THE KAC CONSTRUCTION OF THE CENTRE OF $U(\mathfrak{g})$ FOR LIE SUPERALGEBRAS

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Abstract. In 1984, Victor Kac [K4] suggested an approach to a description of central elements of a completion of $U(\mathfrak{g})$ for any Kac-Moody Lie algebra $\mathfrak{g}$. The method is based on a recursive procedure. Each step is reduced to a system of linear equations over a certain subalgebra of meromorphic functions on the Cartan subalgebra. The determinant of the system coincides with the Shapovalov determinant for $\mathfrak{g}$.

We prove that the Kac approach can also be applied to finite dimensional Lie superalgebras $\mathfrak{g}(A)$ with Cartan matrix A (as claimed in [K4]) and reproduce for them Sergeev's description of the centers of $U(\mathfrak{g})$ [S2]. In order to prove this, one needs to show that the recursive procedure stops after a finite number of steps. The original paper [K4] does not indicate how to check this fact.

Here we give a detailed presentation of the Kac approach and apply it to finite dimensional Lie superalgebras $\mathfrak{g}(A)$. In particular, we deduce the Kac formulas for the Shapovalov determinants and verify the finiteness of the recursive procedure.

1. Preliminaries

1.1. We start from a complex finite-dimensional reductive Lie algebra or contragredient Lie superalgebra $\mathfrak{g}$ (see [K1], [vdL]; these superalgebras are either simple or close to simple). Recall that such an algebra $\mathfrak{g}$ is generated by a commutative even subalgebra (Cartan subalgebra) $\mathfrak{h}$ and symbols e_i and f_i ($i=1,\dots,n=\dim\mathfrak{h}$) of the same parity for each i. The algebra $\mathfrak{h}$ contains linearly independent elements $\alpha_i^\vee$ and the dual space $\mathfrak{h}^*$ contains linearly independent elements α_i satisfying the relations

$$[e_i,f_j]=\delta_{ij}\alpha_i^\vee,\ \ [h,e_i]=\alpha_i(h)e_i,\ \ [h,f_i]=-\alpha_i(h)f_i.$$

On $\mathfrak{h}^*$, there is the standard bilinear form $(\cdot,\cdot)$. An element $\xi\in\mathfrak{h}^*$ is called *isotropic* if $(\xi,\xi)=0$. There is a decomposition

$$\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+,$$

where $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) is the subalgebra of $\mathfrak{g}$ generated by the e_i (resp., f_i). Denote by Δ^+ the set of positive roots and by Δ_0 (resp., Δ_1) the set of even (resp., odd) roots. Set

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$\Delta_i^+ = \Delta^+ \cap \Delta_i$ $(i = 0, 1)$ and set

$$Q := \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \quad Q^+ := \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\alpha_i,$$

where $\mathbb{Z}_{\geq 0}$ stands for the set of non-negative integers. Denote by $\mathbb{Z}_{>0}$ the set of positive integers.

For a $\mathbb{Z}_2$-homogeneous element $x \in \mathfrak{g}$, let $p(x)$ be its parity; in all formulae where this notation is used, x is assumed to be $\mathbb{Z}_2$-homogeneous.

For a given $\mathfrak{g}$-module N, we denote by N_ν $(\nu \in \mathfrak{h}^*)$ the corresponding weight space:

$$N_\nu := \{v \in N \mid hv = \nu(h)v \text{ for any } h \in \mathfrak{h}\}.$$

1.1.1. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an even linear bijection which preserves the elements of $\mathfrak{h}$ and is either an antiautomorphism that is satisfies the sign rule

$$\sigma([x, y]) = (-1)^{p(x)p(y)}[\sigma(y), \sigma(x)], \tag{1}$$

or is a "naive antiautomorphism" that is satisfies the rule

$$\sigma([x, y]) = [\sigma(y), \sigma(x)]. \tag{2}$$

Since α_i $(i = 1, \dots, n)$ are linearly independent, one has $\sigma(e_i) = c_i f_i, \sigma(f_i) = c_i' e_i$ where $c_i, c_i' \in \mathbb{C}^*$ and $c_i c_i' = (-1)^{p(e_i)}$ for σ satisfying (1), $c_i c_i' = 1$ for σ satisfying (2). Observe that $\mathfrak{g}$ admits automorphisms of the following form:

$$\phi(h) = h \text{ for all } h \in \mathfrak{h}, \quad \phi(e_i) = c_i e_i, \quad \phi(f_i) = c_i^{-1} f_i, \text{ where } c_i \in \mathbb{C}^*.$$

Hence, up to an automorphism of $\mathfrak{g}$, σ takes form

$$\sigma|_{\mathfrak{h}} = \mathrm{id}, \quad \sigma(e_i) = f_i, \quad \sigma(f_i) = \begin{cases} (-1)^{p(e_i)} e_i & \text{if } \sigma \text{ satisfies (1)}, \\ e_i & \text{if } \sigma \text{ satisfies (2)}. \end{cases}$$

Notice that σ interchanges $\mathfrak{n}^+$ and $\mathfrak{n}^-$. Moreover,

$$\sigma^2 = \begin{cases} (-1)^{p(x)} x \text{ (and thus } \sigma^4 = \mathrm{id}) & \text{if } \sigma \text{ satisfies (1)}, \\ \mathrm{id} & \text{if } \sigma \text{ satisfies (2)}. \end{cases}$$

1.1.2. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The linear map σ can be uniquely extended to a linear bijection $U(\mathfrak{g}) \to U(\mathfrak{g})$ satisfying the sign rule $\sigma(xy) = (-1)^{p(x)p(y)}\sigma(y)\sigma(x)$ or the rule $\sigma(xy) = \sigma(y)\sigma(x)$ respectively.

We identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$ thanks to the PBW theorem. The projection onto the first summand in the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g}))$ is called the *Harish-Chandra projection*: $\mathrm{HC} : U(\mathfrak{g}) \to S(\mathfrak{h})$. Notice that $\mathrm{HC} \circ \sigma = \mathrm{HC}$.

1.2. Shapovalov forms. The bilinear map $S : U(\mathfrak{g}) \times U(\mathfrak{g}) \to S(\mathfrak{h})$ defined by the formula

$$(a, b) \mapsto \mathrm{HC}(\sigma(a)b)$$

is called the *Shapovalov map* (see [Sh],[KK]).

1.2.1. The Shapovalov map is even and symmetric:

$$S(a, b) = 0 \text{ if } p(a) \neq p(b) \text{ and } S(a, b) = S(b, a).$$

It has the following property known as *contravariance*:

$$S(ca, b) = \begin{cases} (-1)^{p(a)p(c)} S(a, \sigma(c)b) & \text{if } \sigma \text{ satisfies (1)}, \\ S(a, \sigma(c)b) & \text{if } \sigma \text{ satisfies (2)}. \end{cases}$$

For any $\lambda \in \mathfrak{h}^*$, denote by $S(\lambda)$ the *evaluated* Shapovalov map $S(\lambda) : U(\mathfrak{g}) \times U(\mathfrak{g}) \to \mathbb{C}$ given by

$$(a, b) \mapsto \mathrm{HC}(\sigma(a)b)(\lambda).$$

1.2.2. Let $\mathbb{C}(\lambda)$ be a one-dimensional even $\mathfrak{h}$-module corresponding to $\lambda \in \mathfrak{h}^*$. Let $M(\lambda)$ be the Verma module with highest weight λ that is

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda)$$

where $\mathbb{C}(\lambda)$ is viewed as a $\mathfrak{b}$-module via the trivial $\mathfrak{n}^+$-action. In $M(\lambda)$, there is a unique maximal proper submodule which we denote by $\overline{M}(\lambda)$. As $U(\mathfrak{n}^-)$-modules, $M(\lambda) \cong U(\mathfrak{n}^-)$ (the isomorphism is defined up to a scalar factor). The restriction of $S(\lambda)$ onto $U(\mathfrak{n})^- \times U(\mathfrak{n})^-$ induces a bilinear form $M(\lambda) \otimes M(\lambda) \to \mathbb{C}$. The contravariance easily implies that the kernel of $S(\lambda)$ coincides with the maximal proper submodule $\overline{M}(\lambda)$ of $M(\lambda)$.

1.2.3. *An interpretation of the Shapovalov maps.* For every left $\mathfrak{g}$-module $M(\lambda)$, the dual module $M(\lambda)^* := \mathrm{Hom}_{\mathbb{C}}(M(\lambda), \mathbb{C})$ has a natural structure of a *right* $\mathfrak{g}$-module. To convert $M(\lambda)^*$ into a left $\mathfrak{g}$-module, for any $g \in \mathfrak{g}$, $f \in M(\lambda)^*$ and $v \in M(\lambda)$, we set

$$(gf)(v) = \begin{cases} (-1)^{p(g)p(f)} f(\sigma(g)v) & \text{if } \sigma \text{ satisfies (1)}, \\ f(\sigma(g)v) & \text{if } \sigma \text{ satisfies (2)}. \end{cases}$$

The module $M(\lambda)^*$ has a submodule

$$M(\lambda)^{\#} := \oplus_{\nu \in Q^+} \mathrm{Hom}_{\mathbb{C}}(M(\lambda)_{\lambda-\nu}, \mathbb{C})$$

which, as $M(\lambda)$, is of highest weight λ thanks to the condition $\sigma|_{\mathfrak{h}} = \mathrm{id}$.

The Shapovalov map $S(\lambda)$ determines a $\mathfrak{g}$-homomorphism $M(\lambda) \to M(\lambda)^{\#}$ which is identical on the highest weight spaces identified with $\mathbb{C}(\lambda)$.

