SHAPOVALOV FORMS FOR POISSON LIE SUPERALGEBRAS

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1. Introduction

Poisson Lie superalgebras are the superalgebras of functions on a symplectic supermanifold. Subquotients of Poisson superalgebras, called superalgebras of Hamiltonian vector fields, appear in the list of simple finite-dimensional Lie superalgebras (see [K]). If the dimension of a supermanifold is even, then a Poisson superalgebra admits a non-degenerate invariant even symmetric form. In particular, there exists a Casimir operator. Poisson superalgebras also have root decomposition in the sense of [PS]. It was noticed in [GL] that in such situation it is possible to define a Shapovalov form using the approach suggested in [KK]. We give a precise formula for determinant of Shapovalov form for finite-dimensional Poisson superalgebra $\mathfrak{po}(0|2n)$ with $n \geq 2$. The case $n = 1$ is well-known since $\mathfrak{po}(0|2)$ is isomorphic to $\mathfrak{gl}(1|1)$. We show that, contrary to the case of classical Lie superalgebras, the Jantzen filtration of a Verma module can be infinite.

One can use another approach to the problem of finding the Shapovalov form. It is well known that there is a deformation G_h of the Poisson superalgebra $\mathfrak{po}(0|2n)$ such that the Lie superalgebra G_h is isomorphic to $\mathfrak{gl}(2^{n-1}|2^{n-1})$ for $h \neq 0$. The Shapovalov form for the latter superalgebra is known (see [KKK]). Since the deformation preserves a Cartan subalgebra and triangular decomposition, one can obtain the Shapovalov for $\mathfrak{po}(0|2n)$ isomorphic to G_0 by evaluating the Shapovalov form for G_h at $h = 0$. However this method seems more difficult. Indeed, several root subspaces are glued together when $h = 0$. The condition on weights of irreducible Verma modules also change dramatically. It seems that the direct approach using the Casimir operator works better. One can illustrate this on a simple example. Indeed, it is much easier to evaluate the Shapovalov form for the Heisenberg algebra than to consider its deformation to $\mathfrak{sl}(2)$ and go back using the result for $\mathfrak{sl}(2)$.

2. Preliminary

2.1. **Poisson superalgebra $\mathfrak{po}(0|n)$.** Let $\Lambda(n)$ be the Grassman superalgebra in $\xi_1, \dots, \xi_n$. The Poisson Lie superalgebra $\mathfrak{po}(0|n)$ can be described as $\Lambda(n)$ endowed with the bracket

$$[f, g] = (-1)^{p(f)+1} \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}.$$

It is easy to see that $[\mathfrak{g}, \mathfrak{g}] = \sum_{i=0}^{n-1} \Lambda^i(n)$. Let $\int : \mathfrak{g} \to \mathbb{C}$ be such a map that $\ker \int = [\mathfrak{g}, \mathfrak{g}]$ and $\int(\xi_1 \dots \xi_n) = 1$. For $f, g \in \Lambda(n)$ define

$$B(f, g) := \int fg.$$

Clearly, B is a non-degenerate invariant bilinear form on $\mathfrak{g}$. If n is even, B gives rise to the quadratic Casimir element.

In this text we consider the even case $\mathfrak{g} := \mathfrak{po}(0|2n)$.

2.2. **Triangular decompositions.** A triangular decomposition of a Lie superalgebra $\mathfrak{g}$ can be constructed as follows (see [PS]). A Cartan subalgebra is a nilpotent subalgebra which coincides with its normalizer. It is proven in [PS] that any two Cartan subalgebras are conjugate by an inner automorphism. Fix a Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{g}$ has a generalized root decomposition

$$\mathfrak{g} := \mathfrak{h} + \oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$$

where Δ is a subset of $\mathfrak{h}^*$ and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | (ad(h) - \alpha(h))^{dim \mathfrak{g}}(x) = 0\}.$$

In case considered in this paper a root space $\mathfrak{g}_\alpha$ is either odd or even. That allows one to define the parity on the set of roots Δ. Denote by $\mathfrak{g}_{\overline{0}}$ (resp., $\mathfrak{g}_{\overline{1}}$) the even (resp., odd) component of $\mathfrak{g}$. Denote by Δ_0 (resp. Δ_1) the set of non-zero weights of $\mathfrak{g}_{\overline{0}}$ (resp., $\mathfrak{g}_{\overline{1}}$) with respect to $\mathfrak{h}$. Then Δ is a disjoint union of Δ_0 and Δ_1.

Now fix $h \in \mathfrak{h}_0^*$ satisfying $\alpha(h) \in \mathbb{R} \subset \{0\}$ for all $\alpha \in \Delta$. Set

$$\Delta^+ := \{\alpha \in \Delta |\ \alpha(h) > 0\},$$
$$\mathfrak{n}^+ := \textstyle\sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the weight space corresponding to α.

Define Δ^- and $\mathfrak{n}^-$ similarly. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a triangular decomposition.

2.3. **Notation.** Denote by Δ_0^+ (resp., Δ_1^+) the set of even (resp., odd) positive roots. Let $Q \subset \mathfrak{h}^*$ be the root lattice that is the $\mathbb{Z}$-span of Δ^+ and let Q^+ be the $\mathbb{Z}_{\geq 0}$-span of Δ^+. Introduce the standard partial ordering on $\mathfrak{h}^*$ by setting $\mu \leq \nu$ if $\nu - \mu \in Q^+$.

Throughout the paper α and β stand for positive roots.

For $\alpha \in \Delta^+$ denote by D_α the matrix of natural pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathfrak{h}$ given by $(x, y) \mapsto [x, y]$.

2.4. **Verma modules.** *From now on suppose that $\mathfrak{h}$ is even and commutative.* Set $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+$. For each $\lambda \in \mathfrak{h}^*$ define $M(\lambda) := U(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} k_\lambda$ where k_λ is a one-dimensional $\mathfrak{b}$-module which is trivial as $\mathfrak{n}^+$-module and corresponds to λ as $\mathfrak{h}$-module. Each Verma module has a unique maximal submodule $\overline{M(\lambda)}$. The corresponding simple module $V(\lambda) := M(\lambda)/\overline{M(\lambda)}$ is called a highest weight simple module.

