weyl group and a free abelian group of finite rank, so it has polynomial growth. By 2.3.2, $W(\lambda)$ is a direct product of Coxeter groups of finite and affine types. In particular, $W(\lambda)$ is of polynomial growth, i.e.

$$\exists C'>0, k\geq 0 \ \text{ such that } \ |\{w\in W(\lambda)| l_\lambda(w)\leq j\}| < C'j^k \ \text{ for all } j\geq 1, \tag{13}$$

and the estimate in 3.2.2 is valid for $Q_{x,y}^\lambda$.

5.7.1. Combining (13), (5) and Lemma 3.2.2 we obtain that there exist $C_1 > C', N > k$ such that for any $z \in W(\lambda)$ one has

$$\sum_{y \in W(\lambda):\ x \leq_\lambda y \leq_\lambda z} |P^\lambda_{x,y}(1) Q^\lambda_{y,z}(1)| < (C_1 l_\lambda(z)^N)^{l_\lambda(z)}.$$

Combining this inequality with Proposition 5.6 and using (13) again, we get for all but finitely many $l > 0$

$$\sum_{z \in W(\lambda): z \geq_\lambda x, l(z)=l} \ \sum_{y \in W(\lambda): x \leq_\lambda y \leq_\lambda z} |P^\lambda_{x,y}(1) Q^\lambda_{x,y}(1)| e^{\mathrm{Re}\langle h, z.\lambda - \lambda\rangle} < (C_1 l^N)^{l+1} e^{-Cl^2}$$

for any $h \in Y_{a,b}$. Since the series $\sum_{l=0}^\infty (C_1 l^{N'})^{l+1} e^{-Cl^2}$ converges, the sum (9) is uniformly bounded for $h \in Y_{a,b}$. This completes the proof of Theorem 5.1.

5.8. *Remark.* We do not know whether the characters of highest weight modules over arbitrary Kac-Moody algebra $\mathfrak{g}$ are meromorphic functions in the interior of the complexified Tits cone Y (see [K], 10.6 for its definition). However, in the rank 2 case it is true, provided that $\Lambda + \rho$ lies in the dual complexified Tits cone. Indeed, the denominator of $\mathrm{ch}_{V(\lambda)}$ equals to $\sum_{w \in W} (-1)^{l(w)} e^{w\rho - \rho}$ by the denominator identity, which converges to a holomorphic function in Y [K, 10.6]. The numerator in this case is always of the form $\sum_w (-1)^{l(w)} e^{w(\Lambda+\rho)-\rho}$, where w runs over a subset of W, since in the rank 2 case $P_{x,y} = 1$ for all $x \leq y$. This series converges to a holomorphic function in Y as well.

6. The poles of $\mathrm{ch}_{V(\lambda)}(h)$

In this section $\lambda \in \mathfrak{h}^*$ is non-critical, i.e. $(\lambda + \rho, \delta) \neq 0$.

By Theorem 5.1, the meromorphic function $\mathrm{ch}_{V(\lambda)}(h)$ in Y has at most simple poles at the hyperplanes $\alpha = 0$, where $\alpha \in \Delta_+^{re}$, and no other singularities. The collection $S_{V(\lambda)} \subset \Delta_+^{re}$ of α's, for which $\mathrm{ch}_{V(\lambda)}(h)$ does have a singularity, is an interesting invariant of $V(\lambda)$. This kind of invariant has been studied in the finite-dimensional case under the name Borho-Jantzen-Duflo τ-invariant (see [BJa],[BJo],[V] and references there).

If $V(\lambda)$ is integrable then $\mathrm{ch}_{V(\lambda)}(h)$ is holomorphic in Y (see [K], 11.10). In this section we will show that the converse is true: if λ is non-critical and $\mathrm{ch}_{V(\lambda)}(h)$ is holomorphic in Y, then $V(\lambda)$ is integrable. We also show that if λ is non-critical and $S_{L(\lambda)} = \Delta_+^{re}$, then $L(\lambda)$ is a Verma module.

