THE PRIME SPECTRUM OF A QUANTUM BRUHAT CELL TRANSLATE

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Abstract. The prime spectra of two families of algebras, S^w and $\check{S}^w$, $w \in W$, indexed by the Weyl group W of a semisimple finitely dimensional Lie algebra $\mathfrak{g}$, are studied in the spirit of [J3]. The algebras S^w have been introduced by A. Joseph (see [J4], Sect. 3). They are q-analogues of the algebras of regular functions on w-translates of the open Bruhat cell of a semisimple Lie group G corresponding to the Lie algebra $\mathfrak{g}$.

We define a stratification of the spectra into components indexed by pairs (y_1, y_2) of elements of the Weyl group satisfying $y_1 \leq w \leq y_2$. Each component admits a unique minimal ideal which is explicitly described. We show the inclusion relation of closures to be that induced by Bruhat order.

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1. Introduction

In this work we study the prime spectra of two families of algebras, S^w and $\check{S}^w$, $w \in W$, indexed by the Weyl group W of a semisimple finitely dimensional Lie algebra $\mathfrak{g}$. The algebras S^w have been introduced by A. Joseph (see [J4], Sect. 3). They are q-analogues of the algebras of regular functions on w-translates of the open Bruhat cell of a semisimple Lie group G corresponding to the Lie algebra $\mathfrak{g}$.

The corresponding classical objects, the algebras of regular functions on different w-translates of the open Bruhat cell, are isomorphic to each other polynomial algebras of rank $\dim \mathfrak{n}^+$.

The q-analogues S^w are much more interesting. For instance, their centres have different Gelfand-Kirillov dimension for different $w \in W$ — see Remark 8.2.2. In particular, S^w are not in general isomorphic for different $w \in W$.

The algebras S^w admit a structure of right $U_q(\mathfrak{g})$ module which comes from the right action of $U_q(\mathfrak{g})$ on the quantum function ring $R_q[G]$. The action of the root torus $T \subseteq U_q(\mathfrak{g})$ on S^w can be naturally extended to an action of the weight torus $\check{T} \supseteq T$. The second family of algebras, $\check{S}^w$, are obtained as the skew-products $\check{S}^w = S^w \# \check{T}$.

The starting point of the construction of the rings S^w is the ring R^+ which is a quantization of the ring of global regular functions on the "base affine space" G/N, see [J4], 1.2. The algebra S^w is obtained as a zero weight space of a localization of R^+. This is why the rings S^w, $\check{S}^w$ are denoted almost everywhere as R^w_0, $\check{R}^w_0$ respectively.

In the case $w = e$ the algebra S^e is isomorphic to the quantized enveloping algebra $U_q(\mathfrak{n}^-)$ of the maximal nilpotent subalgebra $\mathfrak{n}^- \subseteq \mathfrak{g}$ — see [J4], 3.4. The corresponding skew-product algebra $\check{S}^e$ is isomorphic to $\check{U}_q(\mathfrak{b}^-)$.

The prime spectrum of the algebra $\check{S}^e \cong \check{U}_q(\mathfrak{b}^-)$ was described by A. Joseph [J3], Sect.9. It is presented as a disjoint union of locally closed strata $X(w)$ indexed by the elements of the Weyl group. Moreover, the strata $X(w)$ admit an action of a group $\mathbb{Z}_2^l \subseteq \mathrm{Aut}(\check{S}^e)$ and the quotient $X(w)/\mathbb{Z}_2^l$ is isomorphic (as a partially ordered set) to the spectrum of a Laurent polynomial ring.

In this paper we present a similar description (Proposition 5.3.3) of the spectrum of $\check{S}^w$ for arbitrary $w \in W$. In our case the strata $X_w(y,z)$ are indexed by a more complex set: this is the collection

$$W \overset{w}{\diamond} W := \{(y,z) \in W \times W |\ y \leq w \leq z\}$$

where $\leq$ is the Bruhat order. Note that $W \overset{w}{\diamond} W$ inherits an order relation through $(y,z) \succeq (y',z')$ iff $y \leq y',\ z \geq z'$. In Corollary 6.13 we prove that the closure of $X_w(y,z)$ coincides with the union of $X_w(y',z') :\ (y,z) \succeq (y',z')$.

The spectrum of S^w is a union of strata $Y_w(y,z)$ indexed by the same set $W \overset{w}{\diamond} W$ (Proposition 5.3.3). One has also a similar decomposition of a "generic part" $\mathrm{Spec}_+ R^+$

of the spectrum of R^+ (see 5.2,Corollary 5.2.4). Here the strata $X(y,z)$ are indexed by the set

$$W\diamond W := \{(y,z) \in W \times W |\ y \le z\}.$$

The strata $X_w(y,z)$ (resp., $Y_w(y,z)$) are isomorphic for different $w:\ y \le w \le z$ (Proposition 7.2.2). Moreover, $X_w(y,z)$ are all isomorphic to the component $X(y,z)$ of $\operatorname{Spec}_+ R^+$ (Proposition 7.3). It turns out that the component $X(y,z)$ is isomorphic (up to an action of a group $\mathbb{Z}_2^l$) to the spectrum of a Laurent polynomial ring — see Theorem 7.4.4.

The stratum $X(y,z)$ admits a unique minimal element $Q(y,z)$ which we calculate explicitly in Proposition 6.8. We deduce from this that the stratum $Y_w(y,z)$ also admits a unique minimal element $Q(y,z)_w$ which can be expressed through a localization of $Q(y,z)$ (Corollary 6.10.1). Then the unique minimal element of the stratum $X_w(y,z)$ can be written as $Q(y,z)_w\#\check{T}$ — see Corollary 6.10.1. The prime ideals $Q(y,z),\ Q(y,z)_w,\ Q(y,z)_w\#\check{T}$ are completely prime.

In the last Section 8 we calculate the centres of the rings S^w (note that the centres of $\check{S}^w$ are trivial). These are polynomial rings whose dimension depends on $w \in W$.

In the special case $\mathfrak{g} = \mathfrak{sl}_4$ the prime and the primitive spectra of S^w were calculated in [G1]. The results of the first draft of this paper have been announced in [G2].

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2. The rings $S^w,\ \check{S}^w$

2.1. The base field k is assumed to be of characteristic zero and K is an extension of $k(q)$.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantization of $U(\mathfrak{g})$ defined for example in [J1], 3.2.9 whose notation we retain. In this $U_q(\mathfrak{g})$ is a K-algebra generated by $x_i,\ y_i,\ t_i,\ t_i^{-1}\ i = 1,\dots,l$ where l is the rank of $\mathfrak{g}$. Denote the extension of $U_q(\mathfrak{g})$ by the torus $\check{T}$ of weights ([J1], 3.2.10) by $\check{U}_q(\mathfrak{g})$. Consider the subalgebra $U_q(\mathfrak{n}^-)$ generated by the $y_i,\ i = 1,\dots,l$ ([J1], 3.2.10). By [J1], 10.4.9 $U_q(\mathfrak{n}^-)$ admits a structure of a right $U_q(\mathfrak{g})$-module such that:

(1) This module structure is compatible with the algebra structure of $U_q(\mathfrak{n}^-)$ and the coproduct on $U_q(\mathfrak{g})$.

(2) Endowed with this $U_q(\mathfrak{g})$-module structure $U_q(\mathfrak{n}^-)$ is isomorphic to the dual $\delta M(0)$ of the $U_q(\mathfrak{g})$-module Verma ([J1], 5.3) of highest weight zero.

After Lusztig-Soibelman the braid group of $\mathfrak{g}$ acts on $U_q(\mathfrak{g})$ by automorphisms r_w such that if $\tau(\lambda)$ is an element of the torus T and $\overline{w}$ is the image of w in the Weyl group W of $\mathfrak{g}$ then:

$$r_w\tau(\lambda) = \tau(\overline{w}\lambda).$$

Fix an element $\overline{w}$ of the Weyl group and let w be a representative of $\overline{w}$ in the braid group. The automorphism r_w acts on the category of $U_q(\mathfrak{g})$-modules by transport of structure. Denote $(\delta M(0))^{r_w}$ by S^w. As noted in [J1], 10.4.9 the $\check{T}$-character of S^w is given by the formula

$$\operatorname{ch} S^w = w\left(\prod_{\beta\in\Delta^-}(1-e^\beta)^{-1}\right) = \prod_{\beta\in w\Delta^-}(1-e^\beta)^{-1}.$$

Suppose ψ is an automorphism of $U_q(\mathfrak{g})$ such that the module $(\delta M(0))^\psi$ has the same character as S^w. Then the module $N = (\delta M(0))^{r_w^{-1}\psi}$ has the same character as $\delta M(0)$. Since N is obtained from $\delta M(0)$ by transport of structure the following property of $\delta M(0)$ holds also for N: if v_0 is a vector of weight zero and v is a vector of N then v_0 belongs to the submodule generated by v. Hence the dual module δN is generated by a highest weight vector. Yet it is also has the same character as the Verma module $M(0)$, so δN is isomorphic to $M(0)$, N is isomorphic to $\delta M(0)$ and $(\delta M(0))^\psi$ is isomorphic to $(\delta M(0))^{r_w}$. Hence the $U_q(\mathfrak{g})$-module S^w depends only on the class $\overline{w}$ of w in the Weyl group W of $\mathfrak{g}$.

According to [J1], 10.2.9, S^w admits the structure of a $U_q(\mathfrak{g})$-algebra and this further extends to a $\check{U}_q(\mathfrak{g})$ -algebra structure. Moreover one checks that the $\check{U}_q(\mathfrak{g})$ -algebra structure on the module S^w is uniquely determined up to a scalar by its module structure and the requirement that a non-zero vector of weight zero is the identity of the ring (see also [K], prop. 3.2). The automorphism r_w is an algebra automorphism but it does not preserve the coalgebra structure of $U_q(\mathfrak{g})$. Thus one should not expect that the algebras S^w are isomorphic for different elements $\overline{w} \in W$. Rather we obtain a collection of $U_q(\mathfrak{g})$-algebras parametrized by W which are generally distinct. Trying to understand the possible isomorphisms between them was a main motivation for our present work. Our results suggest that S^w is isomorphic to $S^{w'}$ iff $W \overset{w}{\diamond} W$ and $W \overset{w'}{\diamond} W$ are isomorphic as ordered sets.

2.2. Let w_0 be the longest element of the Weyl group. Consider the involution ψ of the algebra $U_q(\mathfrak{g})$ defined by

$$\psi(x_i) = -y_i \qquad \psi(t_i) = t_i^{-1}.$$

Then by the character formula of 2.1 one has

$$\operatorname{ch}(S^w)^\psi = \operatorname{ch} S^{ww_0}.$$

By the reasoning of 2.1 the modules $(S^w)^\psi$ and S^{ww_0} are isomorphic and hence are isomorphic as algebras. The map ψ is an algebra automorphism and coalgebra antiautomorphism. The last implies that the $U_q(\mathfrak{g})$-algebras S^w and $(S^w)^\psi$ have opposite algebra structures. Hence the algebras S^w and S^{ww_0} are opposites.

2.3. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and let $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be the corresponding set of simple roots. Let $Q(\pi) = \mathbb{Z}\pi$, $Q^{\pm}(\pi) = \pm\mathbb{N}\pi$, $P(\pi)$ (resp., $P^+(\pi)$) be the set of weights (resp., dominant weights) and $\{\omega_i\}_{i=1}^l$ be the set of fundamental weights. Define an order relation on $P(\pi)$ by $\mu \geq \nu$ if $\mu - \nu \in Q^+(\pi)$. Let τ be the isomorphism of the additive group $Q(\pi)$ to themultiplicative group T defined by $\tau(\alpha_i) = t_i,\ i = 1, \dots, l$. We can extend τ to the isomorphism of $P(\pi)$ onto $\check{T}$.

For each $\lambda \in P^+(\pi)$ let $V(\lambda)$ be the $U_q(\mathfrak{g})$ module with highest weight λ and $c^\lambda_{\xi,v}$: $\xi \in V(\lambda)^*$, $v \in V(\lambda)$ be the element $a \mapsto \xi(av)$ of $U_q(\mathfrak{g})^*$. Let $R_q[G]$ be the Hopf subalgebra of $U_q(\mathfrak{g})^*$ generated as a vector space by these elements. By [J1], 9.1.1 $R_q[G]$ admits a structure of a $U_q(\mathfrak{g})$-bialgebra.

Let u_λ be a highest weight vector of $V(\lambda)$ and $V^+(\lambda)$ denote the subspace of $R_q[G]$ generated by the $c^\lambda_{\xi,u_\lambda}$: $\xi \in V(\lambda)^*$. Then $R^+ := \oplus_{\lambda \in P^+(\pi)} V^+(\lambda)$ is a subalgebra of $R_q[G]$. Moreover R^+ is a right $U_q(\mathfrak{g})$-submodule and left T-submodule of $R_q[G]$. The left T-action defines a $P^+(\pi)$-grading on R^+. Indeed the weight subspace of weight λ is just $V^+(\lambda)$. Hence $V^+(\lambda)$ is invariant with respect to the right action of $U_q(\mathfrak{g})$ and the multiplication satisfies the Cartan multiplication rule:

$$V^+(\mu)V^+(\lambda) = V^+(\lambda + \mu).$$

Let $\Omega(V^+(\lambda))$ denote the set of weights of $V^+(\lambda)$ for the right T-action counted with their multiplicites. (This is just the set of weights of $V(\lambda)$).

For each $w \in W$ let $\xi_{w\lambda}$ be a vector of the weight $w\lambda$ in $V(\lambda)^*$ viewed as a right $U_q(\mathfrak{g})$ module and write $c^\lambda_{\xi_{w\lambda},u_\lambda}$ (resp., $c^\lambda_{\xi,u_\lambda}$) simply as c^λ_w (resp., c^λ_ξ). The elements c^λ_w are defined up to scalars. By [J1], 9.1.10 these scalars can be chosen so that $c^\mu_w c^\nu_w = c^{\mu+\nu}_w$ for any $\mu, \nu \in P^+(\pi)$ and $c_w = \{c^\lambda_w : \lambda \in P^+(\pi)\}$ becomes an Ore set in R^+ . Extend c^μ_w to $\mu \in P(\pi)$ through $c^{\mu-\nu}_w = c^\mu_w (c^\nu_w)^{-1}\ \ \forall \mu, \nu \in P^+(\pi)$.

Consider the localized algebra $R^w := R^+[c_w^{-1}]$; by [J1], 4.3.12 the right action of $U_q(\mathfrak{g})$ extends to R^w. Since each of c^λ_w is homogeneous it follows that the $P^+(\pi)$-grading on R^+ extends to a $P(\pi)$-grading on R^w; again the homogeneous components are invariant with respect to the right action of $U_q(\mathfrak{g})$. It implies that the zero weight subspace R^w_0 of R^w with repsect to the left action of T is a $U_q(\mathfrak{g})$-subalgebra of R^w and as suggested in [J4], 3.1, it may be viewed as a q-analogue of the algebra of regular functions on the w-translate of the open Bruhat cell. Since R^+ is a domain of finite Gelfand-Kirillov dimension it admits a skew-field of fractions and this contains the $R^w : w \in W$. Again $c^{-\lambda}_w V^+(\lambda) \hookrightarrow c^{-(\lambda+\nu)}_w V^+(\lambda + \nu)\ \ \forall \lambda, \nu \in P^+(\pi)$. Thus one may write

$$R^w_0 = \sum_{\lambda \in P^+(\pi)} c^{-\lambda}_w V^+(\lambda) \cong \varinjlim_{\lambda \in P^+(\pi)} c^{-\lambda}_w V^+(\lambda). \tag{1}$$

This implies that the rings of fractions of R^w_0 coincide for different w.

By [J1], 10.4.8 S^w and R^w_0 are isomorphic as a $U_q(\mathfrak{g})$-algebras.

Denote by $\check{R}_0^w$ the skew-product of R_0^w and the fundamental torus $\check{T}$ through the action of $\check{T}$ on $\check{U}_q(\mathfrak{g})$-module R_0^w— see 3.2.

2.4. By [J4], 6.4, 6.6, R^+ and S^w are left and right noetherian. By [MCR], 2.9 it follows that $\check{R}_0^w$ is also noetherian.

2.5. Set $w = e$. Then S^e is isomorphic to $U_q(\mathfrak{n}^-)$ as a $\check{U}_q(\mathfrak{g})$-algebra. Consider the subalgebra $\check{U}_q(\mathfrak{b}^-)$ of $\check{U}_q(\mathfrak{g})$ which is the skew-product of $U_q(\mathfrak{n}^-)$ and the fundamental torus $\check{T}$. The algebra $\check{U}_q(\mathfrak{b}^-)$ can be also considered as the skew-product of S^e and $\check{T}$ through the action of $\check{T}$ on $\check{U}_q(\mathfrak{g})$-module S^e. By [J1], 10.1.11 it follows that the isomorphism 2.3 of $S^e \simeq U_q(\mathfrak{n}^-)$ with R_0^e extends to an isomorphism of $\check{U}_q(\mathfrak{b}^-)$ with R^e.

