to Victor Ginzburg on his 50th birthday

ON REPRESENTATIONS OF THE AFFINE SUPERALGEBRA $\mathfrak{q}(n)^{(2)}$

MARIA GORELIK † AND VERA SERGANOVA

ABSTRACT. In this paper we study highest weight representations of the affine Lie superalgebra $\mathfrak{q}(n)^{(2)}$. We prove that any Verma module over this algebra is reducible and calculate the character of an irreducible $\mathfrak{q}(n)^{(2)}$- module with a generic highest weight. This formula is analogous to the Kac-Kazhdan formula for generic irreducible modules over affine Lie algebras at the critical level.

1. INTRODUCTION

1.1. Affine Lie algebras can be defined as Kac-Moody algebras having finite growth. Recall that the Cartan matrix of an affine Lie algebra is symmetrizable [K2]. An affine Lie algebra can be realized in terms of a finite-dimensional semisimple Lie algebra and its finite order automorphism [K2], Ch. VI-VIII.

The superalgebra generalization of Kac-Moody algebras was introduced in [K1]; a detailed treatment of this notion can be found in [Wk2]. Call a Kac-Moody superalgebra *affine* if it has a finite growth and *symmetrizable* if it has a symmetrizable Cartan matrix. In [vdL],[S] the affine symmetrizable Lie superalgebras are described in terms of finite-dimensional Kac-Moody superalgebras and their finite order automorphisms.

A symmetrizable affine Lie superalgebra has a Casimir element. As a consequence, a Verma module $M(\lambda)$ is irreducible if its highest weight λ is generic. More precisely, $M(\lambda)$ is irreducible unless λ belongs to the union of countably many hyperplanes. Among these hyperplanes one is rather special: $M(\lambda)$ has an infinite length for all λ on this hyperplane. This hyperplane is usually called the set of critical weights.

Non-symmetrizable affine Lie superalgebras were classified in a recent paper [HS]. This classification includes two degenerate superalgebras, one family of constant rank and one series $\mathfrak{q}(n)^{(2)}$. The superalgebras of the series are twisted affinizations of "strange" Lie superalgebras $\mathfrak{q}(n)$.

1.2. In this paper we study the structure of Verma modules over the twisted affine superalgebra $\mathfrak{q}(n)^{(2)}, n \geq 3$. We will see that any Verma module over $\mathfrak{q}(n)^{(2)}$ is reducible.

1991 *Mathematics Subject Classification.* 17B67.

Key words and phrases. affine Lie superalgebra, highest weight representation, Shapovalov form.

† Incumbent of the Frances and Max Hersh career development chair. Supported in part by TMR Grant No. FMRX-CT97-0100.

We prove that a simple highest weight $\mathfrak{q}(n)^{(2)}$-module $L(\lambda)$ with a "generic" highest weight λ has the following character formula:

$$\text{ch}\, L(\lambda) = e^{\lambda} \prod_{\alpha\in\hat{\Delta}^+_{re,\bar{0}}} (1-e^{-\alpha})^{-1} \prod_{\alpha\in\hat{\Delta}^+_{\bar{1}}} (1+e^{-\alpha}), \tag{1}$$

where $\hat{\Delta}^+_{re,\bar{0}}$ is the set of positive real even roots of $\mathfrak{q}(n)^{(2)}$ and $\hat{\Delta}^+_{\bar{1}}$ is the set of positive odd roots of $\mathfrak{q}(n)^{(2)}$. Here genericity means that λ lies in the complement to countably many hypersurfaces in the weight space. This formula is analogous to the Kac-Kazhdan character formula, see below.

1.2.1. Let $\hat{\mathfrak{g}}$ be a complex affine Lie algebra. A Verma module with a generic weight at non-critical level is irreducible. A Verma module at the critical level is always reducible; if λ is a generic weight at the critical level then the character of a simple module $L(\lambda)$ is given by the formula $\text{ch}\, L(\lambda) = e^{\lambda}\prod_{\alpha\in\hat{\Delta}^+_{re}}(1-e^{-\alpha})^{-1}$, where $\hat{\Delta}^+_{re}$ is the set of positive real roots. This formula was conjectured by V. G. Kac and D. A. Kazhdan in [KK] and was proven in different cases by different methods in [Wk1], [Wl], [H], [GW], [FF], [F], [Ar], [M]. For an arbitrary affine Lie algebras (including the twisted case) the formula was proven by J.-M. Ku [Ku] and recently reproven by M. Szczesny [Sz]. Moreover, as it is shown in [Ku], there is a bijection between submodules of a Verma module with a "generic" highest weight at the critical level and graded ideals of a polynomial algebra in countable number of variables, in fact, the Jantzen filtration of a Verma module corresponds to the adic filtration of a polynomial algebra.