1.3. The choice of a PBW basis in $U(\mathfrak{n}^-)$.

1.3.1. Define the *Kostant partition function* $\tau:\ Q \to \mathbb{Z}_{\geq 0}$ by setting $\tau(Q \setminus Q^+) = 0$ and

$$\operatorname{ch} U(\mathfrak{n}^-) = \frac{\prod\limits_{\alpha\in\Delta_1^+}(1+e^{-\alpha})}{\prod\limits_{\alpha\in\Delta_0^+}(1-e^{-\alpha})} =: \sum_{\eta\in Q^+} \tau(\eta) e^{-\eta}.$$

For any $\nu \in Q^+$, the vector $\mathbf{k} = \{k_\alpha\}_{\alpha\in\Delta^+}$ is called *a partition* of ν if $\nu = \sum\limits_{\alpha\in\Delta^+} k_\alpha \alpha$, where $k_\alpha \in \mathbb{Z}_{\geq 0}$ for $\alpha \in \Delta_0^+$, and $k_\alpha \in \{0,1\}$ for $\alpha \in \Delta_1^+$. Set $|\mathbf{k}| = \sum\limits_{\alpha\in\Delta^+} k_\alpha$. Denote by $\mathcal{P}(\nu)$ the set of all partitions of ν. According to PBW-theorem, $\mathcal{P}(\nu)$ has $\tau(\nu)$ elements.

1.3.2. The standard partial ordering on $\mathfrak{h}^*$ is given by the formula

$$\xi \geq \xi' \iff \xi - \xi' \in Q^+.$$

Fix a total ordering on Δ^+ compatible with the above partial ordering. For each $\alpha \in \Delta^+$, let f_α be a non-zero element of $\mathfrak{g}_{-\alpha}$. Then $\{f_\alpha \mid \alpha \in \Delta^+\}$ forms a basis of $\mathfrak{n}^-$. For every $\mathbf{k} \in \mathcal{P}(\nu)$, define the monomial

$$\mathbf{f}^{\mathbf{k}} := \prod f_\alpha^{k_\alpha},$$

where the order of factors is given by the total ordering fixed above: the factors corresponding to smaller roots come first. The set $\{\mathbf{f}^{\mathbf{k}} \mid \mathbf{k} \in \mathcal{P}(\nu)\}$ forms a PBW basis of $U(\mathfrak{n}^-)_{-\nu}$.

1.4. Obviously $S(a,b) = 0$ if $a, b \in U(\mathfrak{g})$ are weight elements of distinct weights. For each $\nu \in Q^+$, denote by S_ν the restriction of S to $U(\mathfrak{n})^-_{-\nu} \times U(\mathfrak{n})^-_{-\nu} \to S(\mathfrak{h})$. Written with respect to the above PBW basis, the matrix of S_ν takes form

$$S_\nu := \big(\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{k}})\mathbf{f}^{\mathbf{s}})\big)_{\mathbf{k},\mathbf{s}\in\mathcal{P}(\nu)}.$$

This symmetric square matrix with entries in $S(\mathfrak{h})$ is called the *Shapovalov matrix*. For each $\lambda \in \mathfrak{h}^*$, the evaluated matrix $S_\nu(\lambda)$ is a complex matrix and

$$\operatorname{corank} S_\nu(\lambda) = \dim \overline{M}(\lambda)_{\lambda-\nu}.$$

In particular,

$$M(\lambda) \text{ is simple } \iff \det S_\nu(\lambda) \neq 0 \text{ for any } \nu \in Q^+.$$

1.4.1. Let σ, σ' be two linear maps described in 1.1.1. Recall that $\sigma'(f_i) = c_i \sigma(f_i)$ ($c_i \in \mathbb{C}^*$) and thus $\sigma'(\mathbf{f}^{\mathbf{k}}) = c_{\mathbf{k}} \sigma(\mathbf{f}^{\mathbf{k}})$ for some $c_{\mathbf{k}} \in \mathbb{C}^*$. Thus the Shapovalov matrix constructed via σ' can be obtained from the Shapovalov matrix constructed via σ by multiplying the rows by invertible scalars. In particular, the determinants differ by the multiplication by an invertible scalar.

We choose σ to be the map satisfying (2) and given by

$$\sigma(e_i) = f_i, \quad \sigma(f_i) = e_i, \quad \sigma|_{\mathfrak{h}} = \mathrm{id}\,.$$

1.5. Denote by $\deg u$ the degree of an element of $u \in S(\mathfrak{g})$, considered as a polynomial with the usual grading (the degree of each indeterminate is equal to 1). In particular, deg is defined on elements of $S(\mathfrak{h}) = U(\mathfrak{h})$. It is easy to see that

$$\deg \mathrm{HC}(x_1 \dots x_r y_1 \dots y_s) \leq \min(r, s) \text{ for any } x_1, \dots, x_r \in \mathfrak{n}^+ \text{ and } y_1, \dots, y_s \in \mathfrak{n}^-.$$

As a result, the $(\mathbf{k}, \mathbf{k}')$-entry of S_ν is of degree $\leq \min(|\mathbf{k}|, |\mathbf{k}'|)$ for any $\mathbf{k}, \mathbf{k}' \in \mathcal{P}(\nu)$. For $|\mathbf{k}| = |\mathbf{k}'|$, the $(\mathbf{k}, \mathbf{k}')$-entry has degree $|\mathbf{k}|$ if and only if $\mathbf{k} = \mathbf{k}'$, see 3.3. As a consequence, the polynomial $\det S_\nu$ is not identically zero and

$$\deg \det S_\nu = \sum_{\mathbf{k} \in \mathcal{P}(\nu)} |\mathbf{k}|.$$

1.5.1. The inequalities $\det S_\nu \neq 0$ for all $\nu \in Q^+$ imply (see, for instance, [J] 7.1.9)

$$\bigcap_{\lambda \in \mathfrak{h}^*} \operatorname{Ann} M(\lambda) = 0. \tag{3}$$

For another proof of this fact, see [LM] (Cor. D).

2. The Kac theorem

2.1. Assumptions. Suppose that

(i) For any $\nu \in Q^+$, the zeroes of the Shapovalov determinant $\det S_\nu$ belong to the union of a countably many hyperplanes $\gamma_1, \gamma_2, \dots$ in $\mathfrak{h}^*$.

(ii) For each m, there is an open in γ_m set $\breve{\gamma}_m$ such that $\breve{\gamma}_m \cap \gamma_i = 0$ for all $i \neq m$ and for all $\lambda \in \breve{\gamma}_m$ we have
   (a) $\operatorname{corank} S_\nu(\lambda)$ is equal to the order of zero of the polynomial $\det S_\nu$ at λ;
   (b) $\overline{M}(\lambda)$ is a quotient of a Verma module $M(r(\lambda))$ for some weight $r(\lambda)$.

2.1.1. *Remark.* The order m of zero of a polynomial q at λ is defined to be $m = 0$ if $q(\lambda) \neq 0$; it is $m = 1$ if $q(\lambda) = 0$ and there exists a non-zero partial derivative $\frac{dq}{dx}(\lambda) \neq 0$, and so on.

It is clear that

$$\operatorname{corank} S_\nu(\lambda) \leq \text{ the order of zero of } \det S_\nu \text{ at } \lambda.$$

2.1.2. The above assumptions are valid for superalgebras $\mathfrak{g}(A)$; the first assumption easily follows from the existence of quadratic Casimir operator and the second assumption corresponds to the fact that the Jantzen filtration of $M(\lambda)$ has length two for $\lambda \in \breve{\gamma}_m$. For finite-dimensional superalgebras $\mathfrak{g}(A)$ we give details in sect. 3. The similar assumptions are valid in the quantum case (see [J], 4.1).

2.1.3. The property (i) is equivalent to the fact that each Shapovalov determinant $\det S_\nu$ admits a linear factorization: up to a constant factor, we have

$$\det S_\nu = \prod_m (h_m - c_m)^{d_m(\nu)},$$

where $d_m(\nu) \in \mathbb{Z}_{\geq 0}$ and $h_m \in \mathfrak{h}$, $c_m \in \mathbb{C}$. The pairs (h_m, c_m) define the hyperplanes γ_m from (i):

$$\gamma_m = \{\mu \in \mathfrak{h}^* \mid \mu(h_m) - c_m = 0\}.$$

The order of zero of $\det S_\nu$ at λ is equal to the sum

$$\sum_{m \mid \lambda \in \gamma_m} d_m(\nu).$$

Thus (ii)(a) states that $d_m(\nu) = \operatorname{corank} S_\nu(\lambda)$ for all $\lambda \in \breve{\gamma}_m$.

2.2. Theorem (Kac, '84). *Set*

$$X := \bigcup_m \breve{\gamma}_m.$$

The restriction of HC *to* $\mathcal{Z}(\mathfrak{g})$ *is an algebra isomorphism*

$$\mathrm{HC} : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} \{\varphi \in S(\mathfrak{h}) \mid \varphi(\lambda) = \varphi(r(\lambda)) \textit{ for any } \lambda \in X\} \qquad (*)$$

where $r(\lambda)$ *is introduced in the assumption (ii) (b).*

2.3. Observe that a central element $z \in \mathcal{Z}(\mathfrak{g})$ acts on $M(\lambda)$ by $\mathrm{HC}(z)(\lambda)\,\mathrm{id}$. From (3) we conclude that the restriction of HC to $\mathcal{Z}(\mathfrak{g})$ is injective. Obviously, $\varphi \in \mathrm{HC}(\mathcal{Z}(\mathfrak{g}))$ implies that

$$\varphi(\lambda) = \varphi(r(\lambda)) \text{ for any } \lambda \in X.$$

Thus the image $A := \mathrm{HC}(\mathcal{Z}(\mathfrak{g}))$ lies in the right-hand side of $(*)$. Fix an element $\varphi \in A$. We prove the existence of $z \in \mathcal{Z}(\mathfrak{g})$ satisfying $\mathrm{HC}(z) = \varphi$ in two steps.