By [J3], Sect.10 the prime and primitive spectra of $\check{U}_q(\mathfrak{b}^-)$ take the following form

$$\operatorname{Spec} \check{U}_q(\mathfrak{b}^-) = \coprod_{w \in W} X(w) \ ,$$

$$\operatorname{Prim} \check{U}_q(\mathfrak{b}^-) = \coprod_{w \in W} X^{max}(w) \ ,$$

where each $X(w)$ is the spectrum of some Laurent polynomial ring up to an action of $\mathbb{Z}_2^l$ and all prime ideals are completely prime.

Each $X(w)$ has a unique minimal element $Q(w)$ which has the following nice description in the notation of 2.3. Fix $w \in W$. For each $\lambda \in P^+(\pi)$ let $u_{w\lambda} \in V(\lambda)$ be a vector of the weight $w\lambda$. Denote by $V_w^+(\lambda)^\perp$ the orthogonal of the Demazure module $V_w^+(\lambda) := U_q(\mathfrak{b}^+)u_{w\lambda}$ in $V(\lambda)^*$, the latter identified with $V^+(\lambda)$. Then [J1], 10.1.8

$$Q(w) = \sum_{\lambda \in P^+(\pi)} V_w^+(\lambda)^\perp.$$

2.6. An element x of a ring A is called *normal* if $xA = Ax$. If A is prime a non-zero normal element is regular. Each regular normal element determines an automorphism of the ring sending $a \in A$ to the unique element $b \in A$ such that $xa = bx$.

Let A be a ring, x be an element of A and c be a subset of A. Suppose that the multiplicative closures of c and $\{x\}$ are Ore sets in A. In this case we denote the localizations of the ring A at the corresponding multiplicative closures respectively by $A[c^{-1}]$, $A[x^{-1}]$.

3. TWO LEMMAS

3.1. Let S be an algebra graded by a free abelian group H. Then

Lemma. *(i) A graded ideal P is prime iff for any homogeneous $a, b \in S\backslash P$ there exists c such that $acb \notin P$.*

(ii) Take a prime ideal I of S and let J be a maximal homogeneous ideal contained in I. Then J is prime.

Proof. (i) Assume that for any homogeneous $a, b \in S \backslash P$ there exists c such that $acb \notin P$. Take any $a', b' \in S \backslash P$. We can assume that none of the homogeneous components of a' and of b' belong to I. Fix a lexicographic order on H. Denote by a (resp., b) the minimal homogeneous component of a' (resp., b') with respect to the order. Take c such that $acb \notin P$. Then $ac'b \notin P$ for some homogeneous component c' of c. Since the minimal homogeneous component of $a'c'b'$ is just $ac'b$, it follows that $a'c'b' \notin P$ so P is prime as required.

(ii) Observe that J is a linear span of the set of homogeneous elements of I. Take homogeneous $a, b \notin J$. Then $a, b \notin I$ so there exists c such that $acb \notin I$. Hence $acb \notin J$ that, by (i), gives the required assertion. $\square$

3.2. Let S be a K-algebra, $\check{T} \cong \mathbb{Z}^l$ be a torus acting on S by right automorphisms. Denote the action of $t \in \check{T}$ on $s \in S$ by $s.t$. Define an algebra structure on $S \otimes K[\check{T}]$ through

$$(s_1 \otimes t_1)(s_2 \otimes t_2) = (s_1(s_2.t_1^{-1}) \otimes t_1 t_2).$$

The vector space $S \otimes K[\check{T}]$ endowed with the above algebra structure is called the skew-product $S \# \check{T}$. It will be denoted also by $\check{S}$. Denote by $(\operatorname{Spec} S)^{\check{T}}$ the set of $\check{T}$-invariant prime ideals of S.

Lemma. *(i) If $I \in (\operatorname{Spec} S)^{\check{T}}$ then $J := (I \# \check{T})$ is prime in $\check{S}$ and $J \cap S = I$.*

(ii) Assume that $\check{T}$ acts on S by semisimple automorphisms and the set of weights H is a subset of a free abelian group. If $J \in \operatorname{Spec} \check{S}$ then $I := (J \cap S)$ is a prime $\check{T}$ invariant ideal of S.

Proof. (i) The algebra $\check{S}$ admits a natural grading by $\check{T}$ through $\check{S}_t := S \otimes t$. Since J is graded one can use Lemma 3.1 (i). Take homogeneous $a_1, a_2 \in \check{S} \backslash J$. Write $a_i = s_i t_i : s_i \in S,\ t_i \in T,\ i = 1, 2$. Then $s_1, s_2 \in S \backslash I$ so $s_2.t_1^{-1} \in S \backslash I$. Take $g \in S$ such that $s_1 g(s_2.t_1^{-1}) \notin I$. Then $a_1(g.t_1)a_2 = s_1 g(s_2.t_1^{-1})t_1 t_2 \notin J$ as required. The last part is clear.

(ii) The adjoint action of $\check{T}$ defines a H grading on $\check{S}$ and on S. Since $\check{T} \subset \check{S}$ each two-sided ideal of $\check{S}$ is graded so I is also graded. Assume that I is not prime. Then, by Lemma 3.1, there exist homogeneous $a, b \in S \setminus I$ such that $aSb \subseteq I$. Then $a\check{S}b = aS\check{T}b = aSb\check{T} \subseteq I\check{T} \subseteq J$ that contradicts J being prime and completes the proof. $\square$

4. SOME COMMUTATION RELATIONS IN R_0^w

Fix $w \in W$. For a weight vector $a \in R^w$ denote by $\operatorname{lwt} a$ (resp., $\operatorname{rwt} a$) the weight of a wrt the left (resp., right) action of T. If L is a subspace of R^w set $L|^\lambda = \{a \in L : \operatorname{lwt} a = \lambda\}$, $L|_\mu = \{a \in L : \operatorname{rwt} a = \mu\}$. Given weight vector $\xi \in V^+(\lambda)|_\mu$ it is convenient to write $c^\lambda_{\xi, u_\lambda}$ as c^λ_μ.

4.1. Let $J^+_\lambda(\eta)$ (resp., $J^-_\lambda(\eta)$) denote the left ideal of R^+ generated by $c^\lambda_{\eta'}$ with $\eta' < \eta$ (resp., $\eta' > \eta$). In the notation of [J1], 9.1.5 one has $J^\pm_\lambda(\eta) = J^\pm_\lambda(\eta, \lambda) \cap R^+$. By [J1], 9.1.5 $J^\pm_\lambda(\eta)$ are two-sided ideals of R^+.

The commutative relations [J1], 9.1.5 imply that the following relations hold in R^+ :

$$(i) \quad c^\nu_\mu c^\lambda_\eta = q^{(\lambda,\nu)-(\eta,\mu)} c^\lambda_\eta c^\nu_\mu \mod J^+_\lambda(\eta)|^{\lambda+\nu},$$
$$(ii) \quad c^\lambda_\eta c^\nu_\mu = q^{(\lambda,\nu)-(\eta,\mu)} c^\nu_\mu c^\lambda_\eta \mod J^-_\lambda(\eta)|^{\lambda+\nu}.$$

4.2. Let $J^+_\lambda(\eta)_w$ (resp., $J^+_\lambda(\eta)_w$) denote the left ideal of R^w_0 generated by $c^{-\lambda}_w c^\lambda_{\eta'}$ with $\eta' < \eta$ (resp., $\eta' > \eta$).

Lemma. *For any* $\lambda, \nu \in P^+(\pi)$; $\mu \in \Omega(V^+(\nu)), \eta \in \Omega(V^+(\lambda))$ *one has*

$$(i) \quad c^{-\lambda-\nu}_w c^\nu_\mu c^\lambda_\eta = q^{(\lambda,\nu)-(\eta,\mu)} c^{-\lambda-\nu}_w c^\lambda_\eta c^\nu_\mu \quad mod \quad J^+_\lambda(\eta)_w\,,$$
$$(ii) \quad c^{-\lambda-\nu}_w c^\lambda_\eta c^\nu_\mu = q^{(\lambda,\nu)-(\eta,\mu)} c^{-\lambda-\nu}_w c^\nu_\mu c^\lambda_\eta \quad mod \quad J^-_\lambda(\eta)_w.$$

Proof. Consider $a \in J^+_\lambda(\eta)|^{\lambda+\nu}$. By definition of $J^+_\lambda(\eta)$ one can write $a = \sum_i c^{\nu_i}_{\xi_i} c^\lambda_{\eta'_i}$, $\eta'_i < \eta$ for all i. Since $\operatorname{lwt} a = \lambda + \nu$ one can assume that $\nu_i = \nu$ for all i. Therefore

$$c^{-\lambda-\nu}_w a = \sum_i c^{-\lambda-\nu}_w c^\nu_{\xi_i} c^\lambda_{\eta'_i} = \sum_i b_i (c^{-\lambda}_w c^\lambda_{\eta'_i}), \quad \text{where} \quad b_i \in R^w.$$

Since $\operatorname{lwt}(c^{-\lambda-\nu}_w a) = \operatorname{lwt}(c^{-\lambda}_w c^\lambda_{\eta'_i}) = 0$ it follows that $\operatorname{lwt} b_i = 0$ for all i so $b_i \in R^w_0$.

Consequently $c^{-\lambda-\nu}_w J^+_\lambda(\eta)|^{\lambda+\nu} \subseteq J^+_\lambda(\eta)_w$ and similarly $c^{-\lambda-\nu}_w J^-_\lambda(\eta)|^{\lambda+\nu} \subseteq J^-_\lambda(\eta)_w$. Multiply relations (i), (ii) of 4.1 on $c^{-\lambda-\nu}_w$. Then the inclusions above give the relations (i), (ii). □

4.3. For $\nu \in P(\pi)$ consider the inner automorphism ϕ^ν_w of R^w: $a \mapsto c^{-\nu}_w a c^\nu_w$. Since ϕ^ν_w preserves both left and right weight subspaces its restriction on R^w_0 gives an automorphism ϕ^ν_w of R^w_0 which preserves the right weight subspaces. Set $\Phi_w = \{\phi^\nu_w \mid \nu \in P(\pi)\}$.

From [J1], 9.1.4(i), 10.1.11(ii) it follows that for weight vector $a \in R^w_0$ one has $ac^\nu_e = q^{(\nu, \operatorname{rwt} a)} c^\nu_e a$, $ac^\nu_{w_0} = q^{(-w_0\nu, \operatorname{rwt} a)} c^\nu_{w_0} a$. This implies that $c^{-\nu}_w c^\nu_e$, $c^{-\nu}_w c^\nu_{w_0}$ are normal elements of R^w_0 for all $\nu \in P^+(\pi)$.

Take $\mu = w\nu$. Then Lemma 4.2 gives

$$\text{(i)} \quad \phi^\nu_w \left(c^{-\lambda}_w c^\lambda_\eta \right) = q^{(w\nu, \eta - w\lambda)} c^{-\lambda}_w c^\lambda_\eta \mod J^+_\lambda(\eta)_w,$$

Moreover $J^+_\nu(\mu)_w$ is Φ_w-invariant.

$$\text{(ii)} \quad \phi^\nu_w \left(c^{-\lambda}_w c^\lambda_\eta \right) = q^{-(w\nu, \eta - w\lambda)} c^{-\lambda}_w c^\lambda_\eta \mod J^-_\lambda(\eta)_w.$$

Moreover $J^-_\nu(\mu)_w$ is Φ_w-invariant.

Let us show that the $J_\lambda^{\pm}(\eta)_w$ are two-sided ideals. Take $(c_w^{-\lambda} c_{\eta'}^{\lambda})$ with $\eta' < \eta$. As noted in the proof of Lemma 4.2 one has $c_w^{-\lambda-\nu} J_\lambda^{+}(\eta)|^{\lambda+\nu} \subseteq J_\lambda^{+}(\eta)_w$ for any $\nu \in P^+(\pi)$. Then $c_w^{-\lambda-\mu} c_{\eta'}^{\lambda} c_\mu^\nu \in J_\lambda^{+}(\eta)_w$. Therefore $c_w^{-\lambda} c_{\eta'}^{\lambda} c_\mu^\nu c_w^{-\nu} = \phi_w^{-\nu}\left(c_w^{-\lambda-\mu} c_{\eta'}^{\lambda} c_\mu^\nu\right) \in J_\lambda^{+}(\eta)_w$. Since the elements $c_\mu^\nu c_w^{-\nu}$ generate R_0^w it follows that $J_\lambda^{+}(\eta)_w$ is a two-sided ideal of R_0^w. The same reasoning applies to $J_\lambda^{-}(\eta)_w$.

Since the $J_\lambda^{\pm}(\eta)_w$ are two-sided Φ_w-invariant ideals and the $c_w^\lambda : \lambda \in P(\pi)$ generate R^w over R_0^w it follows that $R^w J_\lambda^{\pm}(\eta)_w = J_\lambda^{\pm}(\eta)_w R^w$ and $R^w J_\lambda^{\pm}(\eta)_w R^w \cap R_0^w = J_\lambda^{\pm}(\eta)_w$.

4.4. **Lemma.** *For any* $\lambda, \nu \in P^+(\pi)$; $\mu \in \Omega(V^+(\nu)), \eta \in \Omega(V^+(\lambda))$ *one has*

$$(i)\quad (c_w^{-\nu} c_\mu^\nu)(c_w^{-\lambda} c_\eta^\lambda) = q^{(\lambda,\nu)-(\mu,w\lambda)} c_w^{-\lambda-\nu} c_\mu^\nu c_\eta^\lambda \mod J_\nu^{+}(\mu)_w,$$

$$(ii)\quad (c_w^{-\nu} c_\mu^\nu)(c_w^{-\lambda} c_\eta^\lambda) = q^{(\mu,\eta-w\lambda)} \phi_w^\nu\left(c_w^{-\lambda} c_\eta^\lambda\right)(c_w^{-\nu} c_\mu^\nu) \mod J_\nu^{+}(\mu)_w,$$

$$(iii)\quad (c_w^{-\nu} c_\mu^\nu)(c_w^{-\lambda} c_\eta^\lambda) = q^{-(\lambda,\nu)+(\mu,w\lambda)} c_w^{-\lambda-\nu} c_\mu^\nu c_\eta^\lambda \mod J_\nu^{-}(\mu)_w,$$

$$(iv)\quad (c_w^{-\nu} c_\mu^\nu)(c_w^{-\lambda} c_\eta^\lambda) = q^{-(\mu,\eta-w\lambda)} \phi_w^\nu\left(c_w^{-\lambda} c_\eta^\lambda\right)(c_w^{-\nu} c_\mu^\nu) \mod J_\nu^{-}(\mu)_w.$$

Proof. (i) By 4.3 $c_w^{-\lambda} J_\nu^{+}(\mu)_w c_\eta^\lambda \subseteq J_\nu^{+}(\mu)_w$. Therefore, by 4.3(i), one has

$$(c_w^{-\nu} c_\mu^\nu) c_w^{-\lambda} c_\eta^\lambda = c_w^{-\lambda} \phi_w^{-\lambda}(c_w^{-\nu} c_\mu^\nu) c_\eta^\lambda = q^{(\lambda,\nu)-(\mu,w\lambda)} c_w^{-\lambda-\nu} c_\mu^\nu c_\eta^\lambda \mod J_\nu^{+}(\mu)_w.$$

The proof of (iii) is similar.

By Lemma 4.2(i) one has $c_w^{-\lambda-\nu} c_\mu^\nu c_\eta^\lambda = q^{-(\nu,\lambda)+(\mu,\eta)} c_w^{-\lambda-\nu} c_\eta^\lambda c_\mu^\nu \mod J_\nu^{+}(\mu)_w$.

Taking into account the relation above the formula (i) takes the form

$$(c_w^{-\nu} c_\mu^\nu)(c_w^{-\lambda} c_\eta^\lambda) = q^{(\lambda,\nu)-(\mu,w\lambda)} c_w^{-\lambda-\nu} c_\mu^\nu c_\eta^\lambda = q^{(\mu,\eta)-(w\lambda,\mu)} c_w^{-\lambda-\nu} c_\eta^\lambda c_\mu^\nu =$$
$$q^{(\mu,\eta-w\lambda)} \phi_w^\nu\left(c_w^{-\lambda} c_\eta^\lambda\right)(c_w^{-\nu} c_\mu^\nu) \mod J_\nu^{+}(\mu)_w.$$

The proof of (ii) is similar. □

5. SPECTRAL DECOMPOSITION OF R^+, $\check{R}_0^w$, R_0^w.

5.1. The following construction is similar to [J1], 9.3.8.

Fix $P \in \operatorname{Spec} R_0^w$ or $P \in \operatorname{Spec} \check{R}_0^w$. For each $\nu \in P^+(\pi)$ set

$$C_P(\nu) := \{\mu \in \Omega(V(\nu)) \mid \exists \xi \in V(\nu)^*|_\mu : \quad (c_w^{-\nu} c_\xi^\nu) \notin P\}.$$

Obviously $w\nu \in C_P(\nu)$. Denote by $D_P^+(\nu)$ (resp., $D_P^-(\nu)$) the set of minimal (resp., maximal) elements of $C_P(\nu)$.