It turns out that the adic filtration comes to the picture as follows. Let K be the canonical central element of $\hat{\mathfrak{g}}$. Denote by $\mathcal{H}$ (resp., by $\mathcal{H}_-$) the sum of positive (resp., negative) imaginary root spaces of $\hat{\mathfrak{g}}$. Then $\mathcal{H}_-\oplus\mathbb{C}K\oplus\mathcal{H}$ is a countably dimensional Heisenberg algebra. A vacuum module of the Heisenberg algebra is irreducible if the central charge is non-zero. Let V^0 be the vacuum module with zero central charge. Recall that $\hat{\mathfrak{g}}$ is $\mathbb{Z}$-graded (by the eigenvalues of D) and $\mathcal{H}_-$ is its homogeneous subspace. Identify V^0 with the universal enveloping algebra $\mathcal{U}(\mathcal{H}_-)$, which is a polynomial algebra in a countable number of variables, and view V^0 as a graded module via this identification. It is easy to see that the Jantzen filtration on V^0 corresponds to the adic filtration on $\mathcal{U}(\mathcal{H}_-)$. In this interpretation a Verma module $M(\lambda)$ with a generic highest weight at the critical level looks like V^0: there exists an isomorphism HC_+ from the space of singular vectors $M(\lambda)^{\hat{\mathfrak{n}}}$ to V^0, which induces a bijection between the submodules of $M(\lambda)$ and the homogeneous submodules of V^0; moreover, this bijection is compatible with the Jantzen filtrations.

A similar fact holds for affine Lie superalgebras with symmetrizable Cartan matrices if $\mathcal{H}_-\oplus\mathbb{C}K\oplus\mathcal{H}_+$ is a Heisenberg algebra (i.e., for all $\hat{\mathfrak{g}}\neq A(2k,2l)^{(4)}$), see [G].

1.2.2. Let us go back to $\mathfrak{q}(n)^{(2)}$ for $n\geq 3$. In this case the subalgebra $\tilde{\mathcal{H}}$ generated by imaginary root spaces is not isomorphic to a Heisenberg algebra. It turns out that the even part $\tilde{\mathcal{H}}_{\bar{0}}$ of $\tilde{\mathcal{H}}$ is a centre of $\tilde{\mathcal{H}}$. As a result, a Verma module over $\tilde{\mathcal{H}}$ is always reducible. We show (Theorem 4.1) that a Verma $\mathfrak{q}(n)^{(2)}$-module $M(\lambda)$ with a generic highest weight λ looks like a Verma $\tilde{\mathcal{H}}$-module $\mathcal{M}(\lambda)$: there exists a linear map HC_+ from the space of singular vectors $M(\lambda)^{\hat{\mathfrak{n}}}$ to $\mathcal{M}(\lambda)$, which extends to a bijection between the submodules of $M(\lambda)$ and the graded submodules of $\mathcal{M}(\lambda)$. This implies the character formula (1).

It is possible to compute (as it is done in [G]) the Jantzen filtration on $M(\lambda)$ for a generic weight λ; the result is $M(\lambda)^0 = M(\lambda)$, $M(\lambda)^k = M'(\lambda)$ for any $k > 0$, where $M'(\lambda)$ is the maximal proper submodule of $M(\lambda)$. The structure of the Jantzen filtration on $\mathcal{M}(\lambda)$ is similar, and therefore the bijection between the submodules in $M(\lambda)$ and in $\mathcal{M}(\lambda)$ is compatible with the Jantzen filtrations.

1.2.3. Recall that for affine Lie superalgebras with symmetrizable Cartan matrices all Shapovalov determinants are non-zero polynomials, which admit a linear factorization (i.e. all their irreducible factors are linear).

For $\mathfrak{q}(n)^{(2)}$ this does not hold. The fact that any Verma module is not simple means that there exists a Shapovalov determinant $\det S_\nu$ which is identically equal to zero. Let δ be a minimal imaginary root. It turns out that $\det S_\nu = 0$ iff $\nu \geq 2\delta$. Another interesting feature of $\mathfrak{q}(n)^{(2)}$ is that $\det S_\delta$ has an irreducible factor of degree $n-1$: this follows from the fact that the leading term of $\det S_\delta$ is divisible by the irreducible polynomial $h_1 \dots h_n(\frac{1}{h_1} + \dots \frac{1}{h_n})$ which has degree $n-1$.

1.3. **Content of the paper.** In Section 2 we describe $\mathfrak{q}(n)^{(2)}$, its subalgebra $\tilde{\mathcal{H}}$ and define the projection HC_+.

In Section 3 we describe Verma modules over $\tilde{\mathcal{H}}$.

In Section 4 we formulate the main result of the paper.

In Section 5 we give some results on Verma modules over an arbitrary Kac-Moody superalgebra, which are used in Section 7.1. In particular, we show why the formula $[M(\lambda) : L(\lambda-\nu)] = \dim M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\nu}$ implies that any submodule of $M(\lambda)$ is generated by singular vectors, see Theorem 4.1 (ii).