6.1. **Proposition.** *Fix $\lambda \in \mathcal{C}$. Let $\alpha \in \Pi(\lambda)$ be such that $M(s_\alpha.\lambda)$ is a submodule of $M(\lambda)$. If $V(\lambda)$ is a subquotient of $M(\lambda)/M(s_\alpha.\lambda)$ then $\alpha \notin S_{V(\lambda)}$.*

Proof. Clearly, we may assume that $M(s_\alpha.\lambda) \neq M(\lambda)$. One has

$$\mathrm{ch}_{M(\lambda)/M(s_\alpha.\lambda)} = e^\lambda \frac{1 - e^{-k\alpha}}{\prod_{\beta \in \Delta_+}(1 - e^{-\beta})} = e^\lambda (1 + e^{-\alpha} + \ldots + e^{-(k-1)\alpha}) \prod_{\beta \in \Delta_+ \setminus \{\alpha\}} (1 - e^{-\beta})^{-1}$$

for some $k > 0$. We claim that $\mathrm{ch}_{M(\lambda)/M(s_\alpha.\lambda)}(h)$ absolutely converges in the domain

$$Y_{>;\alpha} := \{h \in \mathfrak{h} | \ \forall \beta \in \Delta_+ \setminus \{\alpha\} \quad \mathrm{Re}\, \beta(h) > 0\}.$$

Indeed, in $Y_{>;\alpha}$ the absolute convergence of the infinite product $\prod_{\beta\in\Delta_+\setminus\{\alpha\}}(1 - e^{-\beta(h)})^{-1}$ is equivalent to the absolute convergence of the infinite sum $\sum_{\beta\in\Delta_+\setminus\{\alpha\}} e^{-\beta(h)}$, which converges absolutely in $Y_{>;\alpha}$ by the argument of 5.3.

Since $\mathrm{ch}_{V(\lambda)}(h)$ is majorized by $\mathrm{ch}_{M(\lambda)/M(s_\alpha.\lambda)}(h)$, $\mathrm{ch}_{V(\lambda)}(h)$ is holomorphic in $Y_{>;\alpha}$.

Consider the case when $\alpha \in \Pi$. Consider $\mathfrak{h}$ as a vector space over $\mathbb{R}$. The domain Y is bounded by the hyperplanes $\mathrm{Re}\, \beta(h) = 0$, $\beta \in \Pi$ so $Y_{>;\alpha}$ strictly contains Y. Therefore $Y_{>;\alpha}$ contains an open ball B which meets the hyperplane $\alpha = 0$. Since $\mathrm{ch}_{V(\lambda)}(h)$ is holomorphic in B, it does not have a pole at the hyperplane $\alpha = 0$ so $\alpha \notin S_{V(\lambda)}$ as required.

Now consider the case when $\alpha \in \Pi(\lambda) \setminus \Pi$. Let X be the Dynkin diagram of $\Pi(\lambda)$ and let $\mathfrak{g}_\lambda$ be the Kac-Moody algebra corresponding to X. By Proposition 2.3.1 (iii), $\mathfrak{g}_\lambda$ is the direct product of simple finite dimensional and affine Lie algebras. Let $\mathfrak{h}_\lambda$ be the Cartan subalgebra of $\mathfrak{g}_\lambda$ and $\Pi' \subset \mathfrak{h}_\lambda^*$ be the set of simple roots. We will reduce the assertion that $\mathrm{ch}_{V(\lambda)}(h)$ does not have a pole at the hyperplane $\alpha = 0$ to a similar assertion for a highest weight module over $\mathfrak{g}_\lambda$.

Extend the natural identification of Π' and $\Pi(\lambda)$ to the linear map $\iota : \mathbb{C}\Pi' \to \mathbb{C}\Pi(\lambda)$. Let ρ_λ be the standard Weyl vector for $\Delta(\lambda)_+$; introduce $*$-action of $W(\lambda)$ on $\mathfrak{h}_\lambda^*$ and on $\mathfrak{h}^*$ by the formulas

$$w * \mu = w(\mu + \rho_\lambda) - \rho_\lambda, \quad w * \mu = w(\mu + \iota(\rho_\lambda)) - \iota(\rho_\lambda).$$