2.3.1. The first step is done in 2.4, 2.5. Fix any $\nu \in Q^+$; to each pair $\mathbf{k}, \mathbf{m} \in \mathcal{P}(\nu)$ we assign a polynomial $\varphi_{\mathbf{k},\mathbf{m}} \in S(\mathfrak{h})$ so that $\varphi_{0,0} = \varphi$ and the sums

$$z_{\leq \nu} := \sum_{0 \leq \mu \leq \nu} z_\mu,$$
$$z_\mu := \sum_{\mathbf{k},\mathbf{m} \in \mathcal{P}(\mu)} \mathbf{f}^{\mathbf{k}} \varphi_{\mathbf{k},\mathbf{m}} \sigma(\mathbf{f}^{\mathbf{m}})$$

satisfy the condition

$$z_{\leq \nu} v = \varphi(\lambda) v \quad \text{for any } \lambda \in \mathfrak{h}^* \text{ and } \mu \in Q^+ \text{ such that } \mu \leq \nu, \text{ and any } v \in M(\lambda)_{\lambda - \mu}. \tag{4}$$

For a given weight ν we realize the elements $\Phi := (\varphi_{\mathbf{k},\mathbf{m}})$ as a solution of the matrix equation $\Phi S_\nu = C$ where S_ν is the corresponding Shapovalov matrix and the entries of C are elements of $S(\mathfrak{h})$ which are expressed in terms of z_μ for $\mu < \nu$. Thanks to a non-degeneracy of S_ν, the equation has a unique solution over the fraction field $\operatorname{Fract} S(\mathfrak{h})$.

Then we prove that the entries of Φ lie in $S(\mathfrak{h})$. Thus we obtain $z_{\leq\nu}$ by a recursive procedure.

2.3.2. In the second step (performed in 2.6) we verify the finiteness of the recursive procedure i.e., that $z_\nu = 0$ for almost all values of ν. This means that the sum

$$z := \sum_{\nu\in Q^+} z_\nu$$

is finite.

For any $\lambda \in \mathfrak{h}^*$, $\nu \in Q^+$ and $v \in M(\lambda)_{\lambda-\nu}$, we have

$$zv = z_\nu v.$$

Therefore $z - \varphi(\lambda)$ annihilates $M(\lambda)$ for any $\lambda \in \mathfrak{h}^*$. Then, for all $u \in U(\mathfrak{g})$, we have

$$zu - uz \in \bigcap_{\lambda\in\mathfrak{h}^*} M(\lambda)$$

and thus $z \in \mathcal{Z}(\mathfrak{g})$. Clearly, $\mathrm{HC}(z) = \varphi$, as required.

To verify the finiteness we show that

$$|\mathbf{k}| + \deg \varphi_{\mathbf{k},\mathbf{m}} \leq \deg \varphi.$$

This inequality implies $\varphi_{\mathbf{k},\mathbf{m}} = 0$ for $|\mathbf{k}| > \deg \varphi$ that is $z_\nu = 0$ for almost all values of ν.

2.4. We proceed by induction on $\nu \in Q^+(\pi)$. Recall that $\varphi_{0,0} = \varphi$. Assume that for all $0 \leq \mu < \nu$ we have constructed $\varphi_{\mathbf{k},\mathbf{m}}$ for any partitions $\mathbf{k}, \mathbf{m} \in \mathcal{P}(\mu)$. Therefore the sum

$$z_{<\nu} := \sum_{\mu<\nu} z_\mu$$

is known. Observe that, for $\mu < \nu$ and $v \in M(\lambda)_{\lambda-\mu}$, we have

$$z_{\leq\nu} v = z_{\leq\mu} v = z_{<\nu} v.$$

Therefore, for such v, the equality $z_{\leq\nu} = \varphi(\lambda)v$ follows from the induction hypothesis. Hence the property (4) is equivalent to

$$z_{\leq\nu} v = \varphi(\lambda)v \quad \text{for any } \lambda \in \mathfrak{h}^* \text{ and } v \in M(\lambda)_{\lambda-\nu}. \tag{5}$$

Denote by v_λ the highest weight vector of $M(\lambda)$. The set $\{\mathbf{f}^{\mathbf{s}} v_\lambda \mid \mathbf{s} \in \mathcal{P}(\nu)\}$ forms a basis of $M(\lambda)_{\lambda-\nu}$. The equation (5) is equivalent to the system whose equations are labeled by partitions of ν: for each partition $\mathbf{s} \in \mathcal{P}(\nu)$ the corresponding equation is

$$z_{\leq\nu} \mathbf{f}^{\mathbf{s}} v_\lambda = \mathbf{f}^{\mathbf{s}} \varphi v_\lambda. \tag{6}$$

For any $u \in U(\mathfrak{g})$, we have

$$u(\mathbf{f}^{\mathbf{s}} v_\lambda) = P_+(u\mathbf{f}^{\mathbf{s}}) v_\lambda,$$

where P_+ is the projection $U(\mathfrak{g}) \to U(\mathfrak{n}^- \oplus \mathfrak{h})$ with the kernel $U(\mathfrak{g})U(\mathfrak{n}^+)$. Formula (6) holds for all $\lambda \in \mathfrak{h}^*$ if and only if

$$P_+(z_{\leq\nu}\mathbf{f}^{\mathbf{s}}) = \mathbf{f}^{\mathbf{s}}\varphi.$$

Using the formula $z_{\leq\nu} = z_{<\nu} + z_\nu$ we obtain for each partition $\mathbf{s} \in \mathcal{P}(\nu)$ the equation

$$P_+\big(\sum_{\mathbf{k},\mathbf{m}\in\mathcal{P}(\nu)} \mathbf{f}^{\mathbf{k}}\varphi_{\mathbf{k},\mathbf{m}}\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}}\big) = \mathbf{f}^{\mathbf{s}}\varphi - P_+(z_{<\nu}\mathbf{f}^{\mathbf{s}})$$

which can be rewritten as

$$\sum_{\mathbf{k},\mathbf{m}\in\mathcal{P}(\nu)} \mathbf{f}^{\mathbf{k}}\varphi_{\mathbf{k},\mathbf{m}}\,\mathrm{HC}\big(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}}\big) = \mathbf{f}^{\mathbf{s}}\varphi - P_+(z_{<\nu}\mathbf{f}^{\mathbf{s}}) \tag{7}$$

since $P_+(u_-u_0) = u_-\,\mathrm{HC}(u)$ for any $u_- \in U(\mathfrak{n}^-)$ and $u \in U(\mathfrak{g})^{\mathfrak{h}}$. Recall that

$$\mathrm{HC}\big(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}}\big) = (S_\nu)_{(\mathbf{m},\mathbf{s})}.$$

Both sides of the equation (7) are elements of $L := U(\mathfrak{n}^-)_{-\nu}S(\mathfrak{h})$. Note that L is a right free $S(\mathfrak{h})$-module and the elements $\mathbf{f}^{\mathbf{k}}$, where $\mathbf{k} \in \mathcal{P}(\nu)$, form a system of free generators. By the induction hypothesis, the right-hand side of (7) is known. Writing

$$\mathbf{f}^{\mathbf{s}}\varphi - P_+(z_{<\nu}\mathbf{f}^{\mathbf{s}}) = \sum_{\mathbf{k}\in\mathcal{P}(\nu)} \mathbf{f}^{\mathbf{k}}c_{\mathbf{k},\mathbf{s}},$$

where $c_{\mathbf{k},\mathbf{s}} \in S(\mathfrak{h})$, we obtain, for each pair $\mathbf{k},\mathbf{s} \in \mathcal{P}(\nu)$, the equation

$$\sum_{\mathbf{m}\in\mathcal{P}(\nu)} \varphi_{\mathbf{k},\mathbf{m}}(S_\nu)_{\mathbf{m},\mathbf{s}} = c_{\mathbf{k},\mathbf{s}}.$$

Hence we get a system of linear equations which in the matrix form is

$$\Phi S_\nu = C, \tag{8}$$

where $\Phi := (\varphi_{\mathbf{k},\mathbf{m}})$ is the matrix of indeterminates and $C := (c_{\mathbf{k},\mathbf{s}})$ is a known matrix. Since $\det S_\nu \neq 0$ the system (8) has a unique solution

$$\Phi = CS_\nu^{-1},$$

where Φ is a matrix with entries in Fract $S(\mathfrak{h})$, the field of fraction of $S(\mathfrak{h})$. Our next goal is to show that the entries of Φ actually lie in $S(\mathfrak{h})$.

2.5. We want to prove that the $\varphi_{\mathbf{k},\mathbf{m}}$ are polynomials. We have $\Phi = CS_\nu^{-1}$ and $\det S_\nu(\xi) \neq 0$ for $\xi \in \mathfrak{h}^* \setminus (\cup\gamma_m)$. Thus the $\varphi_{\mathbf{k},\mathbf{m}}$ are rational functions holomorphic on $\mathfrak{h}^* \setminus (\cup_m\gamma_m)$. Since $\cup\breve{\gamma}_m$ is open in $\cup_m\gamma_m$, it suffices to show that the functions $\varphi_{\mathbf{k},\mathbf{m}}$ are holomorphic in a neighbourhood of any $\lambda \in \cup\breve{\gamma}_m$. Kac proved the following lemma.