In Section 6 we show that for a generic weight λ any singular vector of $M(\lambda)$ is of the weight $\lambda - 2s\delta$ for some $s \in \mathbb{Z}_{\geq 0}$.

In Section 7.1 we describe a suitable set $\Lambda \subset \hat{\mathfrak{h}}^*$ of generic roots and reduce Theorem 4.1 (i), (ii) to assertions (A), (B).

In Section 8 we explicitly construct the singular vectors in $M(\lambda)$ for a generic $\lambda \in \Lambda$; this establishes the assertion (B).

In Section 9 we deduce the assertion (A) and Theorem 4.1 (iii); this completes the proof of Theorem 4.1.

In Section 10 we prove a technical lemma, which is used in Section 8.

1.4. We would like to thank the referee for very useful suggestions.

2. Preliminaries and notation

Our base field is $\mathbb{C}$. For a Lie superalgebra $\mathfrak{g}$ we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra and by $\mathcal{S}(\mathfrak{g})$ its symmetric algebra. If M is a $\mathfrak{g}$-module, then $M^{\mathfrak{g}}$ denotes the subspace of $\mathfrak{g}$-invariants in M.

2.1. **Description of $\mathfrak{q}(n)^{(2)}$.** The Lie superalgebra $\mathfrak{q}(n)^{(2)}$ can be constructed as follows. Recall that $\mathfrak{sq}(n)\subset\mathfrak{gl}(n|n)$ consists of the matrices with the block form

$$X_{A,B}:=\begin{pmatrix} A & | & B\\ -- & - & --\\ B & | & A\end{pmatrix}$$

where A is an arbitrary $n\times n$ matrix, B is a traceless $n\times n$ matrix. The even (resp., odd) part consists of the matrices $X_{A,0}$ (resp., $X_{0,B}$).

Let $\mathcal{L}(\mathfrak{sq}(n))=\mathfrak{sq}(n)\otimes\mathbb{C}[t^{\pm 1}]$ be the corresponding loop superalgebra. Then $\mathfrak{q}(n)^{(1)}=\mathcal{L}(\mathfrak{sq}(n))\oplus\mathbb{C}D$, where D acts on $\mathcal{L}(\mathfrak{sq}(n))$ by $[D,x\otimes t^k]=kx\otimes t^k$. Note that $\mathfrak{sq}(n)$, $\mathfrak{q}(n)^{(1)}$ are not Kac-Moody superalgebras since their Cartan subalgebras contain odd elements. For instance, the matrices $X_{A,B}$ with A,B diagonal form a Cartan subalgebra of $\mathfrak{sq}(n)$.

Let ε be an automorphism of $\mathfrak{sq}(n)$ acting by id on $\mathfrak{sq}(n)_{\bar 0}=\mathfrak{gl}(n)$ and by $-\,\mathrm{id}$ on $\mathfrak{sq}(n)_{\bar 1}$. Extend ε to $\mathfrak{q}(n)^{(1)}$ by $\varepsilon(t)=-t$, $\varepsilon(D)=D$. One can define $\mathfrak{q}(n)^{(2)}$ as the quotient of the subalgebra $(\mathfrak{q}(n)^{(1)})^{\varepsilon}$ of elements fixed by ε by the abelian ideal $\sum_{i\neq 0}\mathbb{C}X_{I,0}\otimes t^{2i}$, where I stands for the $n\times n$ identity matrix. This is a Kac-Moody superalgebra with a non-symmetrizable Cartan matrix, see [HS]. The algebras $\mathfrak{sq}(n)$, $\mathfrak{q}(n)^{(1)},\mathfrak{q}(n)^{(2)}$ do not have even invariant bilinear forms, but have the odd ones.

2.1.1. Using the above definition one can identify $\mathfrak{q}(n)^{(2)}$ with the vector space

$$\mathfrak{sl}(n)\otimes\mathbb{C}[t^{\pm 1}]\oplus\mathbb{C}K\oplus\mathbb{C}D,$$

where $\mathfrak{q}(n)^{(2)}_{\bar 0}=\mathfrak{sl}(n)\otimes\mathbb{C}[t^{\pm 2}]\oplus\mathbb{C}K\oplus\mathbb{C}D$, $\mathfrak{q}(n)^{(2)}_{\bar 1}=\mathfrak{sl}(n)\otimes t\mathbb{C}[t^{\pm 2}]$, and the commutator is given by

$$[x\otimes t^k,y\otimes t^m]=\begin{cases}(xy-yx)\otimes t^{k+m}, & \text{if } km \text{ is even},\\ [x\otimes t^k,y\otimes t^m]=\iota(xy+yx)\otimes t^{k+m}+2\delta_{-k,m}\operatorname{tr}(xy)K, & \text{if } km \text{ is odd},\end{cases}$$

where $\iota:\mathfrak{gl}(n)\to\mathfrak{sl}(n)$ is given by $x\mapsto x-\operatorname{tr}(x)I/n$ and I is the $n\times n$ identity matrix.