2.5. **Shapovalov determinants.** For finite dimensional semisimple Lie algebras N. Shapovalov ([Sh]) constructed a bilinear form $\mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{n}^-) \to S(\mathfrak{h})$ whose kernel at a given point $\lambda \in \mathfrak{h}^*$ determines the maximal submodule $\overline{M(\lambda)}$ of a Verma module $M(\lambda)$. In particular, a Verma module $M(\lambda)$ is simple if and only if the kernel of Shapovalov form at λ is equal to zero. The Shapovalov form can be realized as a direct sum of forms S_ν; for each S_ν one can define its determinant (Shapovalov determinant). The zeroes of Shapovalov determinants determine when a Verma module is reducible.

2.5.1. A Shapovalov form for a Lie superalgebra $\mathfrak{g}$ with an even commutative Cartan subalgebra $\mathfrak{h}$ can be described as follows.

Identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$. Let $\mathrm{HC}: U(\mathfrak{g}) \to S(\mathfrak{h})$ be the Harish-Chandra projection i.e., a projection along the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+)$. Define a form $\mathcal{U}(\mathfrak{n}^+) \otimes \mathcal{U}(\mathfrak{n}^-) \to S(\mathfrak{h})$ by setting $S(x,y) := \mathrm{HC}(xy)$. Using the natural identification of a Verma module $M(\lambda)$ with $\mathcal{U}(\mathfrak{n}^-)$, one easily sees that $\overline{M(\lambda)}$ coincides with the "right kernel" of the evaluated form $S(\lambda): \mathcal{U}(\mathfrak{n}^+) \otimes \mathcal{U}(\mathfrak{n}^-) \to k$ i.e., $\overline{M(\lambda)} = \{y \in \mathcal{U}(\mathfrak{n}^-) |\ (x,y)(\lambda) = 0 \text{ for all } x\}$.

Notice that $S(x,y) = 0$ if $x \in \mathcal{U}(\mathfrak{n}^+)_\nu, y \in \mathcal{U}(\mathfrak{n}^-)_{-\mu}$ and $\nu \neq \mu$. Thus $S = \sum_{\nu \in Q^+} S_\nu$ where S_ν is the restriction of S to $\mathcal{U}(\mathfrak{n}^+)_\nu \otimes \mathcal{U}(\mathfrak{n}^-)_{-\nu}$ By the above, $\dim V(\lambda)_{\lambda-\nu} = \operatorname{codim} \ker_r S_\nu(\lambda)$ where $\ker_r$ stands for the "right kernel".

2.5.2. *Assume that* $\dim \mathcal{U}(\mathfrak{n}^+)_\nu = \dim \mathcal{U}(\mathfrak{n}^-)_{-\nu} < \infty$ *for all* $\nu \in Q^+$. Then $\det S_\nu$ is an element of $S(\mathfrak{h})$ defined up to an invertible scalar. One obtains the following criterion of simplicity of a Verma module: $M(\lambda)$ is simple iff $\det S_\nu(\lambda) \neq 0$ for all ν.

2.6. **Case $\mathfrak{g} := \mathfrak{po}(0|2n)$.** The algebra $\mathfrak{po}(0|2n)$ admits a $\mathbb{Z}$-grading

$$\mathfrak{g} = \oplus_{i=-2}^{2n-2} \mathfrak{g}_i$$

which is obtained from the natural grading on $\Lambda(2n)$ by the shift by -2.

2.6.1. We choose generators $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ of $\Lambda(2n)$ in such a way that

$$[f,g] = (-1)^{p(f)+1} \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i}.$$

2.6.2. Set $I := \{1, \dots, n\}$ and for each $J \subset I$ define

$$h_J := \prod_{i \in J} \xi_i \eta_i.$$

The reader can check that the span of h_J is a Cartan subalgebra of $\mathfrak{g}$ which we denote by $\mathfrak{h}$. If $h_J \in \mathfrak{g}_i$ for $i \neq 0$, $\operatorname{ad} h$ is nilpotent. Therefore the set of roots Δ can be realized as a subset of $\mathfrak{h}_0^*$ where the embedding $\mathfrak{h}_0^* \subset \mathfrak{h}_{\overline{0}}^*$ comes from the decomposition $\mathfrak{h}_{\overline{0}} = \mathfrak{h}_{-2} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_2 \oplus \ldots \oplus \mathfrak{h}_{2n-2}$.

Using the standard notation for $\mathfrak{g}_0 = \mathfrak{so}(n)$, one obtains

$$\Delta = \{\pm\varepsilon_{i_1} \pm \ldots \pm \varepsilon_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.$$

Clearly, $\mathfrak{h}_{-2}$ is spanned by $h_\emptyset \in \Lambda^0(2n)$ and coincides with the centre of $\mathfrak{g}$. Define triangular decompositions as in 2.2.

2.6.3. **Example.** Take $h := 2^{n-1}\varepsilon_1^* + 2^{n-2}\varepsilon_2^* + \ldots + \varepsilon_n^*$. Then

$$\Delta^+ = \{\varepsilon_{i_1} \pm \varepsilon_{i_2} \pm \ldots \pm \varepsilon_{i_k} : i_1 < i_2 < \ldots < i_k\}.$$

Simple roots are

$$\pi := \{\varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_n, \varepsilon_2 - \ldots - \varepsilon_n, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}.$$

2.6.4. **Example.** For $n = 3$ take $h := 4\varepsilon_1^* + 3\varepsilon_2^* + 2\varepsilon_3^*$. Then

$$\pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, -\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}.$$

2.6.5. The group of signed permutations of $\{1, \ldots, n\}$ is a subgroup of $\mathrm{Aut}\,\mathfrak{g}$: a non-signed permutation acts as permutation of indexes and the permutation $i \mapsto -i$ corresponds to the interchange $\xi_i \leftrightarrow \eta_i$. As a consequence, any root is simple with respect to a suitable triangular decomposition (such a decomposition can be obtained from one in the first example by the action of a signed permutation).