For $\nu \in \mathfrak{h}_\lambda^*$ denote by $M'(\nu)$ (resp., by $L'(\nu)$) the Verma (resp., simple) $\mathfrak{g}_\lambda$-module of highest weight ν. Let $\pi : \mathfrak{h}^* \to \mathbb{C}\Pi(\lambda) = \mathrm{Im}\, \iota$ be the orthogonal projection (i.e., $\ker \pi = \{\mu \in \mathfrak{h}^* | \ (\mu, \mathbb{C}\Pi(\lambda)) = 0\}$). Set $\nu := \iota^{-1}\pi(\lambda + \rho) - \rho_\lambda$. Then $\lambda + \rho - \iota(\nu + \rho_\lambda)$ lies in $\ker \pi$. Since $W(\lambda)$ stabilizes the elements of $\ker \pi$ one has

$$\lambda - w.\lambda = \iota(\nu - w * \nu) \ \text{ for any } w \in W(\lambda).$$

In particular, writing $\lambda - s_\alpha.\lambda = k\alpha$ we obtain $\nu - s_\alpha * \nu = k\alpha$. By the assumption, $M(s_\alpha.\lambda)$ is a proper submodule of $M(\lambda)$ so k is a positive integer. Therefore $M'(s_\alpha * \nu)$ is a submodule of $M'(\nu)$.

Observe that for any $w \in W(\lambda)$ one has

$$e^{-\lambda} \prod_{\beta\in\Delta_+\setminus\Delta(\lambda)} (1 - e^{-\beta})\, \mathrm{ch}_{M(w.\lambda)} = \iota\big(e^{-\nu}\, \mathrm{ch}_{M'(w.\nu)}\big),$$

where $\iota(e^\xi) := e^{\iota(\xi)}$. From Theorem 4.3.3 ([KT00]) we obtain for any $x \in W(\lambda)$

$$e^{-\lambda} \prod_{\beta\in\Delta_+\setminus\Delta(\lambda)} (1 - e^{-\beta})\, \mathrm{ch}_{L(x.\lambda)} = \iota\big(e^{-\nu}\, \mathrm{ch}_{L'(x.\nu)}\big).$$

Recall that if $L(\lambda')$ is a subquotient of $M(\lambda)$ then $\lambda'\in W(\lambda).\lambda$. Let $a_w,\ b_w,\ c_w$ be the multiplicities of $L(w.\lambda)$ in $V(\lambda)$, in $M(s_\alpha.\lambda)$ and in $M(\lambda)$ respectively. Clearly, $a_w\leq c_w-b_w$. One has $\operatorname{ch}_{V(\lambda)}=\sum_{w\in W(\lambda)}a_w\operatorname{ch}_{L(w.\lambda)}$ so

$$e^{-\lambda}\prod_{\beta\in\Delta_+\setminus\Delta(\lambda)}(1-e^{-\beta})\operatorname{ch}_{V(\lambda)}=\sum_{w\in W(\lambda)}a_w\iota\big(e^{-\nu}\operatorname{ch}_{L'(w.\nu)}\big). \tag{14}$$

For $h\in Y$ set

$$f(h):=e^{-\lambda(h)}\prod_{\beta\in\Delta_+\setminus\Delta(\lambda)}(1-e^{-\beta(h)})\operatorname{ch}_{V(\lambda)}(h).$$

Recall that $f(h)$ is a meromorphic function on Y.

Denote by $(\mathbb{C}\Pi_\lambda)^\perp$ the orthogonal to $\mathbb{C}\Pi_\lambda$, i.e.

$$(\mathbb{C}\Pi_\lambda)^\perp=\{h\in\mathfrak{h}|\ \beta(h)=0\ \text{ for any } \beta\in\Pi(\lambda)\},$$

and by $(\mathbb{C}\Pi')^\perp$ the orthogonal to $\mathbb{C}\Pi'$, i.e.

$$(\mathbb{C}\Pi')^\perp=\{h\in\mathfrak{h}_\lambda|\ \beta(h)=0\ \text{ for any } \beta\in\Pi'\}.$$

Any $\beta\in\Pi(\lambda)$ defines a functional on $\mathfrak{h}/(\mathbb{C}\Pi_\lambda)^\perp$ and any $\beta\in\Pi'$ defines a functional on $\mathfrak{h}_\lambda/(\mathbb{C}\Pi')^\perp$.