2.5.1. Lemma. *Let $B=(b_{ij})$ and $C=(c_{ij})$ be two $N\times N$ matrices, where b_{ij} and c_{ij} are functions in $z_1,\dots,z_n$ holomorphic on a neighbourhood U of the origin. Set $V:=U\cap\{z_1=0\}$. Let B be invertible on $U\setminus V$ and for all $\lambda\in V$ one has (a) the order of zero of $\det B$ at λ is equal to $\dim\operatorname{Ker} B(\lambda)$, (b) $\operatorname{Ker} B(\lambda)\subset\operatorname{Ker} C(\lambda)$. Then CB^{-1} is holomorphic on U.*

Proof. Since B is invertible on $U\setminus V$, it suffices to show that CB^{-1} has no pole on the hyperplane $z_1=0$. Denote by R the local ring obtained from the ring of holomorphic functions in $z_1,\dots,z_n$ by localization at the principal prime ideal (z_1). Consider B and C as matrices over R. There exists an invertible matrix D_1 over R such that the matrix $B':=D_1B$ is upper-triangular and the diagonal entries are of the form z_1^i where $i\geq 0$.

We can choose an open set $U'\subset U$ such that the entries of D_1 and D_1^{-1} are holomorphic in U', and $V':=U'\cap V$ is open in V. Then, for all $\lambda\in V'$, the order of zero of $\det B'$ at λ is equal to $\dim\operatorname{Ker} B'(\lambda)$. As a consequence, any diagonal entry of B' is either 1 or z_1. Moreover, all entries of the column with diagonal entry z_1 lie in the maximal ideal Rz_1. So there exists an invertible matrix D_2 over R such that the matrix $B'':=D_2B'$ is diagonal and any its diagonal entry is either 1 or z_1.

We can choose an open set $U''\subset U'$ such that the entries of D_2 and D_2^{-1} are holomorphic in U'', and $V'':=U''\cap V$ is open in V. For any $\lambda\in V''$, we have $\operatorname{Ker} B''(\lambda)=\operatorname{Ker} B(\lambda)\subset\operatorname{Ker} C(\lambda)$. This means that if $b'_{ii}=z_1$ then $c_{ji}(\lambda)=0$ for all $\lambda\in V''$ and $j=1,\dots,N$, and hence $c_{ji}\in R(z_1)$, and so $C(B'')^{-1}$ is a matrix over R. Thus $CB^{-1}=C(B'')^{-1}D_2D_1$ is a matrix over R and so CB^{-1} has no pole on the hyperplane $z_1=0$. □

2.5.2. The lemma is proven, so it remains to verify the fulfillment of (a) and (b). Let us view $\mathfrak{h}$ as an affine space. Set $N:=|\mathcal{P}(\nu)|$ and $B:=S_\nu$. Take a hyperplane γ_m and fix $\lambda'\in\breve{\gamma}_m$. Let the hyperplane $z_1=0$ coincides with γ_m. Choose U to be a neighbourhood of λ' such that $U\cap\gamma_i=\emptyset$ for $i\neq m$ and $U\cap\gamma_m\subset\breve{\gamma}_m$.

Set $V:=U\cap\gamma_m$. Since $B=S_\nu$, the first assumption of the lemma is exactly the assumption (ii) (a). To verify the second assumption of the lemma, fix $\lambda\in V$. Let us identify $M(\lambda)$ and $U(\mathfrak{n}^-)$, as $U(\mathfrak{n}^-)$-modules. Consider the matrices $B(\lambda)$ and $C(\lambda)$ as endomorphisms of the N-dimensional vector space $M(\lambda)_{\lambda-\nu}$ with the basis $\mathbf{f}^{\mathbf{s}}$. By 1.2.2 we have

$$\operatorname{Ker} B(\lambda)=\operatorname{Ker} S_\nu(\lambda)=\overline{M}(\lambda)_{\lambda-\nu}.$$

The equality

$$\sum_{\mathbf{k}} c_{\mathbf{k},\mathbf{s}}\mathbf{f}^{\mathbf{k}}=\mathbf{f}^{\mathbf{s}}\varphi-P_+(z_{<\nu}\mathbf{f}^{\mathbf{s}})$$

shows that $C(\lambda)(v)=(\varphi(\lambda)-z_{<\nu})v$ for all $v\in M(\lambda)_{\lambda-\nu}$.

Recall that $\lambda\in V\subset\lambda\in\breve{\gamma}_m$. The assumption (ii) (b) of 2.1, ensures that $\overline{M}(\lambda)$ is a quotient of $M(r(\lambda))$, where $r(\lambda)<\lambda$. Thus, for any $v\in\operatorname{Ker} B(\lambda)=\overline{M}(\lambda)_{\lambda-\nu}$, we have

$$z_{<\nu}v=z_{\leq\mu}v,$$

where $\lambda-\nu=r(\lambda)-\mu$. Then $\mu=\nu+r(\lambda)-\lambda<\nu$ and, by the induction hypothesis, $z_\mu v=\varphi(r(\lambda))v=\varphi(\lambda)v$. Hence $(\varphi(\lambda)-z_{<\nu})v=0$, and so $C(\lambda)v=0$. This establishes the second assumption of the lemma. Finally, $\Phi=CS_\nu^{-1}$ is holomorphic on U.

2.5.3. Corollary. *The functions $\varphi_{\mathbf{k},\mathbf{m}}$ are polynomial.*

2.6. It remains to estimate the degree of the polynomials $\varphi_{\mathbf{k},\mathbf{m}}$. Let us show that

$$|\mathbf{k}|+\deg\varphi_{\mathbf{k},\mathbf{m}}\leq\deg\varphi.$$

We again proceed by induction on $\nu\in Q^+$. For $\nu=0$ the statement is trivial.

2.6.1. Recall that $\Phi=(\varphi_{\mathbf{k},\mathbf{m}})$ is a unique solution of the system of linear equations: the equations are labeled by the pairs $(\mathbf{k},\mathbf{s})$ and take form

$$\sum_{\mathbf{m}}\varphi_{\mathbf{k},\mathbf{m}}b_{\mathbf{m},\mathbf{s}}=c_{\mathbf{k},\mathbf{s}}$$

where $B=S_\nu$. Fix a pair $(\mathbf{k},\mathbf{m})$. One has

$$\varphi_{\mathbf{k},\mathbf{m}}=\frac{\det B'}{\det B},$$

where the matrix B' is obtained from B by replacing the $\mathbf{m}$th row by the $\mathbf{k}$th row of C:

$$B'=(b'_{\mathbf{p},\mathbf{s}}):\quad b'_{\mathbf{p},\mathbf{s}}:=\begin{cases}c_{\mathbf{k},\mathbf{s}} & \text{if } \mathbf{p}=\mathbf{m},\\ b_{\mathbf{p},\mathbf{s}} & \text{if } \mathbf{p}\neq\mathbf{m}.\end{cases}$$

2.6.2. Let us estimate the degree of $c_{\mathbf{k},\mathbf{s}}$. Recall that $S(\mathfrak{g})$ is isomorphic to the graded algebra associated with the canonical filtration of $U(\mathfrak{g})$; let $\mathrm{gr}:U(\mathfrak{g})\to S(\mathfrak{g})$ be the corresponding map.

2.6.3. Recall that

$$\sum_{\mathbf{k}\in\mathcal{P}(\nu)}\mathbf{f}^{\mathbf{k}}c_{\mathbf{k},\mathbf{s}}=\mathbf{f}^{\mathbf{s}}\varphi-\sum_{0\leq\mu<\nu}\sum_{\mathbf{k}',\mathbf{m}'\in\mathcal{P}(\mu)}P_+\big(\mathbf{f}^{\mathbf{k}'}\varphi_{\mathbf{k}',\mathbf{m}'}\sigma(\mathbf{f}^{\mathbf{m}'})\mathbf{f}^{\mathbf{s}}\big). \tag{9}$$

Since

$$P_+(u_1u_+u_2)=P_+(u_1(\mathrm{ad}\,u_+)u_2)\ \text{ for any } u_1,u_2\in U(\mathfrak{n}^-+\mathfrak{h}) \text{ and } u_+\in U(\mathfrak{n}^+)\ ,$$

it follows that the degree of $\mathrm{gr}\,P_+(u_1u_+u_2)\in S(\mathfrak{g})$ is not greater than the degree of $\mathrm{gr}\,u_1u_2$. In particular,

$$\deg P_+\big(\mathbf{f}^{\mathbf{k}'}\varphi_{\mathbf{k}',\mathbf{m}'}\sigma(\mathbf{f}^{\mathbf{m}'})\mathbf{f}^{\mathbf{s}}\big)\ \leq\ \deg\varphi+|\mathbf{s}|$$

since $|\mathbf{k}'|+\deg\varphi_{\mathbf{k}',\mathbf{m}'}\leq\deg\varphi$ by the induction hypothesis. Hence the degree of the right-hand side of formula (9) is $\leq\deg\varphi+|\mathbf{s}|$. Taking into account that $\mathrm{gr}\,\mathbf{f}^{\mathbf{k}}=(\mathrm{gr}\,\mathbf{f})^{\mathbf{k}}$ and $\mathrm{gr}\,c_{\mathbf{k},\mathbf{s}}\in S(\mathfrak{h})$, we conclude:

$$\deg\mathrm{gr}\sum_{\mathbf{k}\in\mathcal{P}(\nu)}\mathbf{f}^{\mathbf{k}}c_{\mathbf{k},\mathbf{s}}=\max_{\mathbf{k}\in\mathcal{P}(\nu):c_{\mathbf{k},\mathbf{s}}\neq0}(|\mathbf{k}|+\deg c_{\mathbf{k},\mathbf{s}}).$$