2.7. **Casimir element C.** Let C be the quadratic Casimir element corresponding to the non-degenerate bilinear form B defined in 2.1. Clearly, C has degree $2n-4$ with respect to the $\mathbb{Z}$-grading defined in 2.6. One easily sees that

$$\mathrm{HC}(C) = \sum_{J \subset I} h_J h_{I \setminus J} + h_C \quad \text{for some } h_C \in \mathfrak{h}_{2n-4}. \tag{1}$$

3. Shapovalov determinants for $\mathfrak{po}(0|2n)$, $n > 2$.

Recall that any root is of the form $\alpha = \sum s_i \varepsilon_i$ where $s_i \in \{-1, 0, 1\}$; for $\alpha \in \Delta^+$ set

$$h_\alpha := \sum_{j \in I} h_j(\alpha) h_{I \setminus \{j\}} = \sum_{j \in I} s_j h_{I \setminus \{j\}}.$$

Notice that $h_\alpha \in \mathfrak{h}^*_{2n-4}$.

In this section we will prove the following formula.

3.1. Theorem.

$$\det S_\nu = \prod_{\alpha \in \Delta_0^+} h_\alpha^{\dim \mathfrak{g}_\alpha \sum_{m=1}^\infty \tau(\nu - m\alpha)} \prod_{\alpha \in \Delta_1^+} h_\alpha^{\dim \mathfrak{g}_\alpha \sum_{\mathrm{m}=1}^\infty (-1)^{\mathrm{m}+1} \tau(\nu - \mathrm{m}\alpha)}.$$

Proof. Fix $\nu \in Q^+$. Recall that $\det S_\nu(\lambda) = 0$ iff $M(\lambda)$ has a primitive vector of weight $\lambda - \mu$ for some $0 < \mu \leq \nu$. Therefore

$$\det S_\nu(\lambda) = 0 \implies \mathrm{HC}(C)(\lambda) = \mathrm{HC}(C)(\lambda - \mu) \text{ for some } 0 < \mu \leq \nu.$$

Combining the formula (1) and the fact that $h(\mu) = 0$ if $h \in \mathfrak{h}$ has a non-zero degree and $\mu \in Q^+$, we obtain

$$\mathrm{HC}(C)(\lambda) - \mathrm{HC}(C)(\lambda - \mu) = 2\sum_{J \subset I} h_J(\mu) h_{I \setminus J}(\lambda) - \sum_{J \subset I} h_J(\mu) h_{I \setminus J}(\mu) = 2\sum_{j \in I} h_j(\mu) h_{I \setminus \{j\}}(\lambda).$$

Therefore

$$\det S_\nu(\lambda) = 0 \implies h_\mu(\lambda) = 0 \text{ for some } 0 < \mu \leq \nu$$

where $h_\mu := \sum_{j \in I} h_j(\mu) h_{I \setminus \{j\}}$. In other words, all zeros of the polynomial $\det S_\nu$ lie in the union of hyperplanes $h_\mu = 0$. Hence $\det S_\nu = \prod_{0 < \mu \leq \nu} h_\mu^{d_\mu(\nu)}$ for some $d_\mu(\nu) \geq 0$. In particular, $\det S_\nu$ is homogeneous and thus coincides with its leading term. Now Theorem 4.2 reduces the assertion to the formula $\det D_\alpha = h_\alpha^{\dim \mathfrak{g}_\alpha}$.

To prove the last formula, recall that D_α is a matrix of the natural pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathfrak{h}$. This matrix does not depend on a triangular decomposition. Since any root is simple with respect to a certain triangular decomposition, we can assume that α is simple that is $D_\alpha = S_\alpha$. Then the above reasoning gives $\det D_\alpha = h_\alpha^{d_\alpha(\alpha)}$. Observe that the entries of D_α lie in $\mathfrak{h}$ and so the degree of $\det D_\alpha$ is $\dim \mathfrak{g}_\alpha$; hence $d_\alpha(\alpha) = \dim \mathfrak{g}_\alpha$ as required. □

3.2. Corollary.

(i) *A Verma module $M(\lambda)$ is simple if and only if $h_\alpha(\lambda) \neq 0$ for all $\alpha \in \Delta^+$.*
(ii) *A Verma module $M(\lambda)$ contains a primitive vector of weight $\lambda - \alpha$ if $h_\alpha(\lambda) = 0$.*

4. The leading term of a Shapovalov determinant.

Let $\mathfrak{g}$ be a Lie superalgebra with a fixed triangular decomposition such that

(i) the Cartan subalgebra is even and commutative;
(ii) $\dim \mathcal{U}(\mathfrak{n}^+)_\nu = \dim \mathcal{U}(\mathfrak{n}^-)_{-\nu} < \infty$ for all $\nu \in Q^+$.

Define Shapovalov determinants as in 2.5.2. In this section we compute the leading term of Shapovalov determinants for such algebras.

4.1. Retain notation of 2.2,2.3. The Kostant partition function $\tau : Q \to \mathbb{Z}_{\geq 0}$ is defined by the formula

$$\mathrm{ch}\,\mathcal{U}(\mathfrak{n}^-) = \prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha}) \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})^{-1} =: \sum_{\eta \in Q} \tau(\eta) e^{-\eta}.$$

Note that $\tau(Q \setminus Q^+) = 0$.

4.2. Theorem. *The leading term of* $\det S_\nu$ *is equal to*

$$\prod_{\alpha\in\Delta_0^+}(\det D_\alpha)^{\sum_{m=1}^{\infty}\tau(\nu-m\alpha)}\prod_{\alpha\in\Delta_1^+}(\det D_\alpha)^{\sum_{m=1}^{\infty}(-1)^{m+1}\tau(\nu-m\alpha)}$$

up to a non-zero scalar.