Notice that the central element K does not lie in $[\mathfrak{q}(n)^{(2)}_{\bar 0},\mathfrak{q}(n)^{(2)}]$, but lies in $[\mathfrak{q}(n)^{(2)}_{\bar 1},\mathfrak{q}(n)^{(2)}_{\bar 1}]$.

2.1.2. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{sl}(n)$. Then

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}D$$

is a Cartan subalgebra of $\hat{\mathfrak{g}}$.

Fix triangular decompositions $\mathfrak{sl}(n) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}$, where

$$\hat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus \mathfrak{sl}(n) \otimes t^{-1}\mathbb{C}[t^{-1}], \quad \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D, \quad \hat{\mathfrak{n}} = \mathfrak{n} \oplus \mathfrak{sl}(n) \otimes t\mathbb{C}[t].$$

2.1.3. Set

$$\tilde{\mathcal{N}}^+ := \mathfrak{n} \otimes \mathbb{C}\left[t^{\pm 1}\right], \quad \tilde{\mathcal{N}}^- := \mathfrak{n}_- \otimes \mathbb{C}\left[t^{\pm 1}\right], \quad \tilde{\mathcal{H}} := \mathfrak{h} \otimes \mathbb{C}\left[t^{\pm 1}\right] \oplus \mathbb{C}K.$$

Observe that $\hat{\mathfrak{g}} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] \oplus \mathbb{C}D$ and that $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \tilde{\mathcal{N}}^+ \oplus \tilde{\mathcal{H}} \oplus \tilde{\mathcal{N}}^-$ is a triangular decomposition of $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$.

2.2. **Root system.** We denote by $\hat{\Delta}^+$ the set of positive roots of $\hat{\mathfrak{g}}$, by $\hat{\Delta}^+_{\bar{0}}$ (resp., $\hat{\Delta}^+_{\bar{1}}$) the set of positive even (resp., odd) roots, and by $\hat{\Delta}^+_{re}$ (resp., $\hat{\Delta}^+_{im}$) the set of positive real (resp., imaginary) roots.

2.2.1. In the standard notation the positive roots of $\mathfrak{sl}(n)$ are $\{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$. The real roots of $\hat{\mathfrak{g}}$ are of the form $\varepsilon_i - \varepsilon_j$, $k\delta \pm (\varepsilon_i - \varepsilon_j)$, where $k \in \mathbb{Z}_{>0}$, $1 \leq i < j \leq n$. The $k\delta + (\varepsilon_i - \varepsilon_j)$ root space of $\hat{\mathfrak{g}}$ is spanned by $E_{ij} \otimes t^k$, where E_{ij} is a generator of the root space $\mathfrak{sl}(n)_{\varepsilon_i - \varepsilon_j}$. The root $k\delta \pm (\varepsilon_i - \varepsilon_j)$ is even (resp., odd) if k is even (resp., odd).

Denote by δ the minimal positive imaginary root. Then $\hat{\Delta}^+_{im} = \mathbb{Z}_{>0}\delta$, where $k\delta$ is an even (resp., odd) root if k is even (resp., odd). For $k \neq 0$ one has $\hat{\mathfrak{g}}_{k\delta} = \mathfrak{h} \otimes t^k$.

2.2.2. Set

$$\hat{Q}^+ := \sum_{\alpha \in \hat{\Delta}^+} \mathbb{Z}_{\geq 0}\alpha.$$

Define a partial ordering on $\hat{\mathfrak{h}}^*$ by setting $\mu \geq \mu'$ if $\mu - \mu' \in \hat{Q}^+$.

2.3. **Verma modules.** For each $\lambda \in \hat{\mathfrak{h}}^*$ let $M(\lambda)$ be the Verma module of the highest weight λ, let v_λ be the canonical generator of $M(\lambda)$ and let $M'(\lambda)$ be the maximal proper submodule of $M(\lambda)$. The module $L(\lambda) := M(\lambda)/M'(\lambda)$ is simple.

We call $v \in M(\lambda)$ *singular* if v is a weight vector and $v \in M(\lambda)^{\hat{\mathfrak{n}}}$.

2.3.1. Verma modules do not admit Jordan-Hölder series, since some Verma modules have an infinite length. However, so-called *local series* introduced in [DGK] are nice substitution for Jordan-Hölder ones. A series of weight modules $N = N_0 \supset N_1 \supset \ldots \supset N_m = 0$ is called *local at* $\nu \in \hat{\mathfrak{h}}^*$ if either $N_i/N_{i+1} \cong L(\lambda_i)$ for some $\lambda_i \geq \nu$ or $(N_i/N_{i+1})_\mu = 0$ for all $\mu \geq \nu$. This allows to define the *multiplicity* $[N : L(\lambda)]$ as the number of i such that $N_i/N_{i+1} \cong L(\lambda)$ for a series local at some $\nu \leq \lambda$; it turns out that the multiplicity does not depend on the choice of local series. Any Verma module has a local series so $[M(\lambda) : L(\lambda')]$ is well-defined.