Fix $\mu \in D_P^+(\nu)$, $a = (c_w^{-\nu} c_\mu^\nu) \notin P$. Then $J_\nu^{+}(\mu)_w \subseteq P$ so, by 4.4(ii), one has

$$a(c_w^{-\lambda} c_\eta^\lambda) = q^{(\mu,\eta-w\lambda)} \phi_w^\nu (c_w^{-\lambda} c_\eta^\lambda) a \mod P.$$

Thus for homogeneous $b \in R_0^w$ one has

$$ab = q^{(\mu, \mathrm{rwt}\, b)} \phi_w^\nu(b) a \mod P.$$

Thus a is a normal element modulo P and hence a non-zero divisor. It follows that if b is homogeneous and $b \in P$ then $\phi_w^\nu(b) \in P$. Thus we have proved the

Lemma. *Any $\check{T}$ invariant prime ideal of R_0^w is Φ_w invariant.*

5.2. Let P^{++} be a set of regular dominant weights. Set

$$R^{++} := \sum_{\nu \in P^{++}} V^+(\nu),$$

$$\mathrm{Spec}_+ R^+ := \{P \in \mathrm{Spec}\, R^+ \ : \ R^{++} \not\subseteq P\}.$$

In this subsection we will define a decomposition of $\mathrm{Spec}_+ R^+$.

5.2.1. Fix $P \in \mathrm{Spec}_+ R^+$. Similar to 5.1 for each $\nu \in P^+(\pi)$ set

$$C_P(\nu) := \{\mu \in \Omega(V(\nu)) |\ \exists \xi \in V(\nu)^*|_\mu : \ c_\xi^\nu \notin P\}.$$

Since $R^{++} \not\subseteq P$ it follows that $C_P(\nu) \neq \emptyset$ for all $\nu \in P^+(\pi)$. Denote by $D_P^+(\nu)$ (resp., $D_P^-(\nu)$) the set of minimal (resp., maximal) elements of $C_P(\nu)$. The reasoning in [J1], 9.3.8 shows that there exists $y_\pm \in W$ such that $D_P^\pm(\nu) = \{y_\pm \nu\}$. Denote by $X(y_-, y_+)$ the set of all $P \in \mathrm{Spec}_+ R^+$ such that $D_P^-(\nu) = \{y_-\nu\}$, $D_P^+(\nu) = \{y_+\nu\}$. Since any $P \in X(y_-, y_+)$ contains $J_\nu^\pm(y_\pm \nu)$ for all $\nu \in P^+(\pi)$, the relations 4.1 imply that $c_{y_-}^\nu, c_{y_+}^\nu$ are normal modulo P.

5.2.2. **Lemma.** *Take $P \in \mathrm{Spec}_+ R^+$. Then for all $\mu \in P(\pi)$ a subspace $P \cap R^+|^\mu$ (resp., $P \cap R^+|_\mu$) is graded wrt the right (resp., left) action of T.*

Proof. It is sufficient to check that for all $a \in (P \cap R^+|^\mu)$ (resp., $a \in (P \cap R^+|_\mu)$) one has $a.T \subset P$ (resp., $T.a \subset P$). Take $y \in W$ such that $D_P^+(\nu) = \{y\nu\}$. Since c_y^ν is normal modulo P we conclude from 4.1(i) that for any weight vector c_η^λ and any $\nu \in P(\pi)$ one has

$$c_\eta^\lambda = q^{(\lambda,\nu)-(\eta, y\nu)} c_\eta^\lambda = \tau(\nu).c_\eta^\lambda.\tau(y\nu) \bmod P.$$

Hence $a = \tau(\nu).a.\tau(y\nu)$ modulo P for all $a \in R^+$, $\nu \in P(\pi)$. If $a \in (P \cap R^+|^\mu)$ then $\tau(\nu).a = q^{(\mu,\nu)} a$ so $a.\tau(y\nu) \in P$. Similarly if $a \in (P \cap R^+|_\mu)$ then $\tau(\nu).a \in P$. This implies the required assertion. □

Remark. The Lemma implies that the set of prime ideals of R^+ which are invariant wrt the left action of T coincides with the set of primes which are invariant wrt the right action of T. Therefore the same assertion holds for the ring R^w. We will denote the corresponding sets of invariant ideals by $(\mathrm{Spec}_+ R^+)^T$, $(\mathrm{Spec}\, R^w)^T$.

5.2.3. Fix $y \in W$. Denote by $V_y^{\pm}(\lambda)^{\perp}$ the orthogonal of the Demazure module $V_y^{\pm}(\lambda) := U_q(\mathfrak{b}^{\pm})u_{y\lambda}$ in $V(\lambda)^*$, the latter identified with $V^+(\lambda)$. Set

$$Q(y)^{\pm} := \sum_{\lambda \in P^+(\pi)} V_y^{\pm}(\lambda)^{\perp}.$$

Observe that $Q(y)^{\pm} \supseteq J_\nu^{\pm}(y_{\pm}\nu)$ for all $\nu \in P^+(\pi)$ so c_y^ν is normal modulo $Q(y)^{\pm}$. Observe also that $c_w \cap Q(y)^+ = \emptyset$ (resp., $c_w \cap Q(y)^- = \emptyset$) if $w \le y$ (resp., $w \ge y$).

By [J1], 10.1.8 $Q(y)^+$ is a completely prime ideal of R^+ (but note a slight difference of notation). A similar assertion holds for $Q(y)^-$. The reasoning in [J1], 10.1.13 shows that

Proposition. *Every $P \in X(y_1, y_2)$ contains $Q(y_1)^-$, $Q(y_2)^+$.*

In particular, $Q(y_2)^+$ (resp., $Q(y_1)^-$) is a unique minimal element of $X(e, y_2)$ (resp., $X(y_1, w_0)$).

5.2.4. The following lemma is a particular case of [J2], 5

Lemma. *Let $P \in X(y_1, y_2)$, $c_y^\lambda \notin P$ for some $\lambda \in P^{++}$, $y \in W$. Then $y_1 \le y \le y_2$.*

Proof. By Proposition 5.2.3 $Q(y_2)^+ \subseteq P$ so $c_y^\lambda \notin Q(y_2)^+$. The definition of $Q(y_2)^+$ implies that $u_{y\lambda} \in V_{y_2}^+(\lambda)$ so $V_y^+(\lambda) \subseteq V_{y_2}^+(\lambda)$. By [J1], 4.4.5 it follows that $y \le y_2$. Similarly $y_1 \le y$. □

In particular, by the definition of $X(y_1, y_2)$, if $P \in X(y_1, y_2)$ then $c_{y_1}^\lambda \notin P$. Therefore $y_1 \le y_2$. Set

$$W \diamond W := \{(y_1, y_2) \in W \times W |\ y_1 \le y_2\}.$$

Corollary.

$$\operatorname{Spec}_+ R^+ = \coprod_{(y_1,y_2) \in W \diamond W} X(y_1, y_2).$$

Remark. It will be shown that each $X(y_1, y_2)$ is non-empty.

5.3. In this subsection we will define decompositions of $\operatorname{Spec} \check{R}_0^w$, $\operatorname{Spec} R_0^w$ which are similar to the above decomposition of $\operatorname{Spec}_+ R^+$.

5.3.1. In order to relate $\operatorname{Spec}_+ R^+$ and $(\operatorname{Spec} R_0^w)^{\check{T}}$ recall that we have embeddings

$$R^+ \overset{l_w}{\hookrightarrow} R^w \overset{\rho_0}{\hookleftarrow} R_0^w \tag{2}$$

where ρ_0 is the obvious embedding and l_w is the localization map. For a two-sided ideal I of R^+ (resp., of R_0^w) denote the ideal $R^w l_w(I) R^w$ (resp., $R^w \rho_0(I) R^w$) of the ring R^w by I^l (resp., by I^ρ).

Let us show that the correspondence $I \mapsto I^\rho$ defines an order preserving injective map

$$\rho : (\operatorname{Spec} R_0^w)^{\check{T}} \to (\operatorname{Spec} R^w)^T.$$

In fact, the torus $\{c_w^\nu\}_{\nu\in P(\pi)}$ acts on R_0^w by automorphisms $\{\phi_w^\nu\}$ and $R^w = R_0^w\#\{c_w\}$. Let P be a $\check{T}$ invariant prime ideal of R_0^w. Then, by Lemma 5.1, P is Φ_w invariant. Then $P^\rho = (P\#\{c_w\})$ is prime by Lemma 3.2(i) and is obviously T invariant. Moreover, $P^\rho \cap R_0^w = P$. This gives an order preserving injection of $(\operatorname{Spec} R_0^w)^{\check{T}}$ into $(\operatorname{Spec} R^w)^T$.

Furthermore, by [J1], A.2.8 and the noetherianity of R^+ (2.4), l_w induces an order preserving bijection $P \mapsto P^l$ (with inverse $Q \mapsto Q \cap R^+$) of $\operatorname{Spec}_w R^+ := \{P \in \operatorname{Spec} R^+ |\ P \cap c_w = \emptyset\}$ onto $\operatorname{Spec} R^w$. Since this bijection maps T invariant prime ideals to T invariant prime ideals, it induces an order preserving injection of $(\operatorname{Spec} R_0^w)^{\check{T}}$ into $(\operatorname{Spec}_w R^+)^T$. We may summarize the above by the following diagram:

$$(\operatorname{Spec}_w R^+)^T \xrightarrow{\sim} (\operatorname{Spec} R^w)^T \xhookleftarrow{\rho} (\operatorname{Spec} R_0^w) \tag{3}$$

Remark. Let $Q \in \operatorname{Spec}_w R^+$ be a T invariant completely prime ideal. Then

$$Q_w := Q^l \cap R_0^w = \sum_{\lambda\in P^+(\pi)} c_w^{-\lambda}(Q \cap V^+(\lambda)).$$

is a $\check{T}$ invariant completely prime ideal of R_0^w so, by Lemma 3.2(i), $\check{Q}_w := (Q_w\#\check{T})$ is a completely prime ideal of $\check{R}_0^w$.

5.3.2. Fix $P \in (\operatorname{Spec} R_0^w)^{\check{T}}$ and set $P' = (P^\rho \cap R^+)$.

Since $P^\rho \cap R_0^w = P$ it follows that $(c_w^{-\nu}c_\xi^\nu) \in P$ iff $c_\xi^\nu \in P'$. Therefore $J_\nu^\pm(\mu)_w \subseteq P$ iff $J_\nu^\pm(\mu) \subseteq P'$. Hence $D_P^\pm(\nu) = D_{P'}^\pm(\nu)$ for all $\nu \in P^+(\pi)$.

Since $P' \in \operatorname{Spec}_w R^+ \subset \operatorname{Spec}_+ R^+$ there exist $y_\pm \in W$ such that $D_P^\pm(\nu) = D_{P'}^\pm(\nu) = \{y_\pm\nu\}$. Since $P' \cap c_w = \emptyset$, we conclude from Lemma 5.2.4 that $y_- \le w \le y_+$.

5.3.3. Fix $P \in \operatorname{Spec} R_0^w$ (resp., $P \in \operatorname{Spec} \check{R}_0^w$) and let P' be a maximal $\check{T}$ invariant ideal contained in P (resp., $P' = P \cap R_0^w$). Then $D_P^\pm(\nu) = D_{P'}^\pm(\nu)$ for all $\nu \in P^+(\pi)$. By Lemma 3.2 $P' \in (\operatorname{Spec} R_0^w)^{\check{T}}$. Hence $D_P^\pm(\nu) = \{y_\pm\nu\}$ for some $y_\pm$ such that $y_- \le w \le y_+$. Set

$$W \overset{w}{\diamond} W := \{(y_1, y_2)|\ y_1 \le w \le y_2\}.$$

Fix $(y_1, y_2) \in W \overset{w}{\diamond} W$ and let $X_w(y_1, y_2)$ (resp., $Y_w(y_1, y_2)$) denote the set of all $P \in \operatorname{Spec} \check{R}_0^w$ (resp., $P \in \operatorname{Spec} R_0^w$) such that $D_P^-(\nu) = \{y_1\nu\}$, $D_P^+(\nu) = \{y_2\nu\}$ for all $\nu \in P^+(\pi)$. We summarize the results above by the

Proposition.

(i) $$\operatorname{Spec} \check{R}_0^w = \coprod_{(y_1,y_2)\in W\overset{w}{\diamond}W} X_w(y_1, y_2).$$

(ii) $$\operatorname{Spec} R_0^w = \coprod_{(y_1,y_2)\in W\overset{w}{\diamond}W} Y_w(y_1, y_2).$$

6. THE STUDY OF THE STRATA

The goal of this section is to show that for each $(y_1, y_2) \in W \diamond W$ the component $X(y_1, y_2)$ of $\operatorname{Spec} R^+$ has a unique minimal element $Q(y_1, y_2)$. Moreover for $y_1 \leq w \leq y_2$ the ideals $Q(y_1, y_2)_w$, $\check{Q}(y_1, y_2)_w$ (notations of Remark 5.3.1) are unique minimals of $Y_w(y_1, y_2), X_w(y_1, y_2)$ respectively.

6.1. **Notations.**

6.1.1. Set $U := U_q(\mathfrak{g})$. For $i = 1, \dots, l$ set $\varphi_i(a) := \max\{n : a.y_i^n \neq 0\}$ (resp., $\varepsilon_i(a) := \max\{n : a.x_i^n \neq 0\}$) for all $a \in R^+$ non-zero; also set $\varphi_i(0) := 0$, $\varepsilon_i(0) := 0$. Note that

$$\varphi_i(ab) = \varphi_i(a) + \varphi_i(b) \quad \text{for non-zero a,b,}$$
$$(\alpha_i, \operatorname{rwt} a) = \varphi_i(a) - \varepsilon_i(a) \quad \text{for any weight vector } a.$$

Let $a \in R^+$ be a non-zero weight vector. Define $a.y_i^* := a.y_i^{\varphi_i(a)}$ (resp., $a.x_i^* := a.x_i^{\varepsilon_i(a)}$). Furthermore for a fixed reduced decomposition $w = s_{i_1} \dots s_{i_r}$ (resp., $ww_0 = s_{j_1} \dots s_{j_p}$) set $a.y_w^* := a.y_{i_1}^* \dots y_{i_r}^*$ (resp., $a.x_w^* := a.x_{j_1}^* \dots x_{j_p}^*$).

Recall that $V^+(\nu) \cong V(\nu)^*$ as right U modules for all $\nu \in P^+(\pi)$. In particular $V^+(\nu)$ has highest weight ν and the corresponding highest weight vector is annihilated by the $y_i : \; i = 1, \dots, l$ rather than by the x_i. Moreover $\varepsilon_i(c_w^\nu) = 0$ (resp., $\varphi_i(c_w^\nu) = 0$) if $s_i w < w$ (resp., if $s_i w > w$). It implies that $c_w^\nu . y_w^* = c_e^\nu$, $c_w^\nu . x_w^* = c_{w_0}^\nu$ up to non-zero scalars.

Fix $i \in \{1, \dots, l\}$. Suppose a, b are weight vectors and set $\varphi_i(a) = n$, $\varepsilon_i(a) = n'$, $\varphi_i(b) = m$, $\varepsilon_i(b) = m'$. Since

$$\triangle(y_i) = y_i \otimes 1 + t_i \otimes y_i, \quad \triangle(x_i) = x_i \otimes t_i^{-1} + 1 \otimes x_i$$

it follows that there exist $P_{m+n}^n \in K^*$ such that $P_{m+n}^n = P_{m+n}^m$ and

$$(ab).y_i^* = P_{m+n}^n q^{(m\alpha_i, \operatorname{rwt} a)} (a.y_i^*)(b.y_i^*)\,, \quad (ab).x_i^* = P_{m'+n'}^{n'} q^{-(n'\alpha_i, \operatorname{rwt} b)} (a.x_i^*)(b.x_i^*).$$

6.1.2. Fix $w \in W$. Using notations of 5.2.3 set

$$Q(y)_w^\pm := \sum_{\nu \in P^+(\pi)} c_w^{-\nu} V_y^\pm(\nu)^\perp.$$

The ideal $Q(y)_w^+$ (resp., $Q(y)_w^-$) does not coincide with whole R_0^w iff $y \geq w$ (resp., $y \leq w$); in this case, by Remark 5.3.1, it is a $\check{T}$ invariant completely prime ideal of R_0^w.

Recall that $\phi_w^\nu : \; a \mapsto c_w^{-\nu} a c_w^\nu$ is an automorphism of R^w and of R_0^w. By Lemma 5.1 $Q(y)_w^\pm$ are Φ_w invariant.

6.1.3. **Definition.** Fix $w \in W$. For $\eta \in wQ^-(\pi)$ call $\lambda \in P^+(\pi)$ sufficiently large for η if the natural embedding $c_w^{-\lambda} V(\lambda)^+|_{w\lambda + \eta} \hookrightarrow R_0^w|_\eta$ is bijective. Since $\dim R_0^w|_\eta < \infty$ the existence of such λ follows from (1).