Therefore

$$\deg c_{\mathbf{k},\mathbf{s}} \leq \deg \varphi + |\mathbf{s}| - |\mathbf{k}|.$$

2.6.4. Recall the construction of matrix B', see 2.6.1, and observe that since $B = S_\nu$, we have $\deg b_{\mathbf{m}',\mathbf{s}'} \leq \min(|\mathbf{m}'|, |\mathbf{s}'|)$ (see 1.5) and so

$$\deg b'_{\mathbf{m},\mathbf{s}} \leq |\mathbf{s}| + N \quad \text{and} \quad \deg b'_{\mathbf{m}',\mathbf{s}} \leq \min(|\mathbf{m}'|, |\mathbf{s}|) \;\; \text{if } \mathbf{m}' \neq \mathbf{m},$$

where $N := \deg \varphi - |\mathbf{k}|$. Hence

$$\deg \det B' \leq N + \sum_{\mathbf{s} \in \mathcal{P}(\nu)} |\mathbf{s}|.$$

By 1.5,

$$\deg \det B = \sum_{\mathbf{s} \in \mathcal{P}(\nu)} |\mathbf{s}|$$

and, finally,

$$\deg \varphi_{\mathbf{k},\mathbf{m}} \leq N = \deg \varphi - |\mathbf{k}|,$$

as required. This completes the proof.

2.7. Description of the anticentre. For a finite dimensional Lie superalgebra $\mathfrak{p}$, the anticentre $\mathcal{A}(\mathfrak{p})$ can be defined as the set of invariants of $\mathcal{U}(\mathfrak{p})$ with respect to a twisted adjoint action: $\mathcal{A}(\mathfrak{p}) := \mathcal{U}(\mathfrak{p})^{\mathrm{ad}'\,\mathfrak{p}}$ where ad' is given by the formula

$$(\mathrm{ad}'\, g)u = gu - (-1)^{p(g)(p(u)+1)} ug.$$

In other words, the anticentre $\mathcal{A}(\mathfrak{p})$ contains the odd elements which commute with all elements of $\mathcal{U}(\mathfrak{p})$ and the even elements which commute with the even elements of $\mathcal{U}(\mathfrak{p})$ and anticommute with the odd ones.

2.7.1. Formula (3) implies that the restriction of HC to $\mathcal{A}(\mathfrak{g})$ is injective. In particular, $\mathcal{A}(\mathfrak{g})$ have no odd elements. In [G1] we showed that $\mathrm{HC}(\mathcal{A}(\mathfrak{g})) = t\,\mathrm{HC}(\mathcal{Z}(\mathfrak{g}_0))$ where $\mathfrak{g}_0$ is the even part of $\mathfrak{g}$ ($\mathfrak{g}_0$ is a reductive Lie algebra) and t is the product of linear factors corresponding to the odd positive roots.

2.7.2. Retain notation of 2.1 and 2.2. For $\lambda \in X$, let m^λ and $\bar{m}^\lambda$ be the highest weight vectors of $M(\lambda)$ and $\overline{M}(\lambda)$, respectively. Set

$$\mathrm{sgn}(\lambda) = \begin{cases} 0 & \text{if } p(m^\lambda) = p(\bar{m}^\lambda), \\ 1 & \text{otherwise.} \end{cases}$$

The following theorem can be proven along the lines of 2.3–2.6.

Theorem. *The restriction of* HC *to* $\mathcal{A}(\mathfrak{g})$ *is an isomorphism*

$$\mathrm{HC} : \mathcal{A}(\mathfrak{g}) \xrightarrow{\sim} \{\varphi \in S(\mathfrak{h}) \mid \varphi(\lambda) = (-1)^{\mathrm{sgn}(\lambda)} \varphi(r(\lambda)) \;\; \textit{for any } \lambda \in X\}.$$

3. The Shapovalov determinants for Lie superalgebras

To check the assumptions (i), (ii) of 2.1, we have to compute the Shapovalov determinants: actually, the assumptions correspond to the main steps of calculation. The first assumption easily follows from the existence of the quadratic Casimir operator (see sec. 3.1). The assumption (ii) (a) is equivalent to the fact that the Jantzen filtration of $M(\lambda)$ is of length 2. To verify the assumption (ii), we choose the set $\breve{\gamma}_m$ in such a way that for all $\lambda \in \breve{\gamma}_m$ both $M(\lambda)$ and its Jantzen filtration have length 2. This is done in sec. 3.5.

The formulas for the Shapovalov determinant for the Lie superalgebras with Cartan matrix were obtained in [K2],[KK]. For $\mathfrak{g} = \mathfrak{osp}(1, 2l)$, a detailed proof is written in [M]; it is similar to that for a semisimple Lie algebra. In presence of isotropic roots we have to modify the proof. We give some details below.

3.1. Every contragredient Lie superalgebra has a quadratic even Casimir element. This is a central element $z \in \mathcal{Z}(\mathfrak{g})$ satisfying

$$\mathrm{HC}(z)(\lambda) = (\lambda + \rho, \lambda + \rho),$$

where ρ can be defined for finite-dimensional $\mathfrak{g}$ (for infinite dimensional $\mathfrak{g}$ another definition is used) from the formula

$$2\rho := \sum_{\alpha \in \Delta_0^+} \alpha - \sum_{\alpha \in \Delta_1^+} \alpha.$$

3.1.1. The existence of z implies that if $M(\lambda)$ has a primitive vector of weight $\lambda - \mu$, then

$$(\lambda + \rho, \lambda + \rho) = (\lambda + \rho - \mu, \lambda + \rho - \mu)$$

that is

$$2(\lambda + \rho, \mu) = (\mu, \mu). \tag{10}$$

Now if $\det S_\nu(\lambda) = 0$, then $M(\lambda)$ has a primitive vector of weight $\lambda - \mu$ for some $\mu \in Q^+$ such that $0 < \mu \le \nu$. In other words, $\det S_\nu(\lambda) = 0$ forces (10) for some $\mu \in Q^+ \setminus \{0\}$. For each $\mu \in Q^+ \setminus \{0\}$, the solutions of (10) for λ form a hyperplane γ_μ in $\mathfrak{h}^*$. Hence all zeroes of $\det S_\nu$ belong to $\bigcup_{\mu \in Q^+ \setminus \{0\}} \gamma_\mu$. Set

$$\mathrm{Irr} := \{\eta \in Q^+ \mid \eta = k\xi \ \text{ for } k \in \mathbb{Z}_{\ge 0},\ \xi \in Q^+ \implies k = 1\}.$$

It is clear that any $\mu \in Q^+ \setminus \{0\}$ has a unique representation in the form $\mu = k\eta$ with $k \in \mathbb{Z}_{>0}$, $\eta \in \mathrm{Irr}$; the equation (10) takes the form

$$2(\lambda + \rho, \eta) - k(\eta, \eta) = 0.$$

This gives the one-to-one correspondence between the hyperplanes γ_μ and the elements of Y, where

$$Y := \{(\eta, k) \mid k \in \mathbb{Z}_{>0}, \eta \in \mathrm{Irr} : (\eta, \eta) \neq 0\} \cup \{(\eta, 1) \mid \eta \in \mathrm{Irr} : (\eta, \eta) = 0\}.$$

3.1.2. As a consequence, if $\mathfrak{g}$ possesses a quadratic Casimir element, each Shapovalov determinant $\det S_\nu$ admits a linear factorization: up to a constant factor, we have

$$\det S_\nu = \prod T_{k\eta}^{d_{k\eta}(\nu)},$$

where $\eta \in \mathrm{Irr}$, $k \in \mathbb{Z}_{>0}$,

$$T_{k\eta}(\lambda) = 2(\lambda + \rho, \eta) - k(\eta, \eta) \tag{11}$$

and $d_{k\eta}(\nu) \in \mathbb{Z}_{\geq 0}$.

3.2. The Jantzen filtration. The notion of *Jantzen filtration* was introduced for semi-simple Lie algebras in [Ja]. In this subsection we recall the definition and apply it for superalgebras as well. Fix a $\rho' \in \mathfrak{h}^*$ so that $(\rho', \mu) \neq 0$ for all $\mu \in Q^+ \setminus \{0\}$.

3.2.1. Let t be an even indeterminate and let $A := \mathbb{C}[t]_{(t)}$ be the localization of $\mathbb{C}[t]$ by the principal ideal (t). The algebra A is a local ring and $F := \mathbb{C}(t)$ is its field of fractions.

We shall extend the field of scalars from $\mathbb{C}$ to A, and sometimes to F and accordingly replace all tensorings over $\mathbb{C}$ by tensorings over A or over F. For any $\mathbb{C}$-vector superspace V, we denote by V_A the A-module $A \otimes V$. Since tensorings are over A, we can (and will) identify $U(\mathfrak{g}_A)$ with $U(\mathfrak{g})_A$.

For any $\xi \in \mathfrak{h}_A^*$, we can (and will) define a $U(\mathfrak{g}_A)$-module — a Verma module — $M(\xi)_A$ in the usual way. For each Verma module $M(\xi)_A$, let

$$\langle \cdot, \cdot \rangle_\xi : \ M(\xi)_A \times M(\xi)_A \to A$$

be the contravariant form defined in the same way as in 1.2.2. Consider the Shapovalov determinant $\det S_\nu \in S(\mathfrak{h})$ as an element of the ring $S(\mathfrak{h}_A)$. Then $\det S_\nu$ thus defined coincides with the Shapovalov determinant $\det S_\nu$ constructed for $U(\mathfrak{g}_A)$ (notice that the $\det S_\nu$ is defined up to an invertible element of A).