4.3. Proof. Denote by $\tilde{\Delta}_0^+,\tilde{\Delta}_1^+$ the corresponding multisets of roots (the multiplicity of α is equal to $\dim\mathfrak{g}_\alpha$). Set $\tilde{\Delta}^+:=\tilde{\Delta}_0^+\cup\tilde{\Delta}_1^+$.

Definition. A vector $\mathbf{k}=\{k_\alpha\}_{\alpha\in\tilde{\Delta}^+}$ is called a *partition of* $\nu\in Q^+$ if

$$\nu=\sum_{\alpha\in\tilde{\Delta}^+}k_\alpha\alpha;\quad k_\alpha\in\mathbb{Z}_{\geq 0}\text{ for }\alpha\in\tilde{\Delta}_0^+\text{ and }k_\alpha\in\{0,1\}\text{ for }\alpha\in\tilde{\Delta}_1^+.$$

Denote by $\mathcal{P}(\nu)$ the set of all partitions of ν. Clearly, $|\mathcal{P}(\nu)|=\tau(\nu)$.

4.3.1. Set $|\mathbf{k}|:=\sum_{\alpha\in\tilde{\Delta}^+}k_\alpha$. Take $\alpha\in\Delta^+$ and let $\alpha^{(i)}:i=1,\dots,\dim\mathfrak{g}_\alpha$ be the corresponding elements of the multiset $\tilde{\Delta}^+$. Denote by $\mathbf{k}_\alpha$ the subpartition $\mathbf{k}_\alpha:=(k_\alpha^{(i)}:i=1,\dots,\dim\mathfrak{g}_\alpha)$.

Define an equivalence relation on $\mathcal{P}(\nu)$ by setting $\mathbf{k}\approx\mathbf{m}$ if $|\mathbf{k}_\alpha|=|\mathbf{m}_\alpha|$ for all α. Thus the equivalence classes are indexed by vectors $\kappa=(\kappa_\alpha:\alpha\in\Delta^+)$ where $\mathbf{k}\in\kappa$ iff $|\mathbf{k}_\alpha|=\kappa_\alpha$ for all α. Set

$$\operatorname{supp}\kappa=\{\alpha\in\Delta^+:\kappa_\alpha\neq 0\}$$

and define $\operatorname{supp}\mathbf{k}$ similarly. Set

$$\mathcal{P}(r,\alpha):=\{\mathbf{k}|\ \operatorname{supp}\mathbf{k}=\{\alpha\},|\mathbf{k}|=r\}$$

and denote by $p(r,\alpha)$ the cardinality of $\mathcal{P}(r,\alpha)$.

4.3.2. Fix a total ordering on $\tilde{\Delta}^+$ compatible with the standard partial ordering on $\mathfrak{h}^*$. Fix bases $\{e_\alpha:\alpha\in\tilde{\Delta}^+\}$ of $\mathfrak{n}^+$ and $\{f_\alpha:\alpha\in\tilde{\Delta}^+\}$ of $\mathfrak{n}^-$ where e_α (resp., f_α) has weight α (resp., $-\alpha$). For every $\mathbf{k}\in\mathcal{P}(\nu)$ set

$$\mathbf{f}^{\mathbf{k}}:=\prod_{\alpha\in\tilde{\Delta}^+}f_\alpha^{k_\alpha}$$

where the factors are arranged with respect to the total ordering: the first factor corresponds to the minimal root. Define $\mathbf{e}^{\mathbf{k}}$ by the similar formula but with factors arranging in the reverse order. The sets $\{\mathbf{f}^{\mathbf{k}}:\ \mathbf{k}\in\mathcal{P}(\nu)\}$ and $\{\mathbf{e}^{\mathbf{k}}:\ \mathbf{k}\in\mathcal{P}(\nu)\}$ form PBW bases of $\mathcal{U}(\mathfrak{n}^-)_{-\nu}$ and $\mathcal{U}(\mathfrak{n}^+)_\nu$ respectively. Let S_ν be the matrix of Shapovalov form written in these bases: its columns and rows are indexed by the partitions $\mathbf{k}\in\mathcal{P}(\nu)\}$ and the $(\mathbf{k},\mathbf{m})$th entry is $\mathrm{HC}(\mathbf{e}^{\mathbf{k}}\mathbf{f}^{\mathbf{m}})$.

4.4. Let A, B be two square matrices. One can naturally define $A \otimes B$ as the matrix of the corresponding linear operator.

On the other hand, view B as a matrix of bilinear form on V and define

$$\begin{array}{l}\tilde{S}^k(B)(v_1 \otimes \ldots \otimes v_k, v'_k \otimes \ldots \otimes v'_1) := \sum_{\sigma \in S_k} \prod_{i=1}^k B(v_i, v'_{\sigma(i)}),\\ \tilde{\Lambda}^k(B)(v_1 \otimes \ldots \otimes v_k, v'_k \otimes \ldots \otimes v'_1) := \sum_{\sigma \in S_k} (-1)^{\operatorname{sgn}\sigma} \prod_{i=1}^k B(v_i, v'_{\sigma(i)}).\end{array}$$

Now define $S^k(B)$ and $\Lambda^k(B)$ as the restrictions of $\tilde{S}^k(B)$ and $\tilde{\Lambda}^k(B)$ to $S^k(V)$ and $\Lambda^k(V)$ respectively.

4.4.1. Let C be an $m \times m$ matrix with entries in $S(\mathfrak{h})$. For each $\sigma \in S_m$ let $\deg(C, \sigma)$ be the degree of $\prod_1^m c_{i\sigma(i)}$; put $\deg(C) := \max_\sigma \deg(C, \sigma)$ and denote by $\det' C$ the term of degree $\deg(C)$ in the polynomial $\det C$. Thus $\det' C$ is either zero or equal to the leading term of $\det C$.