2.4. **The subalgebra $\tilde{\mathcal{H}}$.** The subalgebra $\tilde{\mathcal{H}}$ defined in 2.1.3 is generated by the imaginary root spaces. One has $\tilde{\mathcal{H}}_{\bar{0}} = \sum_{r\in\mathbb{Z}} \mathfrak{h} \otimes t^{2r} \oplus \mathbb{C}K$, $\tilde{\mathcal{H}}_{\bar{1}} = \sum_{r\in\mathbb{Z}} \mathfrak{h} \otimes t^{2r+1}$. Set

$$\mathcal{H}_- := \tilde{\mathcal{H}} \cap \hat{\mathfrak{n}}_- = \mathfrak{h} \otimes t^{-1}\mathbb{C}\left[t^{-1}\right], \qquad \mathcal{S} := \mathcal{U}(\mathcal{H}_{-,\bar{0}}).$$

The subalgebra $\mathcal{H}_{-,\bar{0}} = \mathfrak{h} \otimes t^{-2}\mathbb{C}[t^{-2}]$ is commutative so $\mathcal{S}$ is a polynomial algebra in countably many variables.

2.4.1. It is convenient to identify $\mathfrak{h}$ with the space of traceless diagonal matrices and K with I/n, where I is the $n \times n$ identity matrix. Then $\mathfrak{sl}(n) \oplus \mathbb{C}K$ is identified with $\mathfrak{gl}(n)$ and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes 1 \oplus \mathbb{C}K$ with the space of diagonal matrices; by $h_1, ..., h_n$ we denote the standard basis in the latter ($h_i = E_{ii}$). The subalgebra $\tilde{\mathcal{H}}$ is spanned by $\{h_i : i = 1, \ldots n; (h_i - h_{i+1}) \otimes t^r : \ i = 1, \ldots, n-1, r \in \mathbb{Z}, r \neq 0\}$.

The centre of $\tilde{\mathcal{H}}$ coincides with $\tilde{\mathcal{H}}_{\bar{0}}$. The odd part $\tilde{\mathcal{H}}_{\bar{1}}$ is spanned by $\{(h_i - h_{i+1}) \otimes t^{2r+1},\ i = 1, \ldots, n-1, r \in \mathbb{Z}\}$ and the relations are

$$[x \otimes t^{2r+1}, y \otimes t^{-2r-1}] = 2xy, \quad [x \otimes t^{2r+1}, y \otimes t^{2s+1}] = \iota(2xy) \otimes t^{2(r+s+1)},$$

for $r + s + 1 \neq 0$ (recall that $\iota(x) = x - \mathrm{tr}(x)I/n$).

2.4.2. For $\alpha \in \hat{\Delta}^+$ let D_α be a matrix of the pairing $\hat{\mathfrak{g}}_\alpha \times \hat{\mathfrak{g}}_{-\alpha} \to \hat{\mathfrak{h}}$ given by $(x, y) \mapsto [x, y]$.

Note that $D_\alpha = (h_\alpha)$ if α is real, and that $D_{m\delta}$ is an $(n-1) \times (n-1)$ matrix. By 2.4.1, $D_{2m\delta}$ is the zero matrix and $D_{(2m+1)\delta} = D_\delta$. A straightforward calculation shows that

$$D_\delta = \begin{pmatrix} 2(h_1 - h_2) & -2h_2 & 0 & \ldots & & \\ -2h_2 & 2(h_2 - h_3) & -2h_3 & 0\ldots & & \\ 0 & -2h_3 & 2(h_3 - h_4) & -2h_4 & 0\ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{pmatrix}$$

and so $\det D_\delta = 2^{n-1}h_1 \ldots h_n(\frac{1}{h_1} + \ldots + \frac{1}{h_n})$. Here $\det D_\delta$ is considered as an element in $\mathcal{S}(\hat{\mathfrak{h}})$.

2.5. **Projections** HC **and** HC_+. Denote by HC the Harish-Chandra projection HC : $\mathcal{U}(\hat{\mathfrak{g}}) \to \mathcal{U}(\hat{\mathfrak{h}}) = \mathcal{S}(\hat{\mathfrak{h}})$ with the kernel $\mathcal{U}(\hat{\mathfrak{g}})\hat{\mathfrak{n}} + \hat{\mathfrak{n}}_-\mathcal{U}(\hat{\mathfrak{g}})$ using the decomposition $\mathcal{U}(\hat{\mathfrak{g}}) = \mathcal{U}(\hat{\mathfrak{h}}) \oplus (\mathcal{U}(\hat{\mathfrak{g}})\hat{\mathfrak{n}} + \hat{\mathfrak{n}}_-\mathcal{U}(\hat{\mathfrak{g}}))$.