Similarly, for any A-module V, we denote by V_F the vector superspace $V \otimes_A F$. As above, we consider tensoring in $U(\mathfrak{g}_F)$ over F, and hence can identify $U(\mathfrak{g}_F) = U(\mathfrak{g})_F$. For any $\xi \in \mathfrak{h}_A^*$, the module $M(\xi)_F$ admits the natural structure of $U(\mathfrak{g}_F)$-Verma module of highest weight ξ. Let us show that $M(\lambda + t\rho')_F$ is simple for any $\lambda \in \mathfrak{h}^*$. Indeed, for any $\mu \in Q^+ \setminus \{0\}$, we have

$$(\lambda + t\rho' + \rho, \mu) = (\lambda + \rho, \mu) + t(\rho', \mu) \notin \mathbb{C}$$

since $(\rho', \mu) \neq 0$. In particular, $(\lambda + t\rho' + \rho, \mu) \notin \mathbb{Z}_{\geq 0}(\mu, \mu)$. By 3.1.1, this implies the simplicity of $M(\lambda + t\rho')_F$ (since the reasoning of 3.1.1 can be applied to a contravariant Lie superalgebra over any field).

3.2.2. For any $\lambda \in \mathfrak{h}^*$, define the decreasing filtration on the module $M(\lambda + t\rho')_A$ by setting

$$M(\lambda+t\rho')^i_A := \begin{cases} M(\lambda+t\rho')_A & \text{for } i \in -\mathbb{Z}_{\geq 0} \\ \{m \in M(\lambda+t\rho')_A \mid \langle m, M(\lambda+t\rho')_A\rangle_{\lambda+t\rho'} \subseteq t^i A\} & \text{for } i \in \mathbb{Z}_{>0}. \end{cases}$$

The simplicity of $M(\lambda + t\rho')_F$ implies that the contravariant form $\langle\cdot,\cdot\rangle_{\lambda+t\rho'}$ is non-degenerate. Thus $\bigcap_{i\in\mathbb{Z}} M(\lambda+t\rho')^i_A = 0$. We identify $M(\lambda)$ and $M(\lambda+t\rho')_A/tM(\lambda+t\rho')_A$. For each $i \in \mathbb{Z}_{\geq 0}$, define the $U(\mathfrak{g})$-submodule $M(\lambda)^i$ of $M(\lambda)$ by setting

$$M(\lambda)^i := M(\lambda+t\rho')^i_A/tM(\lambda+t\rho')^{i-1}_A.$$

Since $M(\lambda+t\rho')^{i+1}_A \cap tM(\lambda+t\rho')_A = tM(\lambda+t\rho')^i_A$, it follows that $\{M(\lambda)^i\}_{i\in\mathbb{Z}_{\geq 0}}$ forms a decreasing filtration of $M(\lambda)$ and $\bigcap_{i\in\mathbb{Z}_{\geq 0}} M(\lambda)^i = 0$. This filtration is called the *Jantzen filtration.*

3.2.3. *Remark.* Observe that $\mathrm{HC}(\sigma(a)b)(\lambda+t\rho')$ is a polynomial in one variable t for any $a, b \in U(\mathfrak{n}^-)$ and $\lambda \in \mathfrak{h}^*$. Thus $\mathrm{HC}(\sigma(a)b)(\lambda+t\rho') \in (t)$ if and only if $\mathrm{HC}(\sigma(a)b)(\lambda) = 0$. Hence $M(\lambda)^1$ coincides with the kernel of $\langle\cdot,\cdot\rangle_\lambda$ and $M(\lambda)/M(\lambda)^1 = V(\lambda)$ is the simple module of highest weight λ.

3.2.4. We are going to use the following fact proved in [Ja]. Recall that $A = \mathbb{C}[t]_{(t)}$. Let N be a free A-module of finite rank r endowed by a bilinear form $N \otimes N \to A$, and let $D \in \mathrm{Mat}_{r\times r}(A)$ be the matrix of this bilinear form. Define a decreasing filtration

$$F^j(N) := \{v \in N \mid D(v, v') \in (t^j) \text{ for any } v' \in N\}.$$

Assume that $\det D \neq 0$. We claim that *the order of zero of* $\det D$ *at the origin is equal to*

$$\sum_{j=1}^{\infty} \dim\big(F^j(N)/(F^j(N) \cap tN)\big).$$

Indeed, it is easy to see that N has two systems of generators v_i and v'_i (for $i = 1, \ldots, r$) such that $D(v_i, v'_j) = \delta_{ij}t^{s_i}$ (for $s_i \in \mathbb{Z}_{\geq 0}$). The order of zero of $\det D$ at the origin is $\sum_{i=1}^{r} s_i$ and

$$\dim F^j(N)/(F^j(N) \cap tN) = |\{i \mid s_i \geq j\}|.$$

The equality $\sum_i s_i = \sum_{j=1}^{\infty} |\{i \mid s_i \geq j\}|$ implies the claim. □

3.2.5. Taking $D = S_\nu$, we conclude that, for any $\lambda \in \mathfrak{h}^*$ and any $\nu \in Q^+$, the sum $\sum_{r=1}^{\infty} \dim F^r(M(\lambda)_{\lambda-\nu})$ is equal to the order of zero of $\det S_\nu$ at λ. In particular, we see that the assumption (ii) (a) is equivalent to the condition $M(\lambda)^2 = 0$ for any $\lambda \in X$.

3.2.6. In the notation of sec. 3.1.2 we obtain

$$\sum_{k,\eta:T_{k\eta}(\lambda)=0} d_{k\eta}(\nu)=\sum_{r=1}^{\infty}\dim M(\lambda)^r_{\lambda-\nu} \text{ for any } \lambda\in\mathfrak{h}^* \text{ and any } \nu\in Q^+. \tag{12}$$

3.2.7. Fix a non-isotropic $\mu\in\mathrm{Irr}$ and $k\in\mathbb{Z}_{>0}$. Set

$$R:=\{\lambda\in\mathfrak{h}^*\mid(\lambda+\rho,\mu)=k\ \&\ (\lambda+\rho,\eta)\not\in\mathbb{Z}_{\geq 0}\ \text{ for any }\ \eta\in\mathrm{Irr},\ \eta\neq\mu\}.$$

The set R is dense in the hyperplane $T_{k\mu}=0$. For any $\lambda\in R$, the only element $\xi\in Q^+$ satisfying

$$(\lambda+\rho,\lambda+\rho)=(\lambda+\rho-\xi,\lambda+\rho-\xi)$$

is $\xi=k\mu$. Therefore the module $M(\lambda-k\mu)$ is simple. Moreover, $M(\lambda)$ is either simple or its socle is the sum of several copies of $M(\lambda-k\mu)$. In any case, for all $r\geq 1$, there exists $j_r\geq 0$ such that

$$\mathrm{ch}\, M(\lambda)^r=j_r\,\mathrm{ch}\, M(\lambda-k\mu)$$

(if $M(\lambda)$ is simple, then $j_r=0$ for all $r\geq 1$). Taking (12) into account, we obtain

$$d_{k\mu}(\nu)=j\dim M(\lambda-k\mu)_{\lambda-\nu}=j\tau(\nu-k\mu),$$

where $j:=\sum\limits_{r=1}^{\infty} j_r$. Finally, for any non-isotropic $\mu\in\mathrm{Irr}$, and any $k\in\mathbb{Z}_{>0}$, there exists $j\in\mathbb{Z}_{\geq 0}$ such that

$$d_{k\mu}(\nu)=j\tau(\nu-k\mu)\ \text{ for all } \nu\in Q^+. \tag{13}$$

3.3. The leading term of $\det S_\nu$. For each root $\alpha\in\Delta^+$, denote by $\alpha^\vee$ the corresponding coroot, i.e. $\alpha^\vee\in\mathfrak{h}$ is such that $(\alpha,\xi)=\alpha^\vee(\xi)$ for all $\xi\in\mathfrak{h}^*$. Notice that $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$ is spanned by $\alpha^\vee$.

The following lemma for Lie superalgebras has the same proof as in Lie algebra case.

3.3.1. Lemma. *For any $\mathbf{m},\mathbf{s}\in\mathcal{P}(\nu)$, we have*

(i) $\deg\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}})\leq\min(|\mathbf{m}|,|\mathbf{s}|)$;
(ii) *If $|\mathbf{m}|=|\mathbf{s}|$, we have*

$$\deg\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}})=|\mathbf{m}\,|\Longleftrightarrow\ \mathbf{m}=\mathbf{s};$$

(iii) *Up to a non-zero scalar factor*

$$\mathrm{gr}\,\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{m}})=\prod_{\alpha\in\Delta^+}(\alpha^\vee)^{m_\alpha}.$$

3.3.2. Corollary. *Up to a non-zero scalar factor, the leading term of* $\det S_\nu$ *is equal to*

$$\prod_{\alpha\in\Delta^+}(\alpha^\vee)^{r_\alpha(\nu)},$$

where

$$r_\alpha(\nu):=\sum_{\mathbf{m}\in\mathcal{P}(\nu)} m_\alpha.$$

3.3.3. Set

$$\overline{\Delta}_0^+ := \{\alpha\in\Delta_0^+ \mid \tfrac{1}{2}\alpha\not\in\Delta_0^+\},$$
$$\overline{\Delta}_1^+ := \{\alpha\in\Delta_1^+ \mid 2\alpha\not\in\Delta_0^+\}.$$

It is well-known (see [K1]) that $\overline{\Delta}_1^+=\{\alpha\in\Delta^+ \mid (\alpha,\alpha)=0\}$. We have

$$\Delta^+\cap \mathrm{Irr}=\overline{\Delta}_0^+\cup\Delta_1^+=\overline{\Delta}_0^+\cup\overline{\Delta}_1^+\cup(\Delta_1^+\setminus\overline{\Delta}_1^+).$$

For any $\alpha\in\Delta_1^+$, set

$$\tau_\alpha(\nu):=|\{\mathbf{k}\in\mathcal{P}(\nu)\mid k_\alpha=0\}|.$$

We need the following technical lemma.