4.5. Fix $\alpha \in \Delta^+$. Let $D_{m\alpha}$ be the submatrix of the Shapovalov matrix $S_{m\alpha}$ formed by the entries whose both coordinates lie in $\mathcal{P}(m, \alpha)$. For $m = 1$ this definition gives the same matrix as was defined in 2.3. Observe that $D_{m\alpha} = S_{m\alpha}$ if α is simple. Recall that all entries of D_α has degree one and so $\det' D_\alpha = \det D_\alpha$.

By Lemma 4.6.1 the leading terms of the entries of $D_{m\alpha}$ form the matrix $S^m(D_\alpha)$ if α is even and the matrix $\Lambda^m(D_\alpha)$ if α is odd. Consequently,

$$\det{}' D_{m\alpha} = \begin{cases} \det S^m(D_\alpha) & \text{if } \alpha \text{ is even,} \\ \det \Lambda^m(D_\alpha) & \text{if } \alpha \text{ is odd.} \end{cases}$$

Notice that for any square matrix A one has $\det S^m(A) = c(\det A)^{\frac{ms(S^m(A))}{s(A)}}$ where $c \in \mathbb{Z}_{>0}$ and $s(B)$ stands for the size of a matrix B; $\det \Lambda^m(A)$ has the similar formula. Hence, up to a non-zero constant, one has

$$\det{}' D_{m\alpha} = (\det D_\alpha)^{\frac{mp(m,\alpha)}{\dim \mathfrak{g}_\alpha}}. \tag{2}$$

4.6. By 4.6.2 the degrees of the entries of $\mathbf{k}$th row (resp., column) of a Shapovalov matrix S_ν is not greater than $|\mathbf{k}|$. Moreover, if $|\mathbf{k}| = |\mathbf{m}|$ the degree of $(\mathbf{k}, \mathbf{m})$th entry is less than $|\mathbf{k}|$ if $\mathbf{k} \not\approx \mathbf{m}$. Finally, if $\mathbf{k} \approx \mathbf{m}$ then the leading term of $(\mathbf{k}, \mathbf{m})$th entry coincides with the leading term of $c_{\mathbf{k},\mathbf{m}} := \prod_{\alpha \in \Delta^+} \mathrm{HC}(\mathbf{e}^{\mathbf{k}_\alpha} \mathbf{f}^{\mathbf{m}_\alpha})$; note that $\mathrm{HC}(\mathbf{e}^{\mathbf{k}_\alpha} \mathbf{f}^{\mathbf{m}_\alpha})$ is an entry of the matrix $D_{|\mathbf{k}_\alpha|\alpha}$.

As a consequence, $\deg(S_\nu) = \sum_{\mathbf{k} \in \mathcal{P}(\nu)} |\mathbf{k}|$ and $\det' S_\nu = \det' C_\nu$ where $C_\nu = (c_{\mathbf{k},\mathbf{m}})_{\mathbf{k},\mathbf{m} \in \mathcal{P}(\nu)}$ and $c_{\mathbf{k},\mathbf{m}}$ is given by the above formula for $\mathbf{k} \approx \mathbf{m}$, $c_{\mathbf{k},\mathbf{m}} = 0$ for $\mathbf{k} \not\approx \mathbf{m}$. Thus C_ν is a block matrix with the blocks indexed by the equivalence classes of partitions; the block indexed by $\kappa = (\kappa_\alpha)$ is the tensor product of the matrices $D_{\kappa_\alpha \alpha}$ for all $\alpha \in \Delta^+$.

Observe that $\det(A\otimes B)=(\det A)^{s(B)}(\det B)^{s(A)}$. Using the formula (2) we get

$$\det{}' S_\nu=\prod_{\kappa}\prod_{\alpha\in\Delta^+}\left(\det{}' D_{\kappa_\alpha\alpha}\right)^{\prod_{\beta\neq\alpha}p(\kappa_\beta,\beta)}=\prod_{\alpha\in\Delta^+}(\det D_\alpha)^{d(\alpha)}$$

where

$$d(\alpha)=\sum_{\kappa}\frac{\kappa_\alpha\prod_\beta p(\kappa_\beta,\beta)}{\dim\mathfrak{g}_\alpha}=\frac{1}{\dim\mathfrak{g}_\alpha}\sum_{\mathbf{k}\in\mathcal{P}(\nu)}|\mathbf{k}_\alpha|$$

since $\prod_\beta p(\kappa_\beta,\beta)$ is equal to the cardinality of κ. Now Lemma 4.6.3 completes the proof of Theorem 4.2. □

4.6.1. Lemma. *The leading terms of the entries of $D_{m\alpha}$ form the matrix $S^m(D_\alpha)$ if α is even and the matrix $\Lambda^m(D_\alpha)$ if α is odd.*

4.6.2. Lemma. *Take $\nu\in Q^+$ and $\mathbf{k},\mathbf{m}\in\mathcal{P}(\nu)$. Then*

(i) $\deg\mathrm{HC}(\mathbf{e}^{\mathbf{k}}\mathbf{f}^{\mathbf{m}})\leq\min(|\mathbf{k}|,|\mathbf{m}|)$.
(ii) *Assume that* $\deg\mathrm{HC}(\mathbf{e}^{\mathbf{k}}\mathbf{f}^{\mathbf{m}})=|\mathbf{k}|=|\mathbf{m}|$. *Then*

$$\mathbf{k}\approx\mathbf{m}$$

and the leading term of $\mathrm{HC}(\mathbf{e}^{\mathbf{k}}\mathbf{f}^{\mathbf{m}})$ *is equal to the leading term of*

$$\prod_{\alpha\in\Delta^+}\mathrm{HC}(\mathbf{e}^{\mathbf{k}_\alpha}\mathbf{f}^{\mathbf{m}_\alpha}).$$

Proof is by induction on $\nu\in Q^+$.