6.2. **Lemma.** *Take $\eta \in wQ^-(\pi)$ and choose λ sufficiently large for η. Then $V^+(\lambda)|_{w\lambda+\eta}$ is Φ_w invariant.*

Proof. Identify the vector spaces $R_0^w|_\eta$ and $V^+(\lambda)|_{w\lambda+\eta}$ through the map $a \mapsto c_w^\lambda a$.

An automorphism ϕ_w^ν leaves $R_0^w|_\eta$ invariant. Then for any $a \in R_0^w|_\eta$ one has

$$\phi_w^\nu(c_w^\lambda a) = c_w^{-\nu}(c_w^\lambda a)c_w^\nu = c_w^\lambda \phi_w^\nu(a) \in c_w^\lambda R_0^w|_\eta = V^+(\lambda)|_{w\lambda+\eta}.$$

□

Remark. Actually we showed that the bijection between $R_0^w|_\eta$ and $V^+(\lambda)|_{w\lambda+\eta}$ commutes with the action of Φ_w.

6.3. Fix $\eta \in wQ^-(\pi)$ and choose λ sufficiently large for η. Let us show that the eigenvalues of ϕ_w^ν on $R_0^w|_\eta$ are some integer powers of q. For this we will identify $R_0^w|_\eta$ with $V^+(\lambda)|_{w\lambda+\eta}$ and will study the change of the eigenvalues when we pass from ϕ_w^ν to $\phi_{s_iw}^\nu$.

Let $\overline{K}$ be the algebraic closure of K. Set $\overline{V}^+(\lambda) = V^+(\lambda) \otimes_K \overline{K}$.

6.3.1. **Lemma.** *Fix $\nu, \lambda \in P^+(\pi)$. Suppose $c_\xi^\lambda \in \overline{V}^+(\lambda)$ is a weight vector such that*

$$(a) \quad (\phi_w^\nu)^m(c_\xi^\lambda) \in \overline{V}^+(\lambda) \quad \textit{for all } m \in \mathbb{N},$$

$$(b) \quad (\phi_w^\nu - s \cdot \mathrm{id})^r(c_\xi^\lambda) = 0 \ \ \textit{for some } s \in \overline{K},\ r \in \mathbb{N}.$$

Then

(i) If $i \in \{1, \ldots, l\}$ is such that $s_iw < w$ then

$$(\phi_{s_iw}^\nu)^m(c_\xi^\lambda.y_i^*) \in \overline{V}^+(\lambda) \ \ \textit{for all } m \in \mathbb{N} \ \textit{ and}$$

$$(\phi_{s_iw}^\nu - s' \cdot \mathrm{id})^r(c_\xi^\lambda.y_i^*) = 0 \ \ \textit{where } s' = s \cdot q^{(\mathrm{rwt}\,\xi, w\nu) - (\mathrm{rwt}\,(\xi.y_i^*), s_iw\nu)}.$$

(ii) If $i \in \{1, \ldots, l\}$ is such that $s_iw > w$ then

$$(\phi_{s_iw}^\nu)^m(c_\xi^\lambda.x_i^*) \in \overline{V}^+(\lambda) \ \ \textit{for all } m \in \mathbb{N} \ \textit{ and}$$

$$(\phi_{s_iw}^\nu - s' \cdot \mathrm{id})^r(c_\xi^\lambda.x_i^*) = 0 \ \ \textit{where } s' = s \cdot q^{-(\mathrm{rwt}\,\xi, w\nu) + (\mathrm{rwt}\,(\xi.x_i^*), s_iw\nu)}.$$

Proof. We prove (i) by induction on the nilpotence degree r. Fix i and set $\varphi := \varphi_i,\ y := y_i,\ m := \varphi(c_w^\nu)$. Since $s_iw < w$ it follows from 6.1.1 that $c_w^\nu.y^m = c_{s_iw}^\nu$ up to a non-zero scalar.

Set $c^\lambda_{\xi_1} := (\phi^\nu_w - s\cdot \mathrm{id})(c^\lambda_\xi)$. Then $(\phi^\nu_w - s\cdot \mathrm{id})^{r-1}(c^\lambda_{\xi_1}) = 0$ and also $(\phi^\nu_w)^m(c^\lambda_\xi) \in \overline{V}^+(\lambda)$ for all
$m \in \mathbb{N}$. One has $\phi^\nu_w(c^\lambda_\xi) = sc^\lambda_\xi + c^\lambda_{\xi_1}$ or, in other words,

$$c^\lambda_\xi c^\nu_w = sc^\nu_w c^\lambda_\xi + c^\nu_w c^\lambda_{\xi_1}. \tag{4}$$

If $r=1$ then $\xi_1 = 0$ otherwise $\mathrm{rwt}\,\xi = \mathrm{rwt}\,\xi_1$.

Set $n := \varphi(c^\lambda_\xi)$, $n_1 := \varphi(c^\lambda_{\xi_1})$. Then $\varphi(c^\lambda_\xi c^\nu_w) = m+n$, $\varphi(c^\lambda_{\xi_1} c^\nu_w) = m+n_1$. From the formula (4) it follows that $m+n_1 \le m+n$. Therefore $n_1 \le n$.

Act by y^{m+n} on the both sides of (4). Applying 6.1.1 we get

$$q^{(m\alpha, \mathrm{rwt}\,\xi)}(c^\lambda_\xi . y^*) c^\nu_{s_i w} = q^{(n\alpha, w\nu)}(sc^\nu_{s_i w}(c^\lambda_\xi . y^*) + c^\nu_{s_i w}(c^\lambda_{\xi_1}. y^n)). \tag{5}$$

Note that

$$(\mathrm{rwt}\,\xi, w\nu) - (\mathrm{rwt}\,(\xi . y_i^*), s_i w\nu) = (\mathrm{rwt}\,\xi, w\nu) - (\mathrm{rwt}\,\xi + n\alpha, w\nu + m\alpha) =$$

$$-(n\alpha, s_i w\nu) - (m\alpha, \mathrm{rwt}\,\xi) = (n\alpha, w\nu) - (m\alpha, \mathrm{rwt}\,\xi).$$

Therefore from the formula (5) it follows that

$$(\phi^\nu_{s_i w} - s'\cdot \mathrm{id})(c^\lambda_\xi . y^*) = (s'/s) c^\lambda_{\xi_1} . y^n. \tag{6}$$

Since $\xi_1 = 0$ for $r=1$, the assertion for this case immediately follows from (6).

Suppose $n_1 < n$. Then $c^\lambda_{\xi_1}.y^n = 0$ so the assertion holds. Finally, if $n_1 = n$ then $c^\lambda_{\xi_1}.y^n = c^\lambda_{\xi_1}.y^*$ and $\mathrm{rwt}\,(\xi_1.y^*) = \mathrm{rwt}\,(\xi.y^*)$. The induction hypothesis implies that

$$(\phi^\nu_{s_i w} - s'\cdot \mathrm{id})^{r-1}(c^\lambda_{\xi_1}.y^*) = 0, \quad (\phi^\nu_{s_i w})^m (c^\lambda_{\xi_1}.y^*) \in \overline{V}^+(\lambda) \text{ for all } m \in \mathbb{N}.$$

taking into account (6) we get the required assertion. The proof of (ii) is completely similar. □

6.3.2. By [J1], 9.1.4(i), 10.1.11(ii) one has

$$c^{-\nu}_e c^\lambda_\mu c^\nu_e = q^{(\nu, \mu-\lambda)} c^\lambda_\mu, \qquad c^{-\nu}_{w_0} c^\lambda_\mu c^\nu_{w_0} = q^{-(w_0\nu, \mu - w_0\lambda))} c^\lambda_\mu.$$

So all eigenvalues of the automorphisms ϕ^ν_e, $\phi^\nu_{w_0}$ are integer powers of q. Then from Lemma 6.3.1 it follows, by induction, that for any $w \in W$ all eigenvalues of the automorphisms ϕ^ν_w are integer powers of q.

6.4. Since all eigenvalues of the system of automorphisms Φ_w are integer powers of q it follows that for each common eigenvector $a \in R^w$ there exists $\mu \in Q(\pi)$ such that $\phi^\nu_w(a) = q^{(\mu,\nu)}a$. This element $\mu \in Q(\pi)$ will be called eigenvalue of Φ_w. From this we make the

Definition. For $a \in R^w$ set $\mathrm{wt}_w\, a := \mu \in Q(\pi)$ if $\forall\, \nu\, \exists\, r \in \mathbb{N}:\ (\phi^\nu_w - q^{(\mu,\nu)}\,\mathrm{id})^r a = 0$.

6.4.1. Suppose $a \in R^+$ is homogeneous and $\mathrm{wt}_w\, a$ is defined. Then by Lemma 6.3.1 $\mathrm{wt}_{s_i w}(a.y_i^*)$ (resp., $\mathrm{wt}_{s_i w}(a.x_i^*)$) is defined for $s_i w < w$ (resp., $s_i w > w$) and satisfies to the following relations:

$$\begin{cases} \mathrm{wt}_w\, a + w^{-1}\,\mathrm{rwt}\, a = \mathrm{wt}_{s_i w}(a.y_i^*) + (s_i w)^{-1}\,\mathrm{rwt}(a.y_i^*) & \text{if} \quad s_i w < w \\ \mathrm{wt}_w\, a - w^{-1}\,\mathrm{rwt}\, a = \mathrm{wt}_{s_i w}(a.x_i^*) - (s_i w)^{-1}\,\mathrm{rwt}(a.x_i^*) & \text{if} \quad s_i w > w \end{cases}$$

By induction for any reduced decomposition of w (resp., ww_0)

$$\mathrm{wt}_w\, a + w^{-1}\,\mathrm{rwt}\, a = \mathrm{wt}_e(a.y_w^*) + \mathrm{rwt}(a.y_w^*), \quad \mathrm{wt}_w\, a - w^{-1}\,\mathrm{rwt}\, a = \mathrm{wt}_{w_0}(a.x_w^*) - w_0\,\mathrm{rwt}(a.x_w^*).$$

The relations 6.3.2 imply that

$$\mathrm{wt}_e\, c_\xi^\lambda = \mathrm{rwt}\,(c_e^{-\lambda} c_\xi^\lambda), \qquad \mathrm{wt}_{w_0}\, c_\xi^\lambda = -w_0\,\mathrm{rwt}\,(c_{w_0}^{-\lambda} c_\xi^\lambda).$$

Hence one has the

Proposition. *Take a weight vector* c_ξ^λ *such that* $\mathrm{wt}_w\, c_\xi^\lambda$ *is defined. Then*

$$\mathrm{wt}_w\, c_\xi^\lambda + w^{-1}\,\mathrm{rwt}\,(c_w^{-\lambda} c_\xi^\lambda) = 2\,\mathrm{rwt}\,(c_e^{-\lambda} c_{\xi.y_w^*}^\lambda), \quad \mathrm{wt}_w\, c_\xi^\lambda - w^{-1}\,\mathrm{rwt}\,(c_w^{-\lambda} c_\xi^\lambda) = -2w_0\,\mathrm{rwt}\,(c_{w_0}^{-\lambda} c_{\xi.x_w^*}^\lambda).$$

Consider $a \in R_0^w|_\eta$ such that $\mathrm{wt}_w\, a$ is defined. Note that $\mathrm{wt}_w\, a = \mathrm{wt}_w(c_w^\lambda a)$ for all $\lambda \in P(\pi)$. Choose λ sufficiently large for η (Definition 6.1.3) and set $c_\xi^\lambda := c_w^\lambda a$. Then from the proposition above we get that

$$\left.\begin{aligned} (\mathrm{wt}_w\, a + w^{-1}\eta) = 2\,\mathrm{rwt}\,(c_e^{-\lambda} c_{\xi.y_w^*}^\lambda) \in 2Q^-(\pi) \\ (\mathrm{wt}_w\, a - w^{-1}\eta) = -2w_0\,\mathrm{rwt}\,(c_{w_0}^{-\lambda} c_{\xi.x_w^*}^\lambda) \in 2Q^+(\pi) \end{aligned}\right\} \Longrightarrow w^{-1}\eta \le \mathrm{wt}_w\, a \le -w^{-1}\eta. \tag{7}$$

Note that $w^{-1}\eta \in Q^-(\pi)$.

6.5. Fix $w \in W$. Consider a twisted system of automorphisms $\tilde{\Phi}_w := \{\tilde{\phi}_w^\nu\}$ of R_0^w given by

$$a \mapsto q^{(w^{-1}\,\mathrm{rwt}\, a, \nu)} \phi_w^\nu(a), \text{ on any weight vector } a.$$

Since $J_\nu^+(w\nu)_w \subset Q(w)_w^+$ for any $\nu \in P^+(\pi)$, we conclude from Lemma 4.2(i) that for any weight vector $a \in R_0^w$ one has $\phi_w^\nu(a) = q^{(\nu, -w^{-1}\,\mathrm{rwt}\, a)} a \mod Q(w)_w^+$. Therefore

$$\tilde{\phi}_w^\nu(a) = a \mod Q(w)_w^+ \quad \text{for all } a \in R_0^w. \tag{8}$$

For each $\mu \in Q(\pi)$ denote by $L(w,\mu)|_\eta$ the maximal subspace of $R_0^w|_\eta$ on which all the endomorphisms $(\tilde{\phi}_w^\nu - q^{(\nu,\mu)}\,\mathrm{id})\,, \nu \in P(\pi)$ act nilpotently. Set $L(w,\mu) := \bigoplus_\eta L(w,\mu)|_\eta$. One has

$$L(w,\mu) = \sum \{a \in R_0^w |\ \mathrm{wt}_w\, a = \mu - w^{-1}\,\mathrm{rwt}\, a\}. \tag{9}$$

Then (7) implies that

$$R_0^w = \bigoplus_{\mu \in 2Q^-} L(w,\mu).$$

Observe that $L(w,\mu)L(w,\nu) \subseteq L(w,\mu+\nu)$ so $L(w,0)$ is a subalgebra of R_0^w. Set $L'(w) := \bigoplus_{\mu\neq 0} L(w,\mu)$.

6.5.1. **Lemma.** *(i) One has $Q(w)_w^+ = L'(w)$. In particular $R_0^w = L(w,0) \oplus Q(w)_w^+$.*

(ii) Take a weight vector c_ξ^λ such that $\mathrm{wt}_w\, c_\xi^\lambda$ is defined. Then

$$c_\xi^\lambda \in Q(w)^+ \quad \Longleftrightarrow \quad \mathrm{wt}_w\, c_\xi^\lambda + w^{-1}\xi - \lambda \neq 0.$$

Proof. (i) Fix $\mu \neq 0$ and $\nu \in P^+(\pi)$ such that $(\nu,\mu) \neq 0$. Take $a \in L(w,\mu)$. Since $(\tilde{\phi}_w^\nu - q^{(\nu,\mu)}\,\mathrm{id})^r(a) = 0$ for some $r \in \mathbb{N}$, we conclude from the formula (8) that $a \in Q(w)_w^+$. Hence $L'(w) \subseteq Q(w)_w^+$.

Now suppose that $Q(w)_w^+ \not\subseteq L'(w)$. The formula (8) implies that $Q(w)_w^+$ is $\tilde{\Phi}_w$ invariant. Then there exists a weight vector $a \in Q(w)_w^+$ such that $a \in L(w,0)$. Since each automorphism $\tilde{\phi}_w^{\omega_i}$ acts on $L(w,0)$ nilpotently one can assume that a is an eigenvector that is $\tilde{\phi}_w^\nu(a) = a$ for all $\nu \in P(\pi)$. Choose λ sufficiently large for $\mathrm{rwt}\, a$ and write $a = c_w^{-\lambda} c_\xi^\lambda$.

From Proposition 6.4.1 and the definition of $\tilde{\phi}_w^\nu$ we conclude that $\mathrm{rwt}(c_e^{-\lambda} c^\lambda_{\xi.y_w^*}) = 0$ and so $c^\lambda_{\xi.y_w^*} = c_\lambda^\lambda$ up to a non-zero scalar. Therefore

$$0 \neq \xi.y_w^*(v_\lambda) = \xi.(y_{i_1}^{n_1}\dots y_{i_r}^{n_r})(v_\lambda) = \xi(y_{i_1}^{n_1}\dots y_{i_r}^{n_r} v_\lambda).$$

By [J1], 4.4.6 $(y_{i_1}^{n_1}\dots y_{i_r}^{n_r} v_\lambda) \in V_w^+(\lambda)$ so $\xi(V_w(\lambda)^+) \neq 0$.

However $a = c_w^{-\lambda} c_\xi^\lambda \in Q(w)_w^+$ that is $c_\xi^\lambda \in Q(w)^+$. Hence $\xi(V_w(\lambda)^+) = 0$ giving the required contradiction.