Since $\operatorname{Im}\iota=\mathbb{C}\Pi_\lambda$ from (14) one sees that $f(h)=f(h+h')$ if $h,h+h'\in Y$ and $h'\in(\mathbb{C}\Pi_\lambda)^\perp$. Denote by F the corresponding function on $Y/(\mathbb{C}\Pi')^\perp$:

$$F(h+(\mathbb{C}\Pi')^\perp):=f(h)=e^{-\lambda(h)}\prod_{\beta\in\Delta_+\setminus\Delta(\lambda)}(1-e^{-\beta(h)})\operatorname{ch}_{V(\lambda)}(h).$$

In order to show that $f(h)$ does not have a pole at the hyperplane $\alpha=0$, it is enough to show that $F(h)$ does not have a pole at the hyperplane $\alpha=0$.

Recall that $\operatorname{ch}_{L'(w.\nu)}(h)$ is a meromorphic function on

$$Y':=\{h\in\mathfrak{h}_\lambda|\operatorname{Re}\delta_i(h)>0\ \text{ for } i=1,\dots,k\},$$

where $\delta_1,\dots,\delta_k$ are the minimal imaginary roots of the affine components of $\mathfrak{g}_\lambda$. Observe that $e^{-\nu}\operatorname{ch}_{L'(w.\nu)}$ is a linear combination of e^μ, where $\mu\in\mathbb{Z}\Pi'$. Therefore $e^{-\nu}\operatorname{ch}_{L'(w.\nu)}(h)=e^{-\nu}\operatorname{ch}_{L'(w.\nu)}(h+h')$ if $h,h+h'\in Y'$ and $h'\in(\mathbb{C}\Pi')^\perp$. Denote by F_w the function on $Y'/(\mathbb{C}\Pi')^\perp$ defined by

$$F_w(h+(\mathbb{C}\Pi')^\perp):=e^{-\nu}\operatorname{ch}_{L'(w.\nu)}(h).$$

Identify $\mathfrak{h}/(\mathbb{C}\Pi_\lambda)^\perp$ with $(\mathbb{C}\Pi_\lambda)^*$ and $\mathfrak{h}_\lambda/(\mathbb{C}\Pi')^\perp$ with $(\mathbb{C}\Pi')^*$. Denote by ι^* the map $\mathfrak{h}/(\mathbb{C}\Pi_\lambda)^\perp\to\mathfrak{h}_\lambda/(\mathbb{C}\Pi')^\perp$ which is conjugate to $\iota:\mathbb{C}\Pi'\to\mathbb{C}\Pi_\lambda$. By (14) one has

$$F(h)=\sum_{w\in W(\lambda)}a_wF_w(\iota^*(h)).$$

In order to show that $F(h)$ does not have a pole at $\alpha=0$, we will show that the series in the right-hand side absolutely converges in the domain

$$Y(\lambda)_{>;\alpha}:=\{h\in\mathfrak{h}/(\mathbb{C}\Pi_\lambda)^\perp|\ \operatorname{Re}\beta(h)>0\ \text{ for any } \beta\in\Delta(\lambda)_+\setminus\{\alpha\}\}.$$

Indeed, by Theorem 4.3.3, b_w is equal to the multiplicity of $L'(w*\nu)$ in $M'(s_\alpha*\nu)$ and c_w is equal to the multiplicity of $L'(w*\nu)$ in $M'(\nu)$. Therefore the series $\sum_{w\in W(\lambda)} a_w \operatorname{ch}_{L'(w.\nu)}(h)$ is majorized by the series $\operatorname{ch}_{M'(\nu)/M'(s_\alpha*\nu)}(h)$. Recall that $\alpha\in\Pi(\lambda)$ so, by above, the series $\operatorname{ch}_{M'(\nu)/M'(s_\alpha*\nu)}(h)$ absolutely converges in the domain

$$Y'_{>;\alpha} := \{h\in\mathfrak{h}_\lambda|\ \operatorname{Re}\beta(h)>0\ \text{ for any } \beta\in\Delta(\lambda)_+\setminus\{\alpha\}\}.$$