3.3.4. Lemma.

(i) *For all* $\alpha\in\Delta_0^+$, *we have*

$$\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_\alpha=\sum_{n=1}^{\infty}\tau(\nu-n\alpha).$$

(ii) *For all* $\alpha\in\Delta_1^+$, *we have*

$$\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_\alpha=\tau_\alpha(\nu-\alpha)=\sum_{n=1}^{\infty}(-1)^{n+1}\tau(\nu-n\alpha).$$

Proof. Let us use an induction on the partial order on Q^+. For any $\nu\not\geq\alpha$, the assertions obviously hold since both sides of both equations vanish. Fix $\nu\geq\alpha$ and assume that the assertions hold for all $\mu<\nu$. The map $\mathbf{k}\mapsto(\mathbf{k}-\alpha)$ gives a bijection

$$\{\mathbf{k}\in\mathcal{P}(\nu)\mid k_\alpha\neq 0\}\xrightarrow{\sim}\mathcal{P}(\nu-\alpha).$$

Hence

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_\alpha &= \sum_{\mathbf{k}\in\mathcal{P}(\nu-\alpha)}(k_\alpha+1)=\sum_{n=1}^{\infty}\tau(\nu-(n+1)\alpha)+\mid\mathcal{P}(\nu-\alpha)\mid\\ &\overset{\text{by the assumption}}{=} \sum_{n=2}^{\infty}\tau(\nu-n\alpha)+\tau(\nu-\alpha).\end{aligned}$$

This gives (i). Take $\alpha\in\Delta_1^+$ such that $\nu\geq\alpha$. The map $\mathbf{k}\mapsto(\mathbf{k}-\alpha)$ gives a bijection

$$\{\mathbf{k}\in\mathcal{P}(\nu)\mid k_\alpha=1\}\xrightarrow{\sim}\{\mathbf{k}\in\mathcal{P}(\nu-\alpha)\mid k_\alpha=0\}=\\ \mathcal{P}(\nu-\alpha)\setminus\{\mathbf{k}\in\mathcal{P}(\nu-\alpha)\mid k_\alpha=1\}.$$

Hence

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{P}(\nu)}k_\alpha &=\tau_\alpha(\nu-\alpha)= && |\,cP(\nu-\alpha)|-\sum_{\mathbf{k}\in\mathcal{P}(\nu-\alpha)}k_\alpha\\ &= && \tau(\nu-\alpha)-\sum_{n=1}^{\infty}(-1)^n\tau(\nu-(n+1)\alpha)\\ &\overset{\text{by the assumption}}{=} && \sum_{n=1}^{\infty}(-1)^{n+1}\tau_{n\alpha}(\nu).\end{aligned}$$

This gives (ii). $\square$

3.3.5. For any $\alpha\in\Delta_1^+$, the map used in the proof of (ii) establishes the following useful equality

$$\tau(\nu)-\tau_\alpha(\nu)=\tau_\alpha(\nu-\alpha). \tag{14}$$

In particular, $\sum\limits_{\mathbf{m}\in\mathcal{P}(\nu)}m_\alpha=\tau_\alpha(\nu-\alpha)$.

3.3.6. Substituting the formulas (3.3.4) in Corollary 3.3.2 and using that $(2\alpha)^\vee=2\alpha^\vee$, we conclude that the leading term of the polynomial $\det S_\nu$ is equal, up to a non-zero scalar factor, to

$$\prod_{\alpha\in\overline{\Delta}_0^+}(\alpha^\vee)^{\sum_{n=1}^{\infty}\tau(\nu-n\alpha)}\prod_{\alpha\in\overline{\Delta}_1^+}(\alpha^\vee)^{\tau_\alpha(\nu-\alpha)}\prod_{\alpha\in\Delta_1^+\setminus\overline{\Delta}_1^+}(\alpha^\vee)^{\sum_{n=0}^{\infty}\tau(\nu-(2n+1)\alpha)}.$$

3.4. Comparing the above expression with 3.1.2 we conclude that $d_{k\eta}=0$ if $\eta\not\in\Delta^+$. Furthermore, up to a non-zero scalar factor,

$$\det S_\nu=\prod_{\alpha\in\overline{\Delta}_1^+}T_\alpha^{\tau_\alpha(\nu-\alpha)}\prod_{k=1}^{\infty}\prod_{\alpha\in\overline{\Delta}_0^+\cup\Delta_1^+\setminus\overline{\Delta}_1^+}T_{k\alpha}^{d_{k\alpha}(\nu)},$$

where $T_{k\alpha}$ is given by (11) and

$$\sum_{k=1}^{\infty}d_{k\alpha}(\nu)=\begin{cases}\sum\limits_{n=0}^{\infty}\tau(\nu-(2n+1)\alpha) & \text{for }\alpha\in\Delta_1^+\setminus\overline{\Delta}_1^+\\ \sum\limits_{n=1}^{\infty}\tau(\nu-n\alpha) & \text{for }\alpha\in\overline{\Delta}_0^+.\end{cases} \tag{15}$$

3.4.1. Fix $\alpha \in \overline{\Delta}_0^+ \cup \Delta_1^+ \setminus \overline{\Delta}_1^+$. By (13), for each $k = 1, 2, \ldots$, there exists a $j_k \in \mathbb{Z}_{\geq 0}$ such that

$$d_{k\alpha}(\nu) = j_k \tau(\nu - k\alpha) \text{ for all } \nu \in Q^+.$$

Substituting this $d_{k\alpha}(\nu)$ in (15) we get

$$\sum_{k=1}^{\infty} j_k \tau(\nu - k\alpha) = \begin{cases} \sum_{n=0}^{\infty} \tau(\nu - (2n+1)\alpha) & \text{for } \alpha \in \Delta_1^+ \setminus \overline{\Delta}_1^+ \\ \sum_{n=1}^{\infty} \tau(\nu - n\alpha) & \text{for } \alpha \in \overline{\Delta}_0^+. \end{cases}$$

Consider $\tau_i := \tau(\nu - i\alpha)$ as a function on $\nu \in Q^+$. The functions τ_i for distinct i's are linearly independent since $\tau_i(i\alpha) = 1$ and $\tau_j(i\alpha) = 0$ for $j > i$. Thus, for any $\alpha \in \Delta_1^+ \setminus \overline{\Delta}_1^+$, we have $j_{2n} = 0, j_{2n+1} = 1$ and, for any $\alpha \in \overline{\Delta}_0^+$, we have $j_n = 1$. Hence

$$d_{n\alpha}(\nu) = \begin{cases} \tau(\nu - n\alpha) & \text{for } \alpha \in \Delta_1^+ \setminus \overline{\Delta}_1^+ & \text{and } n \text{ odd} \\ 0 & \text{for } \alpha \in \Delta_1^+ \setminus \overline{\Delta}_1^+ & \text{and } n \text{ even} \\ \tau(\nu - n\alpha) & \text{for } \alpha \in \overline{\Delta}_0^+ & \text{and } n \geq 1. \end{cases}$$

3.4.2. Corollary. *Up to a non-zero scalar factor,*

$$\det S_\nu = \prod_{\alpha \in \overline{\Delta}_1^+} T_\alpha^{\tau_\alpha(\nu-\alpha)} \prod_{k=1}^{\infty} \prod_{\alpha \in \overline{\Delta}_0^+} T_{k\alpha}^{\tau(\nu-k\alpha)} \prod_{k=0}^{\infty} \prod_{\Delta_1^+ \setminus \overline{\Delta}_1^+} T_{(2k+1)\alpha}^{\tau(\nu-(2k+1)\alpha)},$$

where $T_{k\alpha}(\lambda) = 2(\lambda + \rho, \alpha) - k(\alpha, \alpha)$ *for* $k \geq 1$.

3.5. It remains to check the assumption (ii) of 2.1. For each pair $\beta \in \Delta^+$ and $k \in \mathbb{Z}_{\geq 0}$, denote by $\gamma_{j\beta}$ the hyperplane $T_{j\beta} = 0$. Let I be the set of hyperplanes corresponding to the linear factors of the Shapovalov determinants: I consists of the hyperplanes $\gamma\alpha$, where $\alpha \in \overline{\Delta}_1^+$ and the hyperplanes $\gamma_{j\beta}$, where $\beta \in \overline{\Delta}_0^+ \cup \Delta_1^+ \setminus \overline{\Delta}_1^+$ and J is a positive integer which is odd if β is an odd root.