4.6.3. Lemma.

(i) *For any $\alpha\in\Delta_0^+$ one has*

$$\sum_{\mathbf{k}\in\mathcal{P}(\nu)}|\mathbf{k}_\alpha|=\dim\mathfrak{g}_\alpha\sum_{m=1}^{\infty}\tau(\nu-m\alpha).$$

(ii) *For any $\alpha\in\Delta_1^+$ one has*

$$\sum_{\mathbf{k}\in\mathcal{P}(\nu)}|\mathbf{k}_\alpha|=\dim\mathfrak{g}_\alpha\sum_{m=1}^{\infty}(-1)^{m+1}\tau(\nu-m\alpha).$$

Proof. Recall that $|\mathbf{k}_\alpha|=\sum_{i=1}^{\dim\mathfrak{g}_\alpha}k_\alpha^{(i)}$. For each i the formula

$$\sum_{\mathbf{k}\in\mathcal{P}(\nu)}k_\alpha^{(i)}=\sum_{m=1}^{\infty}\tau(\nu-m\alpha)$$

for $\alpha\in\Delta_0^+$ and a similar formula for $\alpha\in\Delta_1^+$ can be obtained by a standard reasonings (see, for instance [G2], 3.3.1). □

5. The case $\mathfrak{po}(0|4)$.

5.1. For the Lie superalgebra $\mathfrak{g} := \mathfrak{po}(0|4)$ all triangular decompositions are conjugated. We fix a triangular decomposition with the following positive roots: $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1, \varepsilon_2$. One easily sees that $\mathrm{HC}(C) = 2(h_\emptyset h_{1,2} + h_1 h_2 - h_1)$ where h_1 stands for $h_{\{1\}}$ and other notations are similar.

5.2. The even roots $\varepsilon_1 \pm \varepsilon_2$ have multiplicity one and $D_{\varepsilon_1 \pm \varepsilon_2} = \pm h_1 + h_2$. The odd roots $\varepsilon_1, \varepsilon_2$ have multiplicity two. To compute D_{ε_2} notice that the weight space $\mathfrak{g}_{\varepsilon_2}$ (resp., $\mathfrak{g}_{-\varepsilon_2}$) has a basis $\{\xi_2, \xi_1\eta_1\xi_2\}$ (resp., $\{\eta_2, \xi_1\eta_1\eta_2\}$). The matrix D_{ε_2} written in these bases takes form

$$D_{\varepsilon_2} = \left(\begin{array}{c|c} h_\emptyset & h_1 \\ \hline h_1 & 0 \end{array}\right)$$

and so $\det D_{\varepsilon_2} = -h_1^2$; similarly $\det D_{\varepsilon_1} = -h_2^2$. By Theorem 4.2, the leading term of $\det S_\nu$ is, up to a non-zero scalar, equal to

$$(h_1 - h_2)^{d(\nu)} (h_1 + h_2)^{d'(\nu)} h_1^{c_2(\nu)} h_2^{c_1(\nu)}$$

where $d(\nu) := \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 - \varepsilon_2))$, $d'(\nu) := \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 + \varepsilon_2))$ and $c_i := 2\sum_{m=1}^\infty (-1)^{m+1} \tau(\nu - m\varepsilon_i)$.

5.3. Arguing as in 3.1, we conclude that all Shapovalov determinants admit linear factorizations and factors of $\det S_\nu$ are of the form $h_2(\mu)h_1 + h_1(\mu)h_2 - h_1(\mu)h_2(\mu) - h_1(\mu)$ where $0 < \mu \leq \nu$. Comparing with the above expression of the leading term we conclude that

$$\det S_\nu = \prod_{k=1}^\infty (h_2 - h_1 + k - 1)^{d_k} (h_2 + h_1 - k - 1)^{d'_k} h_1^{c_2(\nu)} (h_2 - 1)^{c_1(\nu)}$$

where the multiplicities d_k, d'_k are non-negative integers which satisfy the conditions

$$\sum_k d_k = d(\nu) = \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 - \varepsilon_2)), \quad \sum_k d'_k = d'(\nu) = \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 + \varepsilon_2))$$

(in particular, only finitely many multiplicities are non-zero and thus the above product is finite). Now the standard reasoning based on a use of Jantzen filtration gives $d_k = \tau(\nu - k(\varepsilon_1 - \varepsilon_2))$ and $d'_k = \tau(\nu - k(\varepsilon_1 + \varepsilon_2))$. Finally, up to a non-zero scalar, one has

$$\det S_\nu = \prod_{k=1}^\infty \quad (h_2 - h_1 + k - 1)^{\tau(\nu - k(\varepsilon_1 - \varepsilon_2))} (h_2 + h_1 - k - 1)^{\tau(\nu - k(\varepsilon_1 + \varepsilon_2))}$$
$$h_1^{2\sum_{m=1}^\infty (-1)^{m+1} \tau(\nu - m\varepsilon_2)} (h_2 - 1)^{2\sum_{m=1}^\infty (-1)^{m+1} \tau(\nu - m\varepsilon_1)}.$$

6. On the Jantzen filtration of a generic reducible Verma module.

The notion of Jantzen filtration on a Verma module was introduced in [Ja] for semisimple Lie algebras. It can be easily extended to superalgebra case. One has to take into account however that the vector ρ is no longer "regular" in a sense that hypersurfaces $\det S_\nu = 0$ contain straight lines parallel to ρ so that in the construction of the Jantzen filtration one should use a/any regular vector $\rho' \in \mathfrak{h}^*$ instead of ρ— see [G3], 7.1 for details.