Clearly, $\iota^*(Y(\lambda)_{>;\alpha}) = Y'_{>;\alpha}/(\mathbb{C}\Pi')^\perp$. The claim follows. Hence $\operatorname{ch}_{V(\lambda)}(h)$ does not have a pole at $\alpha=0$. □

6.1.1. Retain notation of 4.2.

Corollary. *(i) Let $\lambda\in\mathcal{C}^+$ and let $x\in W(\lambda)$ be the longest element in a coset $xW_0(\lambda)$. If $\alpha\in\Pi(\lambda)$ is such $s_\alpha x >_\lambda x$, then $\alpha\notin S_{L(x.\lambda)}$.*

(ii) Let $\lambda\in\mathcal{C}^-$ and let $x\in W(\lambda)$ be the shortest element in a coset $xW_0(\lambda)$. If $\alpha\in\Pi(\lambda)$ is such $s_\alpha x <_\lambda x$, then $\alpha\notin S_{L(x.\lambda)}$.

(iii) If Λ is non-critical and $S_{L(\Lambda)}=\Delta_+$, then $L(\Lambda)$ is a Verma module.

Proof. The inclusion $M(s_\alpha x.\lambda)\subset M(x.\lambda)$ is equivalent to the inequality $s_\alpha x\geq_\lambda x$ (resp., $s_\alpha x\leq_\lambda x$) for $\lambda\in\mathcal{C}^+$ (resp., for $\lambda\in\mathcal{C}^-$). The inclusion is proper due to the assumption that $x\in W(\lambda)$ is the longest (resp., the shortest) element in a coset $xW_0(\lambda)$.

For (iii) write $\Lambda=x.\lambda$ for $x\in W(\Lambda)$, $\lambda\in\mathcal{C}^+\cup\mathcal{C}^-$, where x is the longest (resp., the shortest) element in a coset $xW_0(\lambda)$ if $\lambda\in\mathcal{C}^+$ (resp., $\lambda\in\mathcal{C}^-$). Combining the assumption $S_{L(\Lambda)}=\Delta_+$ and (i), (ii) we conclude that $\lambda\in\mathcal{C}^-$ and $x=e$. Then $M(\Lambda)=M(\lambda)$ is simple, so $L(\Lambda)=M(\Lambda)$ as required. □

6.1.2. *Remark.* It is not true that $S_{V(\Lambda)}=\Delta_+$ implies that $V(\lambda)$ is a Verma module. For example, consider $\mathfrak{g}=\hat{\mathfrak{sl}}(2)$ and a highest weight module $V(0):=M(0)/M(s_0s_1.0)=M(0)/M(-3\alpha_0-\alpha_1)$. One has

$$\operatorname{ch}_{V(0)} = \frac{1-e^{-3\alpha_0-\alpha_1}}{(1-e^{-\alpha_1})\prod_{k=0}^\infty(1-e^{-k\alpha_0-(k-1)\alpha_1})(1-e^{-k\alpha_0-k\alpha_1})(1-e^{-k\alpha_0-(k+1)\alpha_1})},$$

hence $S_{V(0)}=\Delta_+$.

6.2. **Proposition.** *Let $V(\Lambda)$ be a highest weight module with non-critical highest weight Λ over an affine Lie algebra. Then the character $\operatorname{ch}_{V(\Lambda)}(h)$ is holomorphic in Y if and only if the module $V(\Lambda)$ is integrable (equivalently, if and only if $V(\Lambda)=L(\Lambda)$ and $\Lambda\in P^+$).*

Proof. Recall that a highest weight module $V(\Lambda)$ is integrable if and only if it is simple and $\Lambda\in P^+$, where $P^+=\{\lambda\in\mathfrak{h}^*|\ \langle\lambda+\rho,\alpha\rangle\in\mathbb{Z}_{>0}$ for any $\alpha\in\Pi\}$ (see [K], Chapter 10). Also, by [K], 11.10, $\operatorname{ch}_{L(\Lambda)}(h)$ is holomorphic in Y if $L(\Lambda)$ is integrable.