Retain notation of 2.1.3 and recall the triangular decomposition $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \mathcal{N}^+ + \tilde{\mathcal{H}} + \mathcal{N}^-$. Define the projection $\mathrm{HC}_+ : \mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]) \to \mathcal{U}(\tilde{\mathcal{H}})$ with the kernel $\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])\mathcal{N}^- + \mathcal{N}^+\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])$. The restriction of HC_+ to $\mathcal{U}(\hat{\mathfrak{n}}_-)$ is a projection $\mathcal{U}(\hat{\mathfrak{n}}_-) \to \mathcal{U}(\mathcal{H}_-)$. Define $\mathrm{HC}_+ : M(\lambda) \to \mathcal{U}(\mathcal{H}_-)$ via the natural identification of $M(\lambda)$ with $\mathcal{U}(\hat{\mathfrak{n}}_-)$.

The restriction of HC to $\mathcal{U}(\hat{\mathfrak{g}})^{\hat{\mathfrak{h}}}$ is an algebra homomorphism. Similarly, the restriction of HC_+ to $\mathcal{U}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])^{\mathfrak{h}}$ is an algebra homomorphism.

2.6. **Characters.** For a diagonalizable $\hat{\mathfrak{h}}$-module M we denote by $\Omega(M)$ the set of weights of M and by M_μ the weight space of weight μ.

If M is diagonalizable over $\hat{\mathfrak{h}}$ and all its weight spaces are finite dimensional, one can define the character $\operatorname{ch} M$ as

$$\operatorname{ch} M = \sum_\mu \dim M_\mu e^\mu.$$

2.6.1. For each λ let C_λ be the collection of elements of the form $\sum_{\mu<\lambda} c_\mu e^\mu$, where $c_\mu \in \mathbb{Z}_{\geq 0}$. Set $C := \{\sum_{i=1}^k x_i |\ x_i \in C_{\lambda_i}\}$. Note that $x, y \in C$ implies $xy \in C$. For $x, y \in C$ write $x \geq y$ if $x - y \in C$. In all our examples $\operatorname{ch} M$ belongs to C.

2.6.2. Recall that $\hat{\mathfrak{g}} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] \oplus \mathbb{C}D$. Therefore $M(\lambda)$ and $M(\lambda - r\delta)$ are isomorphic as $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$-modules, and hence $M'(\lambda) \cong M'(\lambda - r\delta)$. As a consequence, $\operatorname{ch} L(\lambda - r\delta) = e^{-r\delta} \operatorname{ch} L(\lambda)$.

2.6.3. *Choice of antiautomorphism.* Call a linear endomorphism σ of a superalgebra a "naive" antiautomorphism if σ is invertible and $\sigma([xy]) = [\sigma(y), \sigma(x)]$. Every Kac-Moody Lie superalgebra has a naive involutive antiautomorphism σ such that $\sigma|_{\hat{\mathfrak{h}}} = \mathrm{id}$, and $\sigma(\hat{\mathfrak{n}}) = \hat{\mathfrak{n}}_-$. Define a naive anti-involution of $\hat{\mathfrak{g}}$ by

$$D \mapsto D, \quad a \otimes t^r \mapsto a' \otimes t^{-r},$$

where a' is the transpose of a matrix $a \in \mathfrak{gl}(n)$.

If M is a $\hat{\mathfrak{g}}$-module with finite-dimensional weight spaces, then the restricted dual space M^* of M has the structure of $\hat{\mathfrak{g}}$-module defined by $(Xf)(m) := f(\sigma(X)m)$ for any $f \in M^*, m \in M, X \in \hat{\mathfrak{g}}$. The corresponding module will be denoted by M^σ. It is easy to see that σ is a contravariant functor on the category of weight modules with well defined characters. Moreover, $\operatorname{ch} M = \operatorname{ch} M^\sigma$ and therefore $L(\lambda)^\sigma \cong L(\lambda)$. It is also clear that $[M : L(\lambda)] = [M^\sigma : L(\lambda)]$.

3. The Lie superalgebra $\tilde{\mathcal{H}}$

In this section we study Verma modules over $\tilde{\mathcal{H}}$.