6.1. Retain notation of 2.4. The Jantzen filtration on $M(\lambda)$ is a decreasing filtration with the following properties:

$$\mathcal{F}^0(M(\lambda)) = M(\lambda), \quad \mathcal{F}^1(M(\lambda)) = \overline{M}(\lambda), \quad \bigcap_{r=0}^{\infty} \mathcal{F}^r(M(\lambda)) = 0,$$

$$d_\nu(\lambda) = \sum_{r\geq 1} \dim \mathcal{F}^r(M(\lambda))_{\lambda-\nu}, \tag{3}$$

where $d_\nu(\lambda)$ is the order of zero of the polynomial $\det S_\nu$ at the point λ (if $\det S_\nu = \prod p_i^{r_i}$ where p_i are irreducible then $d_\nu(\lambda) = \sum_{i:p_i(\lambda)=0} r_i$). The formula (3) is proven in [Ja] and is called "sum fomula".

For $M(\lambda)$ being simple one has $\mathcal{F}^1(M(\lambda)) = 0$. For basic classical (except $\mathfrak{psl}(2|2)$) or Q-type Lie superalgebras the Jantzen filtration has length two, i.e. $\mathcal{F}^2(M(\lambda)) = 0$, if $M(\lambda)$ is a "generic" reducible Verma module. More precisely, $\mathcal{F}^2(M(\lambda)) = 0$ if λ lies on exactly one of irreducible components of a hypersurface $\det S_\nu = 0$. Remarkably, this is far from being true in our case. We demonstrate this phenomenon on some examples below.

Set $\mathfrak{g} := \mathfrak{po}(0|2n)$. In the examples below we assume that

$$\lambda \text{ is a generic point of the hyperplane } h_\alpha = k, \tag{4}$$

where $h_\alpha = k$ is an irreducible component of a hypersurface $\det S_\nu = 0$. For $n > 2$ one has $k = 0$ and genericity means that $h_\beta(\lambda) \neq 0$ for $\beta \in \Delta^+, \beta \neq \alpha$.

Denote by v_λ the highest weight vector of $M(\lambda)$.

6.2. The algebra $\mathfrak{g} = \mathfrak{po}(0|4)$. If $\dim \mathfrak{g}_\alpha = 1$ (i.e., $\alpha = \varepsilon_1 \pm \varepsilon_2$) and λ satisfies (4) one can easily deduce from the sum formula (3) that $\mathcal{F}^2(M(\lambda)) = 0$.

6.2.1. *The case $\alpha := \varepsilon_2$.* Since ε_2 is simple, one has $S_{\varepsilon_2} = D_{\varepsilon_2}$ (see 5.2 for the explicit formula). One has $\dim \mathcal{U}(\mathfrak{n}^-)_{-2\varepsilon_2} = 1$ and the Shapovalov matrix $S_{2\varepsilon_2}$ is equal to h_1^2.

Let λ satisfy (4); since $h_\alpha = h_1$ one has $h_1(\lambda) = 0$. Set $f_{\varepsilon_2} := \xi_1\eta_1\eta_2$. If $h_\emptyset(\lambda) \neq 0$ the vector $f_{\varepsilon_2} v_\lambda$ lies in $\mathcal{F}^2(M(\lambda))$. Now using the "genericity" of λ one can deduce from the sum formula (3) that $f_{\varepsilon_2}v_\lambda$ generates $\mathcal{F}^1(M(\lambda)) = \mathcal{F}^2(M(\lambda))$ and that $\mathcal{F}^3(M(\lambda)) = 0$. Note that a Jordan-Hölder series of $M(\lambda)$ has length two.

If $h_\emptyset(\lambda) = 0$ the term $\mathcal{F}^1(M(\lambda)$ is generated by $M(\lambda)_{\lambda-\alpha}$ ($\mathcal{F}^1(M(\lambda)$ is isomorphic to the sum of two quotients of $M(\lambda-\alpha)$) and $\mathcal{F}^2(M(\lambda)) \cong V(\lambda - 2\alpha)$; one has $\mathcal{F}^3(M(\lambda)) = 0$ as before.

Hence in a generic point of the hyperplane $h_1 = 0$ the Jantzen filtration has length three and $\mathcal{F}^1(M(\lambda)) = \mathcal{F}^2(M(\lambda))$ iff $h_\emptyset(\lambda) = 0$.

6.3. The algebra $\mathfrak{g} = \mathfrak{po}(0|2n)$, $n > 2$.

6.3.1. Claim. *Let α be a simple even root and λ be such that $h_\alpha(\lambda) = 0$. Then the Jantzen filtration of $M(\lambda)$ is infinite.*

Proof. Fix any homogeneous (with respect to the $\mathbb{Z}$-grading) bases in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. The matrix $S_\alpha = D_\alpha$ written in these bases has a column with only non-zero entry: this column corresponds to $f_\alpha \in \mathfrak{g}_{-\alpha}$ having the maximal degree and the non-zero entry corresponds to $e_\alpha \in \mathfrak{g}_\alpha$ having the minimal degree; the non-zero entry is equal to h_α. As a consequence, the matrix $S_{k\alpha}$ also has a column with only non-zero entry: this column corresponds to f_α^k and the entry is h_α^k. This gives $f_\alpha^k v_\lambda \in \mathcal{F}^k(M(\lambda))$. Hence the Jantzen filtration is infinite. $\square$

6.3.2. Notice that a submodule generated by $f_\alpha^k v_\lambda$ is isomorphic to $M(\lambda - k\alpha)$; denote this submodule by M_k. Clearly, $M_k \subset \mathcal{F}^k(M(\lambda))$.

If $\dim \mathfrak{g}_\alpha = 1$, the sum formula (3) implies that $\mathcal{F}^k(M(\lambda)) = M_k$ for λ satisfying (4).

If $\dim \mathfrak{g}_\alpha > 1$ one has $\mathcal{F}^k(M(\lambda)) \neq M_k$ for $k = 1$ or for $k = 2$, since the sum formula gives $\sum_{r\geq 1} \dim \mathcal{F}^r(M(\lambda))_{\lambda-\alpha} = \dim \mathfrak{g}_\alpha$.