Recall that the series $\mathrm{ch}_{V(\lambda)}(h)$ absolutely converges in the domain $Y_>$ to a holomorphic function and $\mathrm{ch}_{V(\lambda)}(h)$ is the analytic continuation of this function to Y. Assume that this function is holomorphic. We claim that the series $\mathrm{ch}_{V(\lambda)}(h)$ absolutely converges in the domain Y. Indeed, write

$$e^{-\lambda(h)}\,\mathrm{ch}_{V(\lambda)}(h)=\sum_{\nu\in Q^+} m_\nu e^{-\nu(h)},\quad \text{where}\quad m_\nu=\dim V(\lambda)_{\lambda-\nu}\in\mathbb{Z}_{\geq 0}.$$

Write $\Pi=\{\alpha_i\}_{i=0}^n$, where the simple roots are arbitrarily numerated, and set $z_i:=e^{-\alpha_i(h)}$ for $i=1,\dots,n$, $z_0:=e^{-\delta(h)}$. For $\nu\in Q^+$ write ν in the form $\nu=k_0\delta+\sum_{i=1}^n k_i\alpha_i$ and observe that $k_0\in\mathbb{Z}_{\geq 0}$, $k_i\in\mathbb{Z}$ for $i=1,\dots,n$; set

$$z^\nu:=\prod_{i=0}^n z_i^{k_i}.$$

Rewrite $e^{-\lambda(h)}\,\mathrm{ch}_{V(\lambda)}(h)$ in new variables and denote the resulting series by $F(z)$:

$$e^{-\lambda(h)}\,\mathrm{ch}_{V(\lambda)}(h)=\sum_{\nu\in Q^+} m_\nu z^\nu=:F(z)$$

By Remark 5.3, the series $F(z)$ converges in the domain

$$Y_>=\{0<|z_i|<1 \text{ for } i=1,\dots,n\ \&\ 0<|z_0|^{n_0}\prod_{i=1}^n |z_i|^{n_i}<1\},$$

where $\delta=\sum_{i=0}^n n_i\alpha_i$ so n_i are non-negative integers. For $i=1,\dots,n$ fix z_i' such that $|z_i'|<1$. Then the series $f(z_0):=F(z_0;z_1',\dots,z_n')$ converges in the domain $\{0<|z_0|<C\}$ for some $C<1$. By above, $f(z_0)$ is a power series so it converges in the open disc $\{|z_0|<C\}$. By the assumption, the sum of $F(z)$ can be analytically extended to a function which is holomorphic in $Y=\{0<|z_0|<1, z_i\neq 0, \text{ for } i=1,\dots,n\}$. Therefore the sum of $f(z_0)$ can be analytically extended to a function $g(z_0)$ which is holomorphic in the ring $\{0<|z_0|<1\}$. By above, the sum of $f(z_0)$ is holomorphic in $\{|z_0|<C\}$. Thus $g(z_0)$ is holomorphic in the open disc $\{|z_0|<1\}$. By Cauchy's theorem, this means that the power series $f(z_0)$ converges to $g(z_0)$ in $\{|z_0|<1\}$; in particular, $f(z_0)$ absolutely converges in $\{|z_0|<1\}$. Hence the series $F(z)$ absolutely converges in the domain Y.

Now for $i=0,2,\dots,n$ fix $z_i'\in\mathbb{R}$ such that $0<z_i'<1$. Set $f(z_1):=F(z_0';z_1;z_2',\dots,z_n')$. By above, $f(z_1)$ converges for any $z_1\neq 0$. Since m_ν are non-negative integers, the series $f(z_1)$ majorizes the series $\sum_{k=0}^\infty m_{k\alpha_1}z_1^k$. Therefore $\sum_{k=0}^\infty m_{k\alpha_1}x^k$ converges for any x. Since the coefficients $m_{k\alpha_1}$ are non-negative integers, we conclude that $m_{k\alpha_1}=0$ for some $k>0$. This means that $\dim V(\lambda)_{\lambda-k\alpha_1}=0$. Since α_1 is an arbitrary simple root, we conclude that for any $\alpha\in\Pi$ one has $\dim V(\lambda)_{\lambda-k\alpha}=0$ for some $k>0$. By [K], Lemma 3.4, this implies that $V(\lambda)$ is integrable. ☐

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