(ii) Recall that $c_\xi^\lambda \in Q(w)^+$ iff $c_w^{-\lambda} c_\xi^\lambda \in Q(w)_w^+$. Then (i) and (9) imply the required assertion. ☐

Remark. The lemma above and the formula (8) imply that $\tilde{\phi}_w^\nu(a) = a$ for all $\nu \in P^+(\pi)$ iff $a \in L(w,0)$.

6.6. **Lemma.** *$Q(y,w)_w := Q(w)_w^+ + Q(y)_w^-$ is a completely prime ideal of R_0^w for all $y \leq w$.*

Proof. By Lemma 5.1 $Q(y)_w^-$ is Φ_w invariant so $\tilde{\Phi}_w$ invariant. By Lemma 6.5.1(i) $L'(w) = Q(w)_w^+$ therefore

$$Q(y,w)_w = L'(w) \oplus (L(w,0) \cap Q(y)_w^-).$$

Consequently,

$$R_0^w/Q(y,w)_w = (L(w,0)\oplus L')/\Big((L(w,0)\cap Q(y)_w^-) \oplus L'\Big) \cong L(w,0)/(L(w,0)\cap Q(y)_w^-).$$

To show that $L(w,0)/(L(w,0)\cap Q(y)^-_w)$ is a domain, observe that, by 6.1.2, $Q(y)^-_w$ is a completely prime ideal of R^w_0. Since $L(w,0)$ is a subalgebra of R^w_0 it follows that $(L(w,0)\cap Q(y)^-_w)$ is a completely prime ideal of $L(w,0)$. □

6.6.1. Similar to 6.5 one can consider a twisted system of automorphisms $\{\tilde{\phi}^\nu_w\}$ of R^w_0 given by $a \mapsto q^{-(w^{-1}\operatorname{rwt} a,\nu)}\phi^\nu_w(a)$, on any weight vector a. Then reasoning similar to 6.5.1— 6.6 shows that $Q(w,y)_w$ is a completely prime ideal of the ring R^w_0 for all $y \geq w$.

6.7. Fix $(y,w)\in W\diamond W$. By 5.2.3 every $P\in X(y,w)$ contains $\tilde{Q}(y,w) := (Q(y)^- + Q(w)^+)$. The ideal $\tilde{Q}(y,w)$ is not in general prime. We describe now an operation which, being applied to $\tilde{Q}(y,w)$, gives a prime ideal.

Recall that for all $z\in W$ the set c_z is an Ore set in R^+. Let I be a two-sided ideal in R^+ such that $I\cap c_z=\emptyset$. We define the saturation of I along c_z by the formula

$$I : c_z = \operatorname{Ker}\left(R^+ \to (R^+/I)[c_z^{-1}]\right).$$

For all $\nu \in P^+(\pi)$ the c^ν_w is normal modulo $\tilde{Q}(y,w)$ and modulo any $P\in X(y,w)$. Therefore $P : c_w = P$. Since the saturation along c_w preserves the inclusion relation of ideals, it follows that $P \supseteq \tilde{Q}(y,w) : c_w$ for all $P\in X(y,w)$. Set

$$Q(y,w) := \tilde{Q}(y,w) : c_w = \{a\in R^+ |\ \exists\lambda\in P^+(\pi)\ s.t.\ c^\lambda_w a \in Q(y)^- + Q(w)^+\}.$$

Therefore $Q(y,w) = R^w Q(y,w)_w \cap R^+$. By Lemma 6.6 $Q(y,w)_w$ is a $\check{T}$ invariant completely prime ideal of R^w_0. By 5.3.1 this implies that $Q(y,w)$ is a T invariant completely prime ideal of R^+.

6.8. **Proposition.** *The T invariant completely prime ideal $Q(y,w)$ of R^+ is the unique minimal element of $X(y,w)$ for all $(y,w)\in W\diamond W$.*

Proof. By 6.7 any $P\in X(y,w)$ contains $Q(y,w)$, which is a T invariant completely prime ideal of R^+. Therefore it is sufficient to show that $Q(y,w)\in X(y,w)$.

Recall that

$$Q(y,w) = \{a\in R^+ |\ \exists\lambda\in P^+(\pi)\ s.t.\ c^\lambda_w a \in Q(y)^- + Q(w)^+\}.$$

Since $c_w \cap Q(y,w)=\emptyset$, it suffices to check that $c^\nu_y \notin Q(y,w)$ for all $\nu\in P^+(\pi)$. We prove this by induction. Namely, from the pair $(y,w)\in W\diamond W$ such that $c^\nu_y\in Q(y,w)$ we will construct a pair $(s_iy,w')\in W\diamond W$ such that $s_iy>y$ and $c^\nu_{s_iy}\in Q(s_iy,w')$. Note that $(w_0,z)\in W\diamond W$ forces $z=w_0$. Since $c^\nu_{w_0}\notin Q(w_0,w_0)$ we will thus obtain a contradiction. The required assertion is proved in 6.8.1— 6.8.5 below.

6.8.1. Suppose that there exists $\nu \in P^+(\pi)$ such that $c^\nu_y \in Q(y,w)$. Then $c^{-\nu}_w c^\nu_y \in Q(y,w)_w$. Set $\eta := (y\nu - w\nu)$. By 6.5 and the proof of Lemma 6.6 one can write

$$c^{-\nu}_w c^\nu_y = \sum_{j=0}^m b_j, \tag{10}$$

where the b_j are weight vectors of the weight η, the values $\mathrm{wt}_w\, b_j$ are defined and pairwise distinct, $b_0 \in L(w,0) \cap Q(y)^-_w$ and $b_j \in Q(w)^+_w$ for $j = 1, \dots, m$.

Choose μ sufficiently large for η (see Definition 6.1.3) such that $\mu > \nu$ and set $\lambda := \mu - \nu$. For $i = 0, \dots, m$ set $f_i := c^\mu_w b_i$. Then multiplying the relation (10) by c^μ_w we get

$$c^\lambda_w c^\nu_y = \sum_{j=0}^m f_j, \tag{11}$$

where the f_j are weight vectors of the weight $w\lambda + y\nu$, $f_0 \in Q(y)^-$ and $f_j \in Q(w)^+$ for $j = 1, \dots, m$. Note that $\mathrm{wt}_w\, f_i = \mathrm{wt}_w\, b_i$.

6.8.2. Fix i such that $s_i y > y$. Then $k[x_i] V^-_{s_i y}(\mu) = V^-_y(\mu)$, so $f_0 \in Q(y)^-$ implies $f_0.x_i^r \in Q(s_i y)^-$ for any $r \in \mathbb{N}$. Set $x = x_i$, $\varepsilon_i = \varepsilon$.

6.8.3. Assume that $s_i w < w$.

Since $Q(w)^+$ is $U_q(\mathfrak{n}^+)$ invariant the relation (11) implies that $(c^\lambda_w c^\nu_y).x^* \in Q(s_i y)^- + Q(w)^+$. Since $c^\lambda_w c^\nu_{s_i y} = (c^\lambda_w c^\nu_y).x^*$ up to a non-zero scalar it follows that $c^\lambda_w c^\nu_{s_i y} \in Q(s_i y)^- + Q(w)^+$.

6.8.4. Assume that $s_i w > w$. Then up to a non-zero scalar one has

$$c^\lambda_{s_i w} c^\nu_{s_i y} = (c^\lambda_w c^\nu_y).x^* = \sum_{j=0}^m f_j.x^n, \quad \text{for some } \ n \in \mathbb{N}. \tag{12}$$

Let us check, using Lemma 6.5.1(ii), that $f_j.x^n \in Q(s_i w)^+$ for $j = 1, \dots, m$. Then, by 6.8.2, it implies that

$$c^\lambda_{s_i w} c^\nu_{s_i y} \in Q(s_i y)^- + Q(s_i w)^+.$$

For $a \in R^+, z = w$ or $z = s_i w$ we set $q_z(a) := \mathrm{wt}_z\, a + z^{-1} \mathrm{rwt}\, a - \mathrm{lwt}\, a$ provided the right-hand side is defined. If $q(a)$ is defined then, by Lemma 6.5.1(ii), $a \in Q(z)^+$ iff $q_z(a) \neq 0$. By 6.4.1 $\mathrm{wt}_{s_i w}(f_j.x^*)$ is defined and

$$\mathrm{wt}_w\, f_j - w^{-1} \mathrm{rwt}\, f_j = \mathrm{wt}_{s_i w}(f_j.x^*) - (s_i w)^{-1} \mathrm{rwt}(f_j.x^*). \tag{13}$$

Since $\mathrm{lwt}(f_j.x^*) = \mathrm{lwt}\, f_j$ this implies that

$$q_{s_i w}(f_j.x^*) - q_w(f_j) = 2((s_i w)^{-1} \mathrm{rwt}(f_j.x^*) - w^{-1} \mathrm{rwt}\, f_j).$$

Assume that $f_j.x^n = f_j.x^*$ for some $j \neq 0$. Then

$$(s_i w)^{-1} \mathrm{rwt}(f_j.x^*) - w^{-1} \mathrm{rwt}\, f_j = (s_i w)^{-1} \mathrm{rwt}(c^\lambda_{s_i w} c^\nu_{s_i y}) - w^{-1} \mathrm{rwt}(c^\lambda_w c^\nu_y) = 0$$

so $q_{s_i w}(f_j.x^*) = q_w(f_j)$. Since $f_j \in Q(w)^+$ it follows that $f_j.x^* \in Q(s_i w)^+$.

Now let us show that $f_j.x^n \neq 0$ iff $f_j.x^n = f_j.x^*$. Observe that, by 6.8.1, the values $\mathrm{wt}_w f_j$ are pairwise distinct for $j = 0, \dots m$ so the left-hand sides of the equality (13) are also pairwise distinct for $j = 0, \dots m$. This implies that the elements $\{f_j.x^*\}_{j=0}^m$ are linearly independent. Set $n' := \max_{0 \leq j \leq m} \varepsilon(f_j)$. Then

$$(c_w^\lambda c_y^\nu).x^{n'} = \sum_{j=0}^m f_j.x^{n'} = \sum_{j:\varepsilon(f_j)=n'} f_j.x^* \neq 0.$$

Compairing with the relation (12) we get $n' = n$ and $f_j.x^n \neq 0$ iff $f_j.x^n = f_j.x^*$ as required.

6.8.5. Set $s_i \star w = \max(s_i w, w)$. Since $y \leq w$ it follows ([J1], A.1.7) that $s_i y \leq s_i \star w$.

Recall our assumption that $c_w^\lambda c_y^\nu \in Q(y)^- + Q(w)^+$ for some pair $(y, w) : \ y \leq w$. Suppose $y \neq w_0$ so there exists i such that $s_i y > y$. Then we conclude by 6.8.3, 6.8.4 that

$$c_{s_i \star w}^\lambda c_{s_i y}^\nu \in Q(s_i y)^- + Q(s_i \star w)^+ \ ,$$

so the assumption holds for the pair $(s_i y, s_i \star w)$, where $y < s_i y \leq s_i \star w$. By induction the assumption holds for the pair (w_0, w_0) : $c_{w_0}^\lambda c_{w_0}^\nu \in (Q(w_0)^- + Q(w_0)^+)$.

However $Q(w_0)^+ = (0)$, $c_{w_0} \cap Q(w_0)^- = \emptyset$. Hence $c_{w_0}^\lambda c_{w_0}^\nu \notin (Q(w_0)^- + Q(w_0)^+)$ which gives a contradiction. $\square$

Remark. Using 6.6.1 we could prove equally that the ideal $Q(y, w)' := \tilde{Q}(y, w) : c_y$ is the unique minimal element of the component $X(y, w)$. Therefore $Q(y, w) = Q(y, w)'$.

6.9. **Example.** The present example illustrates that in general

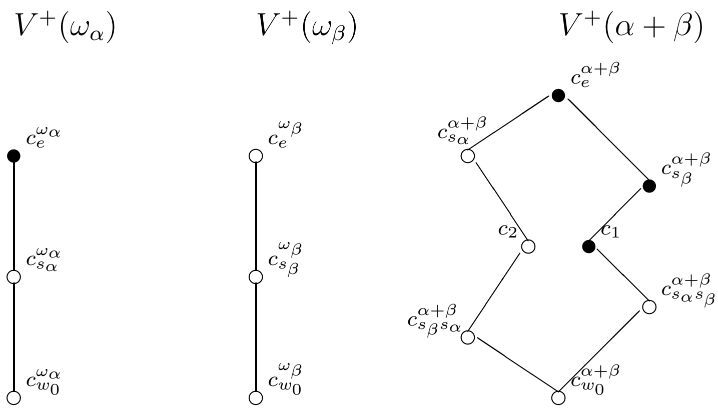
$$Q(s_\alpha, s_\alpha s_\beta) \neq Q(s_\alpha)^- + Q(s_\alpha s_\beta)^+.$$

Put $\mathfrak{g} = \mathfrak{sl}_3$. The diagrams below show the intersection of prime ideals $Q = Q(s_\alpha)^-$, $Q(s_\alpha s_\beta)^+$ of the ring R^+ with the right modules $V = V^+(\omega_\alpha)$, $V^+(\omega_\beta)$, $V^+(\omega_\alpha + \omega_\beta) = V^+(\alpha + \beta)$.

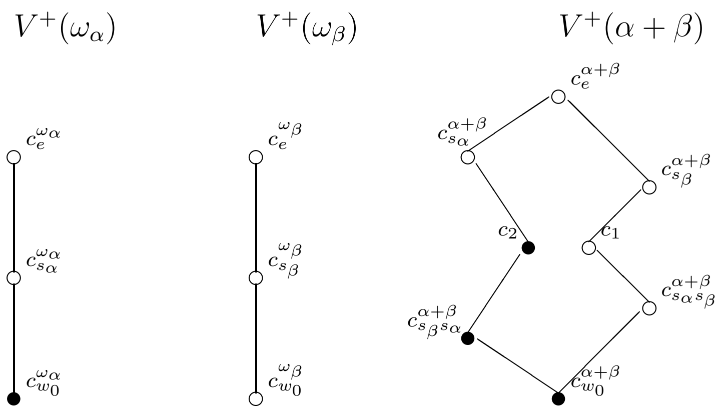
Observe that $V^+(\alpha + \beta)|_0$ is two dimensional. It is spanned by a vector c_1 orthogonal to the zero weight vector in $U_q(\mathfrak{b}^-) u_{s_\alpha(\alpha+\beta)}$ and a vector c_2 orthogonal to the zero weight vector in $U_q(\mathfrak{b}^+) u_{s_\alpha s_\beta(\alpha+\beta)}$, where $u_{s_\alpha(\alpha+\beta)}$, $u_{s_\alpha s_\beta(\alpha+\beta)}$ are the extreme weight vectors of $V^+(\alpha + \beta)$ of the corresponding weights.

In the diagram describing the pair Q, V we mark with black colour the weight vectors of V belonging to $Q \cap V$.

The ideal $Q(s_\alpha)^-$.



The ideal $Q(s_\alpha s_\beta)^+$.



Note that

$$c_{s_\alpha}^{\omega_\alpha} c_{s_\beta}^{\omega_\beta} \in Kc_1 + Kc_2 \subset \tilde{Q}(s_\alpha, s_\alpha s_\beta) = Q(s_\alpha)^- + Q(s_\alpha s_\beta)^+.$$

By Remark 6.8 $Q(s_\alpha, s_\alpha s_\beta) = \tilde{Q}(s_\alpha, s_\alpha s_\beta) : c_{s_\alpha}$ so $c_{s_\beta}^{\omega_\beta} \in Q(s_\alpha, s_\alpha s_\beta)$. Yet this weight vector does not belong to either $Q(s_\alpha)^-$ nor $Q(s_\alpha s_\beta)^+$ and hence not to their sum. Hence $Q(s_\alpha, s_\alpha s_\beta) \neq Q(s_\alpha)^- + Q(s_\alpha s_\beta)^+$.

6.10. **Lemma.** *For all* $(y_1, y_2) \in W \overset{w}{\diamond} W$ *one has* $Q(y_1, y_2) \cap c_w = \emptyset$.

Proof. Suppose that $c_w^\nu \in Q(y_1, y_2)$ for some $\nu \in P^+(\pi)$. This means that $c_{y_2}^\lambda c_w^\nu \in (Q(y_1)^- + Q(y_2)^+)$ for some $\lambda \in P^+(\pi)$. Since $y_1 \leq w$ then $Q(y_1)^- \subseteq Q(w)^-$. Therefore $c_{y_2}^\lambda c_w^\nu \in (Q(w)^- + Q(y_2)^+)$ in contradiction to Proposition 6.8. □

The ideal $Q(y_1, y_2)$ is T invariant. Therefore, by Remark 5.3.1,

$$Q(y_1, y_2)_w := \sum_{\lambda \in P^+(\pi)} c_w^{-\lambda}(Q(y_1, y_2) \cap V^+(\lambda))$$

is a $\check{T}$ invariant completely prime ideal of R_0^w and

$$\check{Q}(y_1, y_2)_w := Q(y_1, y_2)_w \# \check{T}$$

is a completely prime ideal of $\check{R}_0^w$.