For example, let α be a simple even root and $\dim \mathfrak{g}_\alpha = 2$. Then

$$S_\alpha = \begin{pmatrix} h' & | & h_\alpha \\ -- & - & -- \\ h_\alpha & | & 0 \end{pmatrix}$$

for some $h' \in \mathfrak{h}^*_{2n-6}$. If $h_\alpha(\lambda) = 0$ and $h' \neq 0$ one has $f_\alpha v_\lambda \in \mathcal{F}^2(M(\lambda))$ that is $M_1 \subset \mathcal{F}^2(M(\lambda))$. However, a natural guess that $M_1 \subset \mathcal{F}^{\dim \mathfrak{g}_\alpha}(M(\lambda))$ is wrong. The example $\dim \mathfrak{g}_\alpha = 4$ shows that in this case $\mathcal{F}^1(M(\lambda))_{\lambda-\alpha} = \mathcal{F}^2(M(\lambda)))_{\lambda-\alpha}$ is a two dimensional subspace and so $\mathcal{F}^3(M(\lambda))_{\lambda-\alpha} = 0$; in particular, M_1 lies in $\mathcal{F}^2(M(\lambda))$ and does not lie in $\mathcal{F}^3(M(\lambda))$.

6.4. Element T. The enveloping algebra of $\mathfrak{g} := \mathfrak{po}(0|2n)$ contains a special even element T which commutes with the even elements of $\mathfrak{g}$ and anticommutes with the odd one, see [G1]. Recall that $U(\mathfrak{g})$ admits the canonical filtration and that the associated graded algebra is $S(\mathfrak{g})$. The algebra $S(\mathfrak{g})$ contains $\Lambda\mathfrak{g}_{\bar{1}}$. It turns out that the image of T in

$S(\mathfrak{g})$ belongs to $\Lambda^{\rm top}\mathfrak{g}_{\bar 1}$. These conditions (commutational relations and $\operatorname{gr} T \in \Lambda^{\rm top}\mathfrak{g}_{\bar 1}$) determines T up to a non-zero scalar. If $\mathfrak{g}$ has a $\mathbb{Z}$-grading then the degree of T is equal to the degree of $\Lambda^{\rm top}\mathfrak{g}_{\bar 1}$.

The element T acts on a Verma module in the following way: it acts by $\mathrm{HC}(T)(\lambda)\,\mathrm{id}$ on the $\mathbb{Z}_2$-homogeneous component containing a highest weight vector and by $-\mathrm{HC}(T)(\lambda)\,\mathrm{id}$ on another $\mathbb{Z}_2$-homogeneous component.

6.4.1. Take $n > 2$. By Corollary 3.2 (ii), $M(\lambda)$ contains a primitive vector of weight $\lambda-\alpha$ if $h_\alpha(\lambda)=0$. One can deduce from this statement that the polynomial $\mathrm{HC}(T)$ is divisible by h_α for $\alpha\in\Delta_1^+$.

Conjecture: $\mathrm{HC}(T)=\prod_{\alpha\in\Delta_1^+} h_\alpha^{\dim\mathfrak{g}_\alpha}$ up to a non-zero scalar for $n>2$.

6.4.2. Claim. *For $\mathfrak{g} := \mathfrak{po}(0|4)$ one has $\mathrm{HC}(T)=h_1^2(h_2-1)^2$ up to a non-zero scalar.*

Proof. First, let us show that $t := \mathrm{HC}(T)$ is divisible by h_1^2. Set $\alpha := \varepsilon_2$ and let f_1, f_2 (resp., e_1, e_2) be a basis of $\mathfrak{g}_{-\alpha_2}$ (resp., $\mathfrak{g}_\alpha$). Write $T = t+\sum_{i,j=1,2} f_i\phi_{ij}e_j+\sum y_r x_r$, where $y_r\in U(\mathfrak{g}), x_r\in\mathfrak{n}^+_{\mu(r)}$ for some $\mu(r)\neq-\alpha_2$. Let v be a primitive vector. Then $Tv=tv$ and $Tf_rv=-f_rTv$ and

$$Tf_rv = tf_rv+\sum_{i,j=1,2} f_i\phi_{ij}e_jf_rv = tf_rv+\sum_{i=1,2} f_i(\Phi S)_{ir}v$$

where $\Phi=(\phi_{ij})$ and $S:=S_\alpha$ is the Shapovalov matrix written with respect to the above base. Putting $f_1:=\eta_2, f_2:=\xi_1\eta_1\eta_2$ we get

$$tf_1 = f_1(t-\frac{\partial t}{\partial h_2})-f_2\frac{\partial t}{\partial h_{12}},\quad tf_2=f_2(t-\frac{\partial t}{\partial h_2}).$$

Hence

$$\Phi S=\begin{pmatrix} -2t+\frac{\partial t}{\partial h_2} & 0\\ \frac{\partial t}{\partial h_{12}} & -2t+\frac{\partial t}{\partial h_2}\end{pmatrix}$$

Now substituting $S=S_\alpha$ (see 5.2) we conclude that t is divisible by h_1^2 (this reflects the fact that for λ being a generic point of the hyperplane $h_1=0$ one has $\mathcal{F}^2(M(\lambda))_{\lambda-\alpha}\neq 0$).

It remains to show that t is divisible by $(h_2-1)^2$. Take λ such that $\lambda(h_1-h_2)=k\in\mathbb{Z}_{\geq 0}$. Then $M(\lambda)$ has a primitive vector of the weight $\lambda-(k+1)(\varepsilon_1-\varepsilon_2)$ and so $t(\lambda)=t(\lambda-(k+1)(\varepsilon_1-\varepsilon_2))$. As a consequence, t is stable under the involution of the algebra $S(\mathfrak{h})$ which acts by id on $\mathfrak{h}_{-2}+\mathfrak{h}_2$ and acts on $\mathfrak{h}_0$ by mapping h_1 to h_2-1. Since t is divisible by h_1^2, t is divisible by $(h_2-1)^2$ as well. The claim follows. $\square$

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