3.1. Write $\tilde{\mathcal{H}} = \mathcal{H}_- \oplus \mathbb{C}K \oplus \mathfrak{h} \oplus \mathcal{H}$, where $\mathcal{H}_- = \tilde{\mathcal{H}} \cap \hat{\mathfrak{n}}_- = \sum_{r<0} \mathfrak{h} \otimes t^r$, $\mathcal{H} := \tilde{\mathcal{H}} \cap \hat{\mathfrak{n}} = \sum_{r>0} \mathfrak{h} \otimes t^r$. For $\lambda \in \hat{\mathfrak{h}}^*$ let $\mathcal{M}(\lambda)$ be a Verma $\tilde{\mathcal{H}} \oplus \mathbb{C}D$-module that is

$$\mathcal{M}(\lambda) = \mathcal{U}(\tilde{\mathcal{H}}) \otimes_{\mathcal{U}(\hat{\mathfrak{h}}+\mathcal{H})} \mathbb{C}v_\lambda, \quad \text{where } \mathcal{H}v_\lambda = 0,\ hv_\lambda = \lambda(h)v_\lambda \text{ for any } h \in \hat{\mathfrak{h}}.$$

3.1.1. It is clear that $\mathcal{M}$ is diagonalizable over $\hat{\mathfrak{h}}$ and its character is given by

$$\operatorname{ch} \mathcal{M}(\lambda) = e^\lambda \prod_{\alpha \in \Delta_{im,\bar{0}}} (1 - e^{-\alpha})^{-1} \prod_{\alpha \in \Delta_{im,\bar{1}}} (1 + e^{-\alpha}).$$

The subalgebra $\mathcal{S} = \mathcal{U}(\mathcal{H}_{-,\bar{0}})$ lies in the centre of $U(\tilde{\mathcal{H}})$ and acts freely on $\mathcal{M}(\lambda)$. Every ideal J of $\mathcal{S}$ defines the $\tilde{\mathcal{H}}$-submodule $J\mathcal{M}(\lambda)$ in $\mathcal{M}(\lambda)$. On the other hand, any $\tilde{\mathcal{H}}$-submodule $N \subset \mathcal{M}(\lambda)$ defines the ideal $J_N \subset \mathcal{S}$ such that $N \cap \mathcal{S}v_\lambda = J_N v_\lambda$.

Proposition. *Assume that $\lambda \in \hat{\mathfrak{h}}^*$ is such that $\det D_\delta(\lambda) \neq 0$. Then the map $N \mapsto J_N$ defines a bijection between $\tilde{\mathcal{H}}$-submodules in $\mathcal{M}(\lambda)$ and ideals in $\mathcal{S}$.*

Proof. It suffices to check that $N = J_N\mathcal{M}(\lambda)$ and it is clear that $J_N\mathcal{M}(\lambda) \subset N$. Let $V = \mathcal{M}(\lambda)/J_N\mathcal{M}(\lambda)$ and W be the image of N in V under the natural projection. We have to show that $W = 0$. Set $R = \mathcal{S}/J_N$. It is easy to see that V is a free R-module and that $W \cap Rv_\lambda = 0$.

Let $u_1, ..., u_{n-1}$ be a basis in $\mathfrak{h}$, $X_{i,j} := u_i \otimes t^{-2j-1}$, $Y_{i,j} := u_i \otimes t^{2j+1}$. Recall that $[X_{i,j}, Y_{k,j}] = [X_{i,0}, Y_{k,0}]$. Due to the assumption on λ one can choose $u_1, ..., u_{n-1}$ so that $\lambda([X_{i,j}, Y_{k,j}]) = \delta_{i,k}$. Clearly $X_{i,j}$ for $i = 1, ..., n-1$, $j \geq 0$ form a basis in $\tilde{\mathcal{H}}_{-,\bar{1}}$. Introduce the order on $X_{i,j}$ by setting $X_{i,j} \geq X_{k,l}$ if $j \geq l$ or $j = l$ and $i \geq k$. Then $X_{i_1,j_1}...X_{i_k,j_k}v_\lambda$ for all $X_{i_1,j_1} \geq ... \geq X_{i_k,j_k}$ form a basis of V over R. Now assume that $W \neq 0$. Since $W \cap Rv_\lambda = 0$, one can pick up a non-zero $v \in W$ such that the maximal $X_{i,j}$ which appears in the decomposition of v is minimal among all non-zero vectors in W. One can write in the unique way $v = X_{i,j}w + u$ for some non-zero w and u such that in the decomposition of u, w only $X_{k,l} < X_{i,j}$ appear. Then $Y_{i,j}u = Y_{i,j}w = 0$ and $Y_{i,j}v = w$. Then $w \in W$, $w \neq 0$ and all $X_{k,l}$ in the decomposition of w are less than $X_{i,j}$. Contradiction. □

3.2. Now consider $\mathcal{M}(\lambda)$ as a module over $\mathbb{C}D + \tilde{\mathcal{H}}$. Then $\mathcal{M}$ has a unique maximal submodule $\mathcal{M}'(\lambda)$ and a unique simple quotient $\mathcal{L}(\lambda) = \mathcal{M}(\lambda)/\mathcal{M}'(\lambda)$. Moreover, there is a bijection between submodules of $\mathcal{M}(\lambda)$ and graded ideals of $\mathcal{S}$, where the grading on $\mathcal{S}$ is defined by D. The above proposition implies the following