6.10.1. **Corollary.** *(i) For each* $(y_1, y_2) \in W \overset{w}{\diamond} W$ *the component* $Y_w(y_1, y_2)$ *of* Spec R_0^w *has a unique minimal element* $Q(y_1, y_2)_w$ *which is a completely prime* $\check{T}$ *invariant ideal.*

(ii) For each $(y_1, y_2) \in W \overset{w}{\diamond} W$ *the component* $X_w(y_1, y_2)$ *of* Spec $\check{R}_0^w$ *has a unique minimal element* $\check{Q}(y_1, y_2)_w$ *which is completely prime.*

Proof. Since $Q(y_1, y_2)$ is a unique minimal element of $X(y_1, y_2)$, it follows, by 5.3.2, that $Q(y_1, y_2)_w \in Y_w(y_1, y_2)$ and, moreover, it lies in all $\check{T}$ invariant ideals of $Y_w(y_1, y_2)$. By 5.3.3 every $P \in Y_w(y_1, y_2)$ (resp., $P \in X_w(y_1, y_2)$) contains some $\check{T}$ invariant ideal $P' \in Y_w(y_1, y_2)$. Hence $Q(y_1, y_2)_w \subset P$ (resp., $\check{Q}(y_1, y_2)_w \subset P$) as required. □

6.11. Define an order relation on $W \diamond W$ by the formula

$$(y, z) \succeq (y', z') \text{ iff } y \leq y',\ z \geq z'.$$

The definition of $Q(y)^{\pm}$ implies that for $y \leq y'$ one has $Q(y)^- \subseteq Q(y')^-$ (resp., $Q(y)^+ \supseteq Q(y')^+$). Similarly one has

Proposition. *(i)* $Q(y, z) \subseteq Q(y', z')$ *iff* $(y, z) \succeq (y', z')$.

(ii) $Q(y, z)_w \subseteq Q(y', z')_w$ *iff* $(y, z) \succeq (y', z')$.

Proof. (i) Take $Q(y, z) \subseteq Q(y', z')$. Then $c_{y'}^{\lambda},\ c_{z'}^{\lambda} \notin Q(y, z)$ for all $\lambda \in P^+(\pi)$. Lemma 5.2.4 implies that $y \leq y',\ z' \leq z$.

Conversly, take $y \leq y'$. Then

$$Q(y)^- + Q(z)^+ \subseteq Q(y')^- + Q(z)^+ \ \Rightarrow\ Q(y, z) = \tilde{Q}(y, z) : c_z \subseteq \tilde{Q}(y', z) : c_z = Q(y', z).$$

Similarly, by Remark 6.8, one has

$$Q(y', z) = \tilde{Q}(y', z) : c_{y'} \subseteq Q(y', z') : c_{y'} = Q(y', z').$$

Hence (i). The assertion (ii) follows from (i) and 5.3.1. □

6.12. By Propositions 5.3.3, 6.8 and Corollaries 5.2.4, 6.10.1 we have the following decompositions

$$\operatorname{Spec}_+ R^+ = \coprod_{(y_1,y_2)\in W\diamond W} X(y_1,y_2), \quad X(y_1,y_2)^{min} = \{Q(y_1,y_2)\},$$

$$\operatorname{Spec} \check{R}_0^w = \coprod_{(y_1,y_2)\in W\overset{w}{\diamond}W} X_w(y_1,y_2), \quad X_w(y_1,y_2)^{min} = \{\check{Q}(y_1,y_2)_w\},$$

$$\operatorname{Spec} R_0^w = \coprod_{(y_1,y_2)\in W\overset{w}{\diamond}W} Y_w(y_1,y_2), \quad Y_w(y_1,y_2)^{min} = \{Q(y_1,y_2)_w\}.$$

Let us show that the decompositions above are stratifications i.e. that each component $X(y_1,y_2)$ (resp., $Y_w(y_1,y_2)$, $X_w(y_1,y_2)$) is locally closed and its closure $\overline{X}(y_1,y_2)$ (resp., $\overline{Y_w}(y_1,y_2)$, $\overline{X_w}(y_1,y_2)$) with respect to Jacobson topology is a union of components.

One has

$$X(y_1,y_2) = \left\{P \in \operatorname{Spec}_+ R^+ |\ Q(y_1,y_2) \subseteq P,\ c_{y_1}\cap P = \emptyset,\ \ c_{y_2}\cap P = \emptyset\right\}.$$

Hence $\overline{X}(y_1,y_2) = \{P \in \operatorname{Spec}_+ R^+ |\ Q(y_1,y_2) \subseteq P\}$ and $X(y_1,y_2)$ is locally closed.

Proposition 6.11 implies that $X(z_1,z_2) \subseteq \overline{X}(y_1,y_2)$ provided $(y_1,y_2) \succeq (z_1,z_2)$. The inverse is also true. In fact, take $P' \in \overline{X}(y_1,y_2)$. Fix $(z_1,z_2) \in W\diamond W$ such that $P' \in X(z_1,z_2)$. Then $c_{z_i}\cap Q(y_1,y_2) = \emptyset$ for $i = 1,2$. By Lemma 5.2.4 this implies that $y_1 \le z_1 \le z_2 \le y_2$ that is $(y_1,y_2) \succeq (z_1,z_2)$. The same reasoning is suitable for $\overline{X_w}(y_1,y_2)$, $\overline{Y_w}(y_1,y_2)$.

6.13. **Corollary.**

$$\overline{X}(y_1,y_2) = \coprod_{\substack{(z_1,z_2)\in W\diamond W\\ (y_1,y_2)\succeq(z_1,z_2)}} X(z_1,z_2),$$

$$\overline{X_w}(y_1,y_2) = \coprod_{\substack{(z_1,z_2)\in W\overset{w}{\diamond}W\\ (y_1,y_2)\succeq(z_1,z_2)}} X_w(z_1,z_2),$$

$$\overline{Y_w}(y_1,y_2) = \coprod_{\substack{(z_1,z_2)\in W\overset{w}{\diamond}W\\ (y_1,y_2)\succeq(z_1,z_2)}} Y_w(z_1,z_2).$$

7. More about the strata

All rings in this Section are noetherian. Using this and [J1], A.2.8, we will often identify the prime spectrum of the localization $R[c^{-1}]$, c being an Ore subset of R, with the subset

$$\{P \in \operatorname{Spec} R |\ P\cap c = \emptyset\}.$$

7.1. In this Section we will show that the components $Y_w(y_1,y_2)$ of $\operatorname{Spec} R_0^w$ are isomorphic for different $w \in W$ such that $y_1 \le w \le y_2$. Moreover the components $X_w(y_1,y_2)$ of $\operatorname{Spec} \check{R}_0^w$ are isomorphic to the component $X(y_1,y_2)$ of $\operatorname{Spec}_+ R^+$ for all $w \in W$ such that $y_1 \le w \le y_2$. Following [J3] we identify the component $X(y_1,y_2)$ (modulo an action of a group $\mathbb{Z}_2^l$) with the spectrum of a Laurent polynomial ring— see 7.4.2— 7.4.4.

All localizations considered are localizations of domains so the localization maps are injective. We will sometimes denote by the same letter an element of a ring R and its image in a localization (or in a quotient) of R.

7.1.1. **Lemma.** *Take $P \in X(y,w)$. Then $P \cap V^+(\nu) = Q(y,w) \cap V^+(\nu)$ for all $\nu \in P^+(\pi)$.*

Proof. Assume that $P \cap V^+(\nu) \neq Q(y,w) \cap V^+(\nu)$. By Lemma 5.2.2 this implies that there exists a weight vector $c_\zeta^\nu \in P \setminus Q(y,w)$. Choose λ sufficiently large for $(\zeta - w\nu)$ (Definition 6.1.3) such that $\lambda > \nu$. Then Lemma 6.5.1 implies that

$$c_w^{-\nu} c_\zeta^\nu = c_w^{-\lambda} c_\xi^\lambda + c_w^{-\lambda} c_\eta^\lambda, \quad \text{where} \quad c_w^{-\lambda} c_\xi^\lambda \in L(w,0), \quad c_w^{-\lambda} c_\eta^\lambda \in Q(w)^+_w.$$

Then $c_\eta^\lambda \in Q(w)^+$ so $c_\xi^\lambda = (c_w^{\lambda-\nu} c_\zeta^\nu - c_\eta^\lambda) \in P \setminus Q(y,w)$. By Remark 6.5.1 for all $\nu \in P(\pi)$ one has $\tilde{\phi}_w^\nu(c_w^{-\lambda} c_\xi^\lambda) = c_w^{-\lambda} c_\xi^\lambda$, that is

$$c_\xi^\lambda c_w^\nu = q^{(\lambda - w^{-1}\xi, \nu)} c_w^\nu c_\xi^\lambda. \tag{14}$$

Let us show that $c_\xi^\lambda . y_{-\mu} \in P$ for all $\mu \in Q^+(\pi)$ and all elements $y_{-\mu} \in U_q(\mathfrak{b}^-)$ of a weight $(-\mu)$. We prove this by induction on $\mu \in (Q^+(\pi), \le)$. One has

$$\triangle(y_{-\mu}) = y_{-\mu} \otimes 1 + \tau(\mu) \otimes y_{-\mu} + \sum_{0<\eta<\mu} k_\eta \tau(\eta) y_{-\mu+\eta} \otimes y_{-\eta}, \quad k_\eta \in K. \tag{15}$$

Act by $y_{-\mu}$ on the both sides of (14). Applying (15) and induction one obtains

$$(c_\xi^\lambda . y_{-\mu}) c_w^\nu = q^{(\lambda - w^{-1}\xi, \nu) + (\mu, w\nu)} c_w^\nu (c_\xi^\lambda . y_{-\mu}) \bmod P.$$

Using formula (8) we get

$$(c_\xi^\lambda . y_{-\mu}) c_w^\nu = q^{(\lambda - w^{-1}(\xi+\mu), \nu)} c_w^\nu (c_\xi^\lambda . y_{-\mu}) \ \mathit{mod}\ Q^+(w) \subseteq P.$$

Therefore $(1 - q^{2(w\nu,\mu)}) c_w^\nu (c_\xi^\lambda . y_{-\mu}) \in P$ for all $\nu \in P^+(\pi)$. Hence $c_\xi^\lambda . y_{-\mu} \in P$.

Since $c_\xi^\lambda \notin Q(y,w)$ there exists $v \in V_y^-(\lambda) = U_q(\mathfrak{b}^-) u_{y\lambda}$ such that $\xi(v) = 1$. This implies that $c_y^\lambda = c_\xi^\lambda . U_q(\mathfrak{b}^-)$ so $c_y^\lambda \in P$. This contradicts $P \in X(y,w)$. $\square$

7.1.2. **Corollary.**

(i) $(\operatorname{Spec}_+ R^+)^T = \{Q(y,z)\}_{(y,z) \in W \diamond W}$.

(ii) $(\operatorname{Spec} R_0^w)^{\check{T}} = \{Q(y,z)_w\}_{(y,z) \in W \overset{w}{\diamond} W}$.

(iii) $X_w(y,z) = \{P \in \operatorname{Spec} \check{R}_0^w |\ P \cap R_0^w = Q(y,z)_w\}$.

(iv) Take $P \in Y_w(y,z)$. Then a weight vector $c_w^{-\lambda} c_\xi^\lambda$ belongs to P iff $c_\xi^\lambda \in Q(y,z)$.

Proof. The previous lemma implies (i); (ii) obtains from (i), Lemma 3.2 and the diagram (3). (iii), (iv) obtain from (ii) and 5.3.2. $\square$

7.2. For any $y, w, z \in W$ let $R^{y,w,z}$ be the minimal subalgebra of $\operatorname{Fract} R^+$ containing $c_y^{-1},\ c_w^{-1},\ c_z^{-1}$. Both right and left action of T on R^+ extend to $R^{y,w,z}$. Denote the zero component of $R^{y,w,z}$ with respect to the left T-action by $R_0^{y,w,z}$. Then the right action of T on $R_0^{y,w,z}$ extends to the action of $\check{T}$. Denote the corresponding skew-product $R_0^{y,w,z}\#\check{T}$ by $\check{R}_0^{y,w,z}$. It is clear that $\check{R}_0^{w} \subset \check{R}_0^{y,w,z}$.

Now take $y \leq w \leq z$. Recall that

$$Q(y,z)_w = Q(y,z)[c_w^{-1}] \cap R_0^w, \quad Q(y,z)_z = Q(y,z)[c_z^{-1}] \cap R_0^z.$$

Therefore

$$R_0^{y,w,z}Q(y,z)_w \supset Q(y,z)_z, \quad R_0^{y,w,z}Q(y,z)_z \supset Q(y,z)_w.$$

This implies that

$$R_0^{y,w,z}Q(y,z)_w = R_0^{y,w,z}Q(y,z)_z, \quad \check{R}_0^{y,w,z}\check{Q}(y,z)_w = \check{R}_0^{y,w,z}\check{Q}(y,z)_z.$$

For any pair $(w_1, w_2) \in W \times W$ set $c_{w_1,w_2} := \{c_{w_1}^{-\lambda}c_{w_2}^{\lambda}\}_{\lambda \in P^+(\pi)}$.

7.2.1. **Lemma.** *Take $y \leq w \leq z$. There are canonical isomorphisms of the Ore localizations*

$$(R_0^w/Q(y,z)_w)[c_{w,z}^{-1}, c_{w,y}^{-1}] \xrightarrow{\sim} R_0^{y,w,z}/(R_0^{y,w,z}Q(y,z)_w) \xrightarrow{\sim} (R_0^z/Q(y,z)_z)[c_{z,w}^{-1}, c_{z,y}^{-1}], \tag{16}$$

$$(\check{R}_0^w/\check{Q}(y,z)_w)[c_{w,z}^{-1}, c_{w,y}^{-1}] \xrightarrow{\sim} \check{R}_0^{y,w,z}/(\check{R}_0^{y,w,z}\check{Q}(y,z)_w) \xrightarrow{\sim} (\check{R}_0^z/\check{Q}(y,z)_z)[c_{z,w}^{-1}, c_{z,y}^{-1}]. \tag{17}$$

Proof. It is sufficient to check that all the localizations are well-defined. Observe that the image of the set $c_{w,z} \cup c_{w,y}$ in the quotient ring $R_0^w/Q(y,z)_w$ consists of normal elements so $(R_0^w/Q(y,z)_w)[c_{w,z}^{-1}, c_{w,y}^{-1}]$ is well-defined.

Let us check that the image of the set $c_{z,y} \cup c_{z,w}$ in the quotient ring $R_0^z/Q(y,z)_z$ is Ore. Since c_w is Ore in R^+ it follows that for any $c_\xi^\lambda \in R^+, \nu \in P^+(\pi)$ there exist $c_\eta^\mu \in R^+, \nu' \in P^+(\pi)$ such that $c_\xi^\lambda c_w^{\nu'} = c_w^\nu c_\eta^\mu$. By 4.3(i) $c_z^{-\lambda}c_\xi^\lambda$ and $c_\xi^\lambda c_z^{-\lambda}$ coincide up to a power of q modulo $Q(y,z)_z$. Therefore up to a power of q one has

$$(c_z^{-\lambda}c_\xi^\lambda)(c_z^{-\nu'}c_w^{\nu'}) = c_z^{-\lambda}c_\xi^\lambda c_w^{\nu'}c_z^{-\nu'} = c_z^{-\lambda}c_w^\nu c_\eta^\mu c_z^{-\nu'} = (c_z^{-\nu}c_w^\nu)(c_z^{-\mu}c_\eta^\mu) \mod Q(y,z)_z.$$

Hence the image of $c_{z,w}$ is left Ore in $R_0^z/Q(y,z)_z$. Similarly it is right Ore. Since the image of the set $c_{z,y}$ in the quotient ring $R_0^z/Q(y,z)_z$ consists of normal elements and they commute up to powers of q with the elements of the image of $c_{z,w}$, it follows that the image of $c_{z,y} \cup c_{z,w}$ in the quotient ring $R_0^z/Q(y,z)_z$ is Ore. Hence $(R_0^z/Q(y,z)_z)[c_{z,w}^{-1}, c_{z,y}^{-1}]$ is also well-defined. $\square$

7.2.2. **Proposition.** *Take $y \leq w \leq z$.*

(i) The isomorphisms (16) give rise to an order preserving bijection of $Y_w(y,z)$ onto $Y_z(y,z)$.

(ii) The isomorphisms (17) give rise to an order preserving bijection of $X_w(y,z)$ onto $X_z(y,z)$.