3.5.1. For each hyperplane $\gamma_{j\alpha} \in I$ denote by $\breve{\gamma}_{j\alpha}$ its subset defined by

$$\breve{\gamma}_{j\alpha} := \{\xi \in \mathfrak{h}^* \mid 2(\xi + \rho, \eta) \notin \mathbb{Z}_{\geq 0}(\eta, \eta) \quad \text{for any } \eta \in \mathrm{Irr} \setminus \{\alpha\} \ \}.$$

For any $\lambda \in \breve{\gamma}_{j\alpha}$, we have

$$\{\mu \in Q^+ \mid (\lambda + \rho, \lambda + \rho) = (\lambda - \mu + \rho, \lambda - \mu + \rho)\} = \begin{cases} \{0, j\alpha\} & \text{if } (\alpha, \alpha) \neq 0, \\ \mathbb{Z}_{\geq 0}\alpha & \text{if } (\alpha, \alpha) = 0. \end{cases} \tag{16}$$

Notice that $\breve{\gamma}_{j\alpha}$ is obtained from $\gamma_{j\alpha}$ by removing the points of countably many hyperplanes; thus $\breve{\gamma}_{j\alpha}$ is open in $\gamma_{j\alpha}$.

3.5.2. For any $\beta\in\overline{\Delta}_1^+$, let s_β be the linear transformation of $\mathfrak{h}^*$ given by $s_\beta.\xi=\xi-\beta$; for any $\beta\in\Delta^+\setminus\overline{\Delta}_1^+$ let s_β be the corresponding twisted reflection of $\mathfrak{h}^*$:

$$s_\beta.\xi=\xi-2\frac{(\xi+\rho,\beta)}{(\beta,\beta)}\beta.$$

Fix any $\gamma_{j\alpha}\in I$ and take $\lambda\in\breve{\gamma}_{j\alpha}$. The order of zero of $\det S_{j\alpha}$ at λ is equal to 1. Therefore $\operatorname{corank} S_{j\alpha}(\lambda)=1$. Since $\det S_\mu(\lambda)\neq 0$ for $\mu<j\alpha$, it follows that $M(\lambda)$ has a unique primitive vector of weight $\lambda-j\alpha=s_\alpha.\lambda$. In other words, $\overline{M}(\lambda)$ has a unique subquotient isomorphic to $V(s_\alpha.\lambda)$.

3.5.3. Consider the case where $(\alpha,\alpha)\neq 0$. Take $\lambda\in\breve{\gamma}_{j\alpha}$. By (16) $M(s_\alpha.\lambda)$ is simple. Combining (16) and 3.5.2, we conclude that $\overline{M}(\lambda)=M(s_\alpha.\lambda)$ and thus $M(\lambda)$ has length two.

For all $\nu\in Q^+$, we have

$$\operatorname{corank} S_\nu(\lambda)=\dim\overline{M}(\lambda)_{\lambda-\nu}=\dim M(s_\alpha.\lambda)_{\lambda-\nu}=\tau(\nu-j\alpha),$$

and hence $\operatorname{corank} S_\nu(\lambda)$ is equal to the order of zero of $\det S_\nu$ at λ. Hence the Jantzen filtration of $M(\lambda)$ has length two: $M(\lambda)^2=0$.

3.5.4. Now consider the case where $(\alpha,\alpha)=0$, that is $\alpha\in\overline{\Delta}_1^+$. Take $\lambda\in\breve{\gamma}_\alpha$ and note that $\lambda+\alpha\in\breve{\gamma}_\alpha$. By 3.5.2, $\overline{M}(\lambda+\alpha)$ has a subquotient isomorphic to $V(\lambda)$. Therefore

$$\operatorname{corank} S_{\nu+\alpha}(\lambda+\alpha)\geq\operatorname{rank} S_\nu(\lambda)=\tau(\nu)-\operatorname{corank} S_\nu(\lambda).$$

For any $\mu\in Q^+$, the order of zero of $\det S_{\mu+\alpha}$ at $\xi\in\breve{\gamma}_\alpha$ is equal to $\tau_\alpha(\mu)$ and so

$$\tau_\alpha(\mu)\geq\operatorname{corank} S_{\mu+\alpha}(\xi). \tag{17}$$

Combining the two last inequalities (and substituting $\xi:=\lambda+\alpha,\mu:=\nu$ in the last one), we obtain

$$\operatorname{corank} S_\nu(\lambda)\geq\tau(\nu)-\tau_\alpha(\nu)=\tau_\alpha(\nu-\alpha)$$

by (14). Comparison with (17) for $\xi:=\lambda$ and $\mu:=\nu-\alpha$ gives

$$\operatorname{corank} S_\nu(\lambda)=\tau_\alpha(\nu-\alpha) \tag{18}$$

which is equal to the order of zero of $\det S_\nu$ at λ. Hence the Jantzen filtration of $M(\lambda)$ has length two for any $\lambda\in\breve{\gamma}_\alpha$. Moreover (18) implies that, for any $\lambda\in\breve{\gamma}_\alpha$ and $\nu\in Q^+$, we have

$$\dim V(\lambda)_{\lambda-\nu}=\operatorname{rank} S_\nu(\lambda)=\tau(\nu)-\tau_\alpha(\nu-\alpha)=\tau_\alpha(\nu).$$

Using (18) for $\lambda':=\lambda+\alpha$ and $\nu':=\nu+\alpha$ we conclude that

$$\dim V(\lambda)_{\lambda-\nu}=\operatorname{corank} S_{\nu+\alpha}(\lambda+\alpha)=\dim\overline{M}(\lambda+\alpha)_{\lambda-\nu}.$$

Therefore $\operatorname{ch}\overline{M}(\lambda+\alpha)=\operatorname{ch}V(\lambda)$ and so $\overline{M}(\lambda+\alpha)=V(\lambda)$. Hence $M(\lambda+\alpha)$ has length two. Finally, $M(\xi)$ has length two for any $\xi\in\breve{\gamma}_\alpha$.

3.5.5. Corollary. *For any* $\gamma_{j\alpha} \in I$ *and any* $\lambda \in \breve{\gamma}_{j\alpha}$, *we have*

(i) *The module* $M(\lambda)$ *has length two:*

$$0 \subset V(s_\alpha.\lambda) \subset M(\lambda).$$

(ii) *The Jantzen filtration of* $M(\lambda)$ *has length two:* $M(\lambda)^2 = 0$, *and* $M(\lambda)^1 = V(s_\alpha.\lambda)$.

(iii) *For any* $\nu \in Q^+$, *the order of zero of* $\det S_\nu$ *at* λ *is equal to* $\operatorname{corank} S_\nu(\lambda)$.

3.5.6. *Remark.* Take any $\lambda \in \gamma_{j\alpha} \setminus (\underset{\gamma\in I, \gamma\neq\gamma_{j\alpha}}{\cup} \gamma)$. It is easy to see that (ii), (iii) hold for such λ and (i) holds if $(\alpha,\alpha) \neq 0$.

3.6. Corollary 3.5.5 implies the assumption (ii) of 2.1. Hence Kac's Theorem gives

$$\mathrm{HC}(\mathcal{Z}(\mathfrak{g})) = \underset{\gamma_{j\alpha}\in I}{\cap} \{\varphi \in S(\mathfrak{h}) \mid \varphi(\lambda) = s_\alpha.\lambda \quad \text{for any} \quad \lambda \in \breve{\gamma}_{j\alpha}\}.$$

Each set $\breve{\gamma}_{j\alpha}$ is Zariski dense in $\gamma_{j\alpha}$. Therefore

$$\mathrm{HC}(\mathcal{Z}(\mathfrak{g})) = \underset{\gamma_{j\alpha}\in I}{\cap} \{\varphi \in S(\mathfrak{h}) \mid \varphi(\lambda) = s_\alpha.\lambda \ \text{ for any } \ \lambda \in \gamma_{j\alpha}\}.$$

Denote by W the Weyl group of $\mathfrak{g}_0$ and by $S(\mathfrak{h})^W$ the set of W.-invariant functions in $S(\mathfrak{h})$ (the action of W on $S(\mathfrak{h})$ is given by $(s_\alpha.f)(\xi) := f(s_\alpha.\xi)$ for $f \in S(\mathfrak{h}), \xi \in \mathfrak{h}^*$). If α is such that $(\alpha,\alpha) \neq 0$, then the union of the hyperplanes of the form $\gamma_{j\alpha}$ is Zariski dense in $\mathfrak{h}^*$. Hence

$$\underset{\alpha:(\alpha,\alpha)\neq 0}{\cap} \{\varphi \in S(\mathfrak{h}) \mid \varphi(\lambda) = s_\alpha.\lambda \ \text{ for any } \ \lambda \in \gamma_{j\alpha}\} = S(\mathfrak{h})^W.$$

Finally, we obtain

$$\mathrm{HC}(\mathcal{Z}(\mathfrak{g})) = \begin{array}{l} \{\varphi \in S(\mathfrak{h})^W \mid \varphi(\lambda) = \varphi(\lambda - \alpha) \ \text{ for any } \ \alpha \in \overline{\Delta}_1^+ \ \text{ and} \\ \text{any } \lambda \text{ such that } (\lambda+\rho,\alpha) = 0\}. \end{array}$$

4. Remark

Observe that the centres and anticentres of $U(\mathfrak{g})$ for simple finite-dimensional Lie superalgebras we consider here were described by A. Sergeev long ago, see [S1], and for details see [S2],[S3]. For other types of simple finite-dimensional Lie superalgebras $\mathfrak{g}$ there are only conjectural formulas, see [LS]. Observe that for Kac's approach, it is not vital whether $\mathfrak{g}$ possesses a Cartan matrix or not; for instance, a similar approach is applicable for Q-type algebras (sf. [S4]and [G2]). A similar method seems to be instrumental in the quantum case. Notice that the assumption (ii) (a) of 2.1 does not hold for the Poisson superalgebras $\mathfrak{po}(0|2n)$ and for their simple subquotients $H(n)$.

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