Proof. The definition of $Y_w(y,z)$ and Corollary 6.10.1 imply that

$$Y_w(y,z) \cong \operatorname{Spec}(R_0^w/Q(y,z)_w)[c_{w,z}^{-1}, c_{w,y}^{-1}] = \operatorname{Spec} R_0^{y,w,z}/(R_0^{y,w,z}Q(y,z)_w).$$

Taking into account Lemma 7.2.1 and Corollary 7.1.2(iv), we conclude that

$$\begin{array}{l} Y_w(y,z) \cong \operatorname{Spec}(R_0^z/Q(y,z)_z)[c_{z,w}^{-1}, c_{z,y}^{-1}] \cong \\ \{P \in \operatorname{Spec} R_0^z |\ Q(y,z)_z \subset P,\ P \cap (c_{z,w} \cup c_{z,y}) = \emptyset\} = \\ \{P \in \operatorname{Spec} R_0^z |\ Q(y,z)_z \subset P,\ P \cap c_{z,w} = \emptyset\} = Y_z(y,z) \end{array}$$

This gives (i); the proof of (ii) is similar. □

7.3. **Proposition.** *For every triple (y,w,z) such that $y \leq w \leq z$ there is an order preserving bijection of $X_w(y,z)$ onto $X(y,z)$.*

Proof. From the previous proposition we conclude that it is sufficient to check the assertion for the triples (y,z,z). Fix $z \in W$. Using notations of 6.5, denote a subalgebra $L(z,0)\#\check{T}$ of $\check{R}_0^z$ by $\check{L}(z,0)$ and a subalgebra $L(z,0)\#\{c_w^\nu\}_{\nu \in P(\pi)}$ of R^z by $L(z)$. Define a map $\psi :$ $\check{L}(z,0) \to L(z)$ setting $\psi(a) = a$, for $a \in L(z,0)$, $\psi(\tau(\nu)) = c_z^{-z^{-1}\nu}$ for all $\nu \in P(\pi)$. We conclude from 6.5 that ψ is an isomorphism of algebras. Denote by Ψ the corresponding map of $\operatorname{Spec} \check{L}(z,0)$ onto $\operatorname{Spec} L(z)$.

Taking into account that $R^z Q(z)_z^+ = R^z Q(z)^+$ we conclude from Lemmas 6.5.1, 6.6 that

$$\check{R}_0^z = \check{Q}(z)_z^+ \oplus \check{L}(z,0), \quad R^z = R^z Q(z)^+ \oplus L(z).$$

Therefore there are the following bijections

$$\Psi_1 :\ H_1 := \{P \in \operatorname{Spec} \check{R}_0^z |\ \check{Q}(z)_z^+ \subset P\} \ \to\ \operatorname{Spec} \check{L}(z,0), \quad P \mapsto P \cap \check{L}(z,0),$$

with inverse $I \mapsto I \oplus \check{Q}(z)_z^+$;

$$\Psi_2 :\ H_2 := \{P \in \operatorname{Spec} R^z |\ R^z Q(z)^+ \subset P\} \ \to\ \operatorname{Spec} L(z), \quad P \mapsto P \cap L(z),$$

with inverse $I \mapsto I \oplus R^z Q(z)^+$. Hence $(\Psi_2^{-1} \circ \Psi \circ \Psi_1)$ is a bijection of H_1 onto H_2. Identify $X(y,z)$ and its image in $\operatorname{Spec} R^z$ given by the localization map $R^+ \to R^z$. Then

$$H_1 = \coprod_{y \leq z} X_z(y,z), \quad H_2 = \coprod_{y \leq z} X(y,z).$$

Let us show that $(\Psi_2^{-1} \circ \Psi \circ \Psi_1)(X_z(y,z)) = X(y,z)$ for all $y \leq z$. By Corollary 7.1.2(iii) one has

$$X_z(y,z) = \{P \in \operatorname{Spec} \check{R}_0^z |\ P \cap R_0^z = Q(y,z)_z\}.$$

Since $Q(y,z)_z = (Q(y,z)_z \cap L(z,0)) \oplus Q(z)^+$ it follows that

$$\Psi_1(X_z(y,z)) = \{P \in \operatorname{Spec} \check{L}(z,0) |\ P \cap L(z,0) = Q(y,z)_z \cap L(z,0)\}.$$

Observe that $P \cap L(z,0) = \Psi(P) \cap L(z,0)$. Therefore

$$(\Psi \circ \Psi_1)(X_z(y,z)) = \{P \in \operatorname{Spec} L(z) |\ P \cap L(z,0) = Q(y,z)_z \cap L(z,0)\}.$$

Take $J \in X(y,z)$. We conclude from Lemma 7.1.1, Lemma 6.6 that

$$\Psi_2(J) \cap L(z,0) = \sum_{\nu \in P^+(\pi)} c_z^{-\nu}(V^+(\nu) \cap J) = Q(y,z)_z \cap L(z,0).$$

Hence $\Psi_2(X(y,z)) \subseteq \operatorname{Im}(\Psi \circ \Psi_1)(X_z(y,z))$. Since this holds for all $y \leq z$ we conclude that $\Psi_2(X(y,z)) = \operatorname{Im}(\Psi \circ \Psi_1)(X_z(y,z))$ as required. $\square$

7.4. Fix $y \leq w$. Using notations of 7.2, denote $(R_0^w/Q(y,w)_w)[c_{w,y}^{-1}]$ by S and set $\check{S} = S\#\check{T}$. Then the canonical map $\check{R}_0^w \to \check{S}$ defines a bijection of $X_w(y,w)$ onto $\operatorname{Spec} \check{S}$. We calculate $\operatorname{Spec} \check{S}$ in 7.4.1— 7.4.3 below.

7.4.1. For each $\nu \in P(\pi)$, set $z_\nu := c_w^{-\nu} c_y^\nu \tau(y\nu + w\nu) \in \check{S}$. The relations 4.4 imply that $z_\nu s = s z_\nu$ for all $s \in S$. Since $z_\nu \tau(\mu) = q^{(y\nu - w\nu, \mu)} \tau(\mu) z_\nu$ it follows that $z_\nu \in Z(\check{S})$ iff $y\nu = w\nu$. Set

$$P_0(\pi) := \{\nu \in P(\pi) |\ y^{-1}\nu - w^{-1}\nu = 0\}$$

which is a subgroup of $P(\pi)$ so that $P(\pi)/P_0(\pi)$ is torsion-free. Choose a subgroup $P_1(\pi)$ such that $P(\pi) = P_0(\pi) \oplus P_1(\pi)$. Set $T_0 := \tau(P_0(\pi))$, $T_1 := \tau(P_1(\pi))$. Denote the subalgebra $S\#T_0$ of $\check{S}$ by D. Then $\check{S} = D\#T_1$.

Observe that S is noetherian, so by [MCR], 2.9 D is also noetherian.

Lemma. *The map $\psi : J \mapsto J \cap D$ is an order preserving bijection of* $\operatorname{Spec} \check{S}$ *onto* $(\operatorname{Spec} D)^{\check{T}}$.

Proof. Since $P(\pi) = P_0(\pi) \oplus P_1(\pi)$ it follows that $\check{T} = T_0 T_1$. Therefore $(\operatorname{Spec} D)^{\check{T}} = (\operatorname{Spec} D)^{T_1}$. By Lemma 3.2 ψ maps $\operatorname{Spec} \check{S}$ onto $(\operatorname{Spec} D)^{T_1}$ and the map $I \mapsto (I\#T_1)$ is a right inverse of ψ. Let us show that this is also a left inverse of ψ, that is $J = (J \cap D)\#T_1$ for all $J \in \operatorname{Spec} \check{S}$. Fix $J \in \operatorname{Spec} \check{S}$, $a \in J$. Write $a = \sum_\mu a_\mu \tau(\mu) :\ \mu \in P_1(\pi),\ a_\mu \in D$. Recall that the elements z_ν commute with all elements of S and

$$z_\nu \tau(\mu) = q^{(y\nu - w\nu, \mu)} \tau(\mu) z_\nu = q^{(\nu, y^{-1}\mu - w^{-1}\mu)} \tau(\mu) z_\nu.$$

Therefore $z_\nu s = s z_\nu$ for all $s \in D$. Since z_ν is invertible in $\check{S}$ one has

$$z_\nu a z_\nu^{-1} = \sum_\mu a_\mu z_\nu \tau(\mu) z_\nu^{-1} = \sum_\mu q^{(\nu, y^{-1}\mu - w^{-1}\mu)} a_\mu \tau(\mu) \in J.$$

The values $(y^{-1}\mu - w^{-1}\mu)$ are pairwise distinct for different $\mu \in P_1(\pi)$, so $a_\mu \tau(\mu) \in J$ for all $\mu \in P_1(\pi)$. Then $a_\mu \in J \cap D$ and $J = (J \cap D)\#T_1$ as required. $\square$

7.4.2. Let r be the rank of $P_0(\pi)$. Identify $P_0(\pi)/2P_0(\pi)$ with $\mathbb{Z}_2^r$. For each $\tau(\nu) \in T_0$ let $d(\tau(\nu))$ denote the image of ν in $\mathbb{Z}_2^r$. For $s \in S$ set $d(s) := 0$. This defines $\mathbb{Z}_2^r$ grading on D. For $g \in \mathbb{Z}_2^r$ denote the subspace $\{a \in D| \; d(a) = g\}$ by D_g. Denote by Γ the character group of $\mathbb{Z}_2^r$. For each $\gamma \in \Gamma$ define $\theta_\gamma \in \operatorname{Aut} D$ setting $\theta_\gamma|_{D_g} := \gamma(g) \cdot \mathrm{id}$. View Γ as acting on ideals of D via the $\theta_\gamma : \; \gamma \in \Gamma$ and hence on $\operatorname{Spec} D$. Since the θ_γ commute with the action of $\check{T}$ it follows that Γ acts also on $(\operatorname{Spec} D)^{\check{T}}$.

Lemma. *The map taking $I \in (\operatorname{Spec} D_0)^{\check{T}}$ to the minimal primes over DI (with inverse $P \mapsto P \cap D_0$) is a bijection of $(\operatorname{Spec} D_0)^{\check{T}}$ onto the Γ orbits of $(\operatorname{Spec} D)^{\check{T}}$.*

Proof. Since $D = S\#T_0$ it follows that $D = D_0T_0 = T_0D_0$. This implies that DI is a two-sided graded ideal of D for any T invariant ideal I of D_0. The reasoning of [J1], 1.3.9 implies that for any $I \in (\operatorname{Spec} D_0)^{\check{T}}$ the minimal primes Q_i over DI form a single Γ orbit and satisfy $I = Q_i \cap D_0$ for all i.

Let us show that the inverse map is well-defined. Fix $P \in (\operatorname{Spec} D)^{\check{T}}$ and set $I := P \cap D_0$. Assume that I is not prime. Then, by Lemma 3.1, there exist homogeneous $a, b \in D_0 \setminus I$ such that $aD_0b \subseteq I$. Then $aDb = aD_0T_0b = aD_0bT_0 \subseteq IT_0 \subseteq P$ that contradicts P being prime and completes the proof. $\square$

Remark. For $i = 1, \dots, l$ define the element $\sigma_i \in \operatorname{Aut} \check{R}_0^w$ by the formulas

$$\sigma_i|_{R_0^w} = \mathrm{id}; \;\; \sigma_i(\tau(\omega_i)) = -\tau(\omega_i); \;\; \sigma_i(\tau(\omega_j)) = \tau(\omega_j) \text{ for } j \neq i.$$

Consider the group $\mathbb{Z}_2^l \subseteq \operatorname{Aut} \check{R}_0^w$ generated by the automorphisms σ_i. This group acts naturally on D and the image of $\mathbb{Z}_2^l$ in $\operatorname{Aut} D$ identifies with Γ.

7.4.3. Denote the subalgebra of $\check{S}$ generated by the central elements $z_\nu : y\nu = w\nu$ by Z. Take $\mu \in P(\pi)$; then $\mu = y\nu + w\nu$ for some ν such that $y\nu = w\nu$ iff $\mu \in 2P_0(\pi)$. It follows that $Z \subset D_0$ and Z is a Laurent polynomial ring of the rank r. Since $D_0 = S\#\tau(2P_0(\pi))$ it follows that $D_0 \cong S \otimes Z$ as $\check{T}$ algebras (the action of $\check{T}$ on Z is trivial). Since S is noetherian, D_0 is also noetherian.

Lemma. *(i) The map $P \mapsto P \cap Z$ is an isomorphism of $(\operatorname{Spec} D_0)^{\check{T}}$ onto $\operatorname{Spec} Z$.*

(ii) For each $P \in (\operatorname{Spec} D_0)^{\check{T}}$, the quotient D_0/P is a domain.

Proof. Take $P \in (\operatorname{Spec} D_0)^{\check{T}}$. Since P is prime and Z is contained in the centre of D_0 one has $(P \cap Z) \in \operatorname{Spec} Z$.

Take any $I \in \operatorname{Spec} Z$. Since Z is $\check{T}$ invariant then $Q := SI$ a two-sided $\check{T}$ invariant ideal of D_0 contained in P. Identify D_0 with $S \otimes Z$. Then $Q = S \otimes I$ and $D_0/Q \cong S \otimes (Z/I)$ as $\check{T}$ algebras, where the action of $\check{T}$ on Z/I is trivial. Since Z/I is a domain, $G := (Z/I) \setminus \{0\}$ is an Ore subset of $S \otimes (Z/I)$. Set $F := \operatorname{Fract}(Z/I)$ and identify $S \otimes (Z/I)[G^{-1}]$ with $S \otimes F$. The action of $\check{T}$ on $S \otimes (Z/I)$ extends to $S \otimes F$. By definition $S = (R_0^w/Q(y,w)_w)[c_{w,y}^{-1}]$. This is a domain for any choice of the base field $K \supseteq k(q)$. Set (for a moment) $K := F$.

Then we get that $S\otimes F$ is a domain so $S\otimes(Z/I)$ is also a domain. Hence Q is a completely prime ideal of D_0. Since $Q\cap Z=I$ this establishes the surjectivity in (i).

Take $P\in(\operatorname{Spec}D_0)^{\check T}$ and set $I:=(P\cap Z)$. Again set $Q=SI$ and define G,F as above. Denote by $\overline{P}$ the image of P/Q in $S\otimes(Z/I)$ which is a prime $\check T$ invariant ideal. Recall that $P\cap Z=I$ so $\overline{P}\cap G=\emptyset$. Hence $\overline{P}[G^{-1}]$ is a prime $\check T$ invariant ideal of $S\otimes(Z/I)[G^{-1}]=S\otimes F$. Corollary 7.1.2(ii) implies that the zero ideal is the only $\check T$ invariant prime ideal of the ring $S\otimes K'$ for any field K' containing $k(q)$. Hence $\overline{P}=(0)$ that is $P=Q$. This establishes (ii) and injectivity in (i). $\square$

7.4.4. Recall that Z is a subalgebra of $(\check R_0^w/Q(y,w)_w)[c_{w,y}^{-1}]$ generated by the central elements $z_\nu:=c_w^{-\nu}c_y^\nu\tau(2w\nu)$ where $\nu\in P(\pi)$ such that $y\nu=w\nu$. For each $z\in W$ denote by $r(z)$ the rank of the free group $P_z(\pi):=\{\mu\in P(\pi)|\ z\mu=\mu\}$ (one has $r(z)=l-s(z)$, where $s(z)$ denotes the minimal length of an expression for z as a product of reflections). Then $\operatorname{rk}Z=r(w^{-1}y)$. Combining 7.4— 7.4.3 one obtains the

Proposition. *The map $P\mapsto(P/Q(y,w)_w)[c_{w,y}^{-1}]\cap Z$ is an isomorphism of the space of $\mathbb{Z}_2^l$ orbits in $X_w(y,w)$ onto* $\operatorname{Spec}Z$.

Now Propositions 7.2.2, 7.3, 7.4.4 give the

Theorem.

$$\textit{(i)}\ \operatorname{Spec}_+R^+=\coprod_{(y,z)\in W\diamond W}X(y,z),$$

where each $X(y,z)$ is isomorphic up to an action of $\mathbb{Z}_2^l$ to the spectrum of the Laurent polynomail ring of rank $r(y^{-1}z)$.

$$\textit{(ii)}\ \operatorname{Spec}\check R_0^w=\coprod_{(y,z)\in W\overset{w}{\diamond}W}X_w(y,z),$$

where each $X_w(y,z)$ is isomorphic to the component $X(y,z)$ of Spec_+R^+.

8. The Centre of R_0^w

Denote the element $(c_\xi^\lambda)^{-1}$ of $\operatorname{Fract}R^+$ by $c_\xi^{-\lambda}$. Set

$$A:=\{a\in\operatorname{Fract}R^+|\ c_\xi^\lambda a\in R^+\text{ for some }\lambda\in P^+(\pi),\ \xi\in\Omega(V(\lambda)^*)\}.$$

The right action of U_q on R^+ extends to A and $a=c_\xi^{-\lambda}b$ is a weight vector iff $b\in R^+$ is a weight vector.

8.1. **Lemma.** *Let a be a weight vector of A. Then $a\in Z(\operatorname{Fract}R_0^e)$ iff $a\in Kc_e^{-\nu}c_{w_0}^\nu$ for some $\nu\in P(\pi)$ satisfying $w_0\nu=-\nu$.*

Proof. By [J1], 9.1.4(i), 10.1.11(ii) for any $\nu,\lambda,\in P^+(\pi),\mu\in\Omega(V^+(\lambda))$ one has

$$c_\mu^\lambda c_e^\nu=q^{(\nu,\mu-\lambda)}c_e^\nu c_\mu^\lambda,\qquad c_\mu^\lambda c_{w_0}^\nu=q^{-(w_0\nu,\mu-w_0\lambda)}c_{w_0}^\nu c_\mu^\lambda.\tag{18}$$

This implies that $c_{w_0}^{w_0\nu}c_e^{\nu}b = bc_{w_0}^{w_0\nu}c_e^{\nu}$ for any $\nu \in P(\pi)$, $b \in R_0^e$. Hence $c_e^{-\nu}c_{w_0}^{\nu} \in Z(\mathrm{Fract}\, R_0^e)$ if $w_0\nu = -\nu$.

Let us prove the converse. For each $b \in A$ consider the set of pairs $\{(\lambda, \xi) \in P^+(\pi) \times \Omega(V^+(\lambda))|\ c_\xi^\lambda b \in R^+\}$. This set admits a lexicographic preorder $(\lambda, \xi) \le (\lambda', \xi')$ iff $\lambda \le \lambda'$ or $\lambda = \lambda'$ and $\xi \le \xi'$. The expression $b = c_\xi^{-\lambda}d$ $(d \in R^+)$ will be called *a reduced decomposition* if the pair (λ, ξ) is a minimal with respect to the preorder above.

Set
$$B := \{b \in R^+|\ b \notin c_{w_0}^{\omega_i}R^+,\ b \notin c_e^{\omega_i}R^+ \ \text{ for all } i = 1, \dots, l\}.$$
Given $b \in R^+$ write $b = c_{w_0}^{\nu_1}c_e^{\nu_2}b' :\ \nu_1, \nu_2 \in P^+(\pi),\ b' \in B$. Theorem 3 of [J2] implies that $Q(w_0s_i)^+ = c_{w_0}^{\omega_i}R^+$ (similarly $Q(s_i)^- = c_e^{\omega_i}R^+$). Since $Q(w_0s_i)^+, Q(s_i)^-$ are completely prime ideals of R^+ it follows that ν_1, ν_2, b' are uniquely determined. The element b' will be called *the abnormal part* of b.

Let a be a non-zero weight vector of A and let $a \in Z(\mathrm{Fract}\, R_0^e)$. Fix a reduced decomposition $a = c_\xi^{-\lambda}d$. Let $c_{\mu_1}^{\lambda_1},\ c_{\mu_2}^{\lambda_2}$ be the abnormal parts of c_ξ^λ, d respectively. One has
$$a = c_\xi^{-\lambda}d = q^r c_{w_0}^{\nu_1}c_e^{\nu_2}c_{\mu_1}^{-\lambda_1}c_{\mu_2}^{\lambda_2} \quad \text{for some } \nu_1, \nu_2 \in P(\pi), r \in \mathbb{Z}.$$
Set $b := c_{\mu_1}^{-\lambda_1}c_{\mu_2}^{\lambda_2}$. Observe that $b = c_{\mu_1}^{-\lambda_1}c_{\mu_2}^{\lambda_2}$ is a reduced decomposition.

Let c_η^ν be a weight vector of R^+. One has $c_e^{-\nu}c_\eta^\nu a = ac_e^{-\nu}c_\eta^\nu$.

The relations (18) imply that
$$c_\eta^\nu b = q^r bc_\eta^\nu \quad \text{for some } r \in \mathbb{Z}. \tag{19}$$
Moreover one has
$$bc_e^{\omega_i} = q^{(\mathrm{wt}_e\, b, \omega_i)}c_e^{\omega_i}b, \text{ where } \mathrm{wt}_e\, b = \mu_2 - \mu_1 - \lambda_2 + \lambda_1.$$
Act by x_i on the both sides of the relation above. Taking into account that $\mathrm{wt}_e(b.x_i) = \mathrm{wt}_e\, b - \alpha_i$ we obtain
$$q^{(\mathrm{wt}_e\, b, \omega_i)}(1 - q^{-2})c_e^{\omega_i}(b.x_i) = bc_{s_i}^{\omega_i} - q^{(\mathrm{wt}_e\, b, \omega_i) - (\alpha_i, \mathrm{rwt}\, b)}c_{s_i}^{\omega_i}b.$$
Using (19) we conclude that $c_e^{\omega_i}(b.x_i) \in Kbc_{s_i}^{\omega_i}$. One has
$$b.x_i = (c_{\mu_1}^{-\lambda_1}c_{\mu_2}^{\lambda_2}).x_i = c_{\mu_1}^{-\lambda_1}(c_{\mu_2}^{\lambda_2}.x_i) - q^{(\alpha_i, \mu_1 - \mu_2)}c_{\mu_1}^{-\lambda_1}(c_{\mu_1}^{\lambda_1}.x_i)(c_{\mu_1}^{-\lambda_1}c_{\mu_2}^{\lambda_2}) = (c_{\mu_1}^{\lambda_1})^{-2}d \text{ for some } d \in R^+.$$
Therefore
$$c_e^{\omega_i}(c_{\mu_1}^{\lambda_1})^{-2}d \in Kbc_{s_i}^{\omega_i} \quad \Rightarrow \quad c_e^{\omega_i}d \in K(c_{\mu_1}^{\lambda_1})^2 bc_{s_i}^{\omega_i} = Kc_{\mu_1}^{\lambda_1}c_{\mu_2}^{\lambda_2}c_{s_i}^{\omega_i}.$$

Recall that $c_{\mu_1}^{\lambda_1}, c_{\mu_2}^{\lambda_2} \in B$ so $c_{\mu_1}^{\lambda_1}c_{\mu_2}^{\lambda_2}c_{s_i}^{\omega_i} \notin Q(s_i)^-$. Since $c_e^{\omega_i} \in Q(s_i)^-$ it follows that $d = 0$ so $b.x_i = 0$. Replacing $c_e^{\omega_i}$ by $c_{w_0}^{-w_0\omega_i}$ and $Q(s_i)^-$ by $Q(s_iw_0)^+$ we get $b.y_i = 0$. Since $b.x_i = b.y_i = 0$ it follows that $b.t_i = b$.

Let us check that $\lambda_1 = 0$.Assume the converse. Then $c_{\mu_1}^{\lambda_1} \neq c_{w_0}^{\lambda_1}$ since $c_{\mu_1}^{\lambda_1} \in B$. Therefore there exists i such that $c_{\mu_1}^{\lambda_1}.x_i \neq 0$.

Since $b.x_i = 0,\ b.t_i = b$ one has
$$c_{\mu_2}^{\lambda_2}.x_i = (c_{\mu_1}^{\lambda_1} b).x_i = (c_{\mu_1}^{\lambda_1}.x_i)b \quad \Rightarrow \quad b = (c_{\mu_1}^{\lambda_1}.x_i)^{-1}(c_{\mu_2}^{\lambda_2}.x_i).$$

Yet $\mathrm{rwt}(c_{\mu_1}^{\lambda_1}.x_i) < \mathrm{rwt}\, c_{\mu_1}^{\lambda_1}$ this contradicts $b = c_{\mu_1}^{-\lambda_1} c_{\mu_2}^{\lambda_2}$ being a reduced decomposition.

Now, $\lambda_1 = 0$ and therefore $c_{\mu_2}^{\lambda_2}.x_i = c_{\mu_2}^{\lambda_2}.y_i = 0$ for all $i = 1, \dots, l$. Then $\lambda_2 = 0$ so $b \in K^*$. Hence $a \in K^* c_{w_0}^{\nu_1} c_e^{\nu_2}$. Since $a \in Z(\mathrm{Fract}\, R_0^e)$ it follows that $\nu_1 + \nu_2 = 0$. Moreover the relations (18) imply that $(\nu_2, \mu) - (w_0\nu_1, \mu) = 0$ for any $\mu \in Q^-(\pi)$. Hence $a \in K^* c_{w_0}^{\nu_1} c_e^{-\nu_1}$ and $\nu_1 + w_0\nu_1 = 0$ as required. $\square$

8.2. Let θ be the automorphism of the Dynkin diagram defined by the property $w_0\omega_i = -\omega_{\theta(i)}$. One has $\theta^2 = 1$. Set
$$\mathfrak{I} := \{i \in \{1, \dots, l\} \,|\, \theta(i) = i\}, \ \overline{\mathfrak{I}} := \{i \in \{1, \dots, l\} \,|\, \theta(i) > i\}.$$
Set $z_i := c_e^{-\omega_i} c_{w_0}^{\omega_i}$; for $i \in \overline{\mathfrak{I}}$ set $\tilde{z}_i := z_i z_{\theta(i)}$.

One has $z_i \in R_0^e,\ z_i^{-1} \in R_0^{w_0}$. For $w = e$ the centre $Z(R_0^e)$ is the polynomial algebra generated by the set $M := \left\{z_i : \ i \in \mathfrak{I},\ \tilde{z}_i : \ i \in \overline{\mathfrak{I}}\right\}$ — see [J1], 7.1.20. Similarly $Z(R_0^{w_0})$ is the polynomial algebra generated by the set $M^{-1} = \{m^{-1} : \ m \in M\}$. We will show that $Z(R_0^w)$ is the polynomial algebra generated by the set $(M \cup M^{-1}) \cap R_0^w$.

For a more precise description of the set of generators of $Z(R_0^w)$ set
$$\mathfrak{I}_w^- := \{i \in \mathfrak{I} |\ w\omega_i = \omega_i\}, \ \overline{\mathfrak{I}}_w^- := \left\{i \in \overline{\mathfrak{I}} |\ w\omega_i = \omega_i,\ w\omega_{\theta(i)} = \omega_{\theta(i)}\right\},$$

$$\mathfrak{I}_w^+ := \{i \in \mathfrak{I} |\ w\omega_i = w_0\omega_i\}, \ \overline{\mathfrak{I}}_w^+ := \left\{i \in \overline{\mathfrak{I}} |\ w\omega_i = w_0\omega_i,\ w\omega_{\theta(i)} = w_0\omega_{\theta(i)}\right\}.$$

Then $M \cap R_0^w = \left\{z_i : \ i \in \mathfrak{I}_w^-,\ \tilde{z}_i : \ i \in \overline{\mathfrak{I}}_w^-\right\}$ and $M^{-1} \cap R_0^w = \left\{z_i^{-1} : \ i \in \mathfrak{I}_w^+,\ \tilde{z}_i^{-1} : \ i \in \overline{\mathfrak{I}}_w^+\right\}$.

8.2.1. **Proposition.** *The centre $Z(R_0^w)$ is the polynomial algebra generated by the set $C := (M \cup M^{-1}) \cap R_0^w$.*

Proof. Set $z_\nu := c_e^{-\nu} c_{w_0}^{\nu}$ for all $\nu \in P(\pi)$ satisfying $w_0\nu = -\nu$. Observe that $R_0^w \subset A$. Then, in view of Lemma 8.1, it suffices to show that any element $z_\nu \in R_0^w$ can be expressed as a product of elements of C.

Write $\nu = \sum k_i\omega_i$ and set $\mathfrak{A}^- := \{i : k_i < 0\},\ \mathfrak{A}^+ := \{i : k_i > 0\}$. Set
$$\nu_1 := -\sum_{i \in \mathfrak{A}^-} k_i\omega_i, \qquad \nu_2 := \sum_{i \in \mathfrak{A}^+} k_i\omega_i.$$
Then $\nu = \nu_2 - \nu_1,\ \nu_1, \nu_2 \in P^+(\pi)$. Since $w_0\nu = -\nu$ it follows that $k_{\theta(i)} = k_i$ so $\theta(\mathfrak{A}^\pm) = \mathfrak{A}^\pm$ and $w_0\nu_1 = -\nu_1,\ w_0\nu_2 = -\nu_2$. Hence
$$z_\nu = z_{\nu_1}^{-1} z_{\nu_2}; \qquad z_{\nu_1} = \prod_{i \in \mathfrak{I} \cap \mathfrak{A}^-} z_i^{k_i} \prod_{i \in \overline{\mathfrak{I}} \cap \mathfrak{A}^-} \tilde{z}_i^{k_i}, \qquad z_{\nu_2} = \prod_{i \in \mathfrak{I} \cap \mathfrak{A}^+} z_i^{k_i} \prod_{i \in \overline{\mathfrak{I}} \cap \mathfrak{A}^+} \tilde{z}_i^{k_i}.$$

Let us show that $z_i \in R_0^w$ for all $i \in \mathfrak{A}^+$(then also $\tilde{z}_i \in R_0^w$ for $i \in \mathfrak{I} \cap \mathfrak{A}^+$) and $z_i^{-1} \in R_0^w$ for all $i \in \mathfrak{A}^-$.

Observe that $z_i \in R_0^w$ if $w\omega_i = \omega_i$ and $z_i^{-1} \in R_0^w$ if $w\omega_i = w_0\omega_i$. Hence it suffices to check that $w\omega_i = \omega_i$ (resp. $w\omega_i = w_0\omega_i$) for all $i \in \mathfrak{A}^+$ (resp. $i \in \mathfrak{A}^-$).

Since $z_\nu \in R_0^w$ there exists $\lambda \in P^+(\pi)$ such that

$$c_w^{-\lambda} c_\xi^\lambda = z_\nu = c_{w_0}^{-\nu_1} c_{w_0}^{\nu_2} c_e^{\nu_1} c_e^{-\nu_2} \quad \Rightarrow \quad c_\xi^\lambda c_{w_0}^{\nu_1} c_e^{\nu_2} \in K^* c_w^\lambda c_{w_0}^{\nu_2} c_e^{\nu_1}.$$

Take $i \in \mathfrak{A}^+$. Then $c_e^{\nu_2} \in Q(s_i)^-$, whereas $c_{w_0}^{\nu_2} c_e^{\nu_1} \notin Q(s_i)^-$. From the formula above we conclude that $c_w^\lambda \in Q(s_i)^-$ so $w\omega_i = \omega_i$. Similarly $i \in \mathfrak{A}^-$ implies that $w\omega_i = w_0\omega_i$. $\square$

8.2.2. *Remark.* Proposition 8.2.1 implies that the rings R_0^w are in general non-isomorphic: they have centres of different Gelfand-Kirillov dimension. Observe that this dimension is maximal if $w = e, w_0$. If $\mathfrak{g}$ is simple then for all $w \neq e, w_0$ one has $\dim Z(R_0^w) < \dim Z(R_0^e)$.

In fact, fix w is such that $\dim Z(R_0^w) = \dim Z(R_0^e)$. This implies that $\overline{\mathfrak{I}} = \overline{\mathfrak{I}}_w^- \cup \overline{\mathfrak{I}}_w^+$ and $\mathfrak{I} = \mathfrak{I}_w^- \cup \mathfrak{I}_w^+$. Set $\mathfrak{J}_1 := \{i | \ w\omega_i = \omega_i\}$, $\mathfrak{J}_2 := \{i | \ w\omega_i = w_0\omega_i\}$. Then $\mathfrak{J}_1 \cup \mathfrak{J}_2 = \{1, \ldots, l\}$, $\mathfrak{J}_1 \cap \mathfrak{J}_2 = \emptyset$. Observe that $w \in W_2$ where W_2 is a subgroup of W which is generated by $\{s_i : \ i | \ w\omega_i \neq \omega_i\} = \{s_i : \ i \in \mathfrak{J}_2\}$. Similarly, $w_0 w \in W_1$ where W_1 is a subgroup of W which is generated by $\{s_i : \ i | \ (w_0 w)\omega_i \neq \omega_i\} = \{s_i : \ i \in \mathfrak{J}_1\}$. Since $w_0 = (w_0 w)w^{-1}$ it follows that $w_0 \in W_1 W_2$ so $W = W_1 W_2$. Since $\mathfrak{g}$ is simple, one has either $W = W_1$ or $W = W_2$. This means that $w = e$ or $w = w_0$.

9. Appendix: index of notations

Symbols used frequently are given below under the section number where they are first defined.

2.1 $\quad k, K, U_q(\mathfrak{g}), \check{T}, \check{U}_q(\mathfrak{g}), U_q(\mathfrak{n}^-), x_i, y_i, t_i^{\pm 1}, l, W, S^w$

2.2 $\quad w_0$

2.3 $\quad \pi, Q(\pi), Q^\pm(\pi), \omega_i, P(\pi), \geq, P^+(\pi), \tau, V(\lambda), c_{\xi,v}^\lambda,$

$$R_q[G], V^+(\lambda), R^+, \Omega(V^+(\lambda)), c_w^\lambda, c_\xi^\lambda, c_w, R^w, R_0^w, \check{R}_0^w$$

2.6 $\quad A[c^{-1}]$

3.2 $\quad \#$

4 $\quad \mathrm{lwt}, \mathrm{rwt}, \cdot|_\lambda, \ \cdot|^\lambda, \ c_\mu^\lambda$

4.1 $\quad J_\lambda^\pm(\eta)$

4.2 $\quad J_\lambda^\pm(\eta)_w$

4.3 $\quad \phi_w^\nu, \Phi_w$

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