Corollary. *For a generic weight $\lambda \in \hat{\mathfrak{h}}^*$ one has*

(i) $\mathcal{M}(\lambda)^{\mathcal{H}} = \mathcal{S}v_\lambda$;

(ii) $[\mathcal{M}(\lambda):\mathcal{L}(\lambda-s\delta)]=\mathcal{M}(\lambda)^{\mathcal{H}}_{\lambda-s\delta}$ *for all* s*;*
(iii) *any submodule* N *of* $\mathcal{M}(\lambda)$ *is generated by* $\mathcal{M}(\lambda)^{\mathcal{H}}$*;*
(iv) $\operatorname{ch}\mathcal{L}(\lambda)=e^{\lambda}\prod_{\alpha\in\Delta_{im,\overline{1}}}(1+e^{-\alpha})$.

4. Main result

In this paper we will prove the following theorem.

4.1. **Theorem.** *Let* $\lambda\in\hat{\mathfrak{h}}^*$ *be a generic weight.*

(i) *The restriction of* HC_+ *to* $M(\lambda)^{\hat{\mathfrak{n}}}$ *provides a bijection* $\mathrm{HC}_+: M(\lambda)^{\hat{\mathfrak{n}}}\to\mathcal{S}$.
(ii) *One has* $[M(\lambda):L(\lambda-\nu)]=\dim M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\nu}$ *so any submodule of* $M(\lambda)$ *is generated by singular vectors.*
(iii) *A submodule generated by a singular vector is isomorphic to a Verma module* $M(\lambda-s\delta)$ *for some* $s\geq 0$.

4.1.1. *Remark.* In Theorem 4.1 we may replace the projection HC_+ by the projection $\mathrm{HC}_-:\mathcal{U}([\hat{\mathfrak{g}},\hat{\mathfrak{g}}])\to\mathcal{U}(\tilde{\mathcal{H}})$ with the kernel $\mathcal{U}([\hat{\mathfrak{g}},\hat{\mathfrak{g}}])\tilde{\mathcal{N}}^+ +\tilde{\mathcal{N}}^-\mathcal{U}([\hat{\mathfrak{g}},\hat{\mathfrak{g}}])$.

Theorem 4.1 holds for a generic weight at any level $k\in\mathbb{C}$, see Remark 9.3.

4.2. **Character formula (1).** View $\mathcal{S}\subset\mathcal{U}(\hat{\mathfrak{g}})$ as an $\hat{\mathfrak{h}}$-module with respect to the adjoint action; if $I\subset\mathcal{S}$ is $\hat{\mathfrak{h}}$-invariant, we denote by $\operatorname{ch} I$ its character. Note that the set of weights of $\mathcal{S}$ is $\{-2s\delta\}_{s=1}^{\infty}$.

Let λ be a generic weight in the sense of Theorem 4.1. From Theorem 4.1 (i) we see that the weight of a singular vector in $M(\lambda)$ is of the form $\lambda-2s\delta$ ($s\in\mathbb{Z}_{>0}$).

For a submodule N of $M(\lambda)$ set $H(N):=\mathrm{HC}_+(N^{\hat{\mathfrak{n}}})\subset\mathcal{S}$. By Theorem 4.1 (ii), N is generated by $N^{\hat{\mathfrak{n}}}$. Thus H provides a one-to-one correspondence between the submodules of $M(\lambda)$ and the $\hat{\mathfrak{h}}$-invariant ideals of $\mathcal{S}$. Moreover, by Theorem 4.1 (ii),

$$\operatorname{ch} N=\sum_s[N:L(\lambda-2s\delta)]\operatorname{ch} L(\lambda-2s\delta)=\sum_s\dim(N^{\hat{\mathfrak{n}}})_{\lambda-2s\delta}\operatorname{ch} L(\lambda-2s\delta).$$

One has $\operatorname{ch} L(\lambda-2s\delta)=e^{-2s\delta}\operatorname{ch} L(\lambda)$ (see 2.6.2). As a result,

$$\operatorname{ch} N=\operatorname{ch} L(\lambda)\cdot\operatorname{ch}(H(N)).$$

Applying this formula to $N=M(\lambda)$ we get $\operatorname{ch} M(\lambda)=\operatorname{ch} L(\lambda)\cdot\operatorname{ch}\mathcal{S}$, which is equivalent to the character formula (1).

5. Some general properties of Verma modules

Let $\hat{\mathfrak{g}}=\hat{\mathfrak{n}}_-+\hat{\mathfrak{h}}+\hat{\mathfrak{n}}$ be an arbitrary Kac-Moody superalgebra.

5.1. **Proposition.** *If $[M(\lambda) : L(\lambda')] > 0$ then either $M(\lambda)$ has a singular vector of weight λ' or $[M(\lambda) : L(\lambda'')]$, $[M(\lambda'') : L(\lambda')] > 0$ for some λ'' such that $\lambda' < \lambda'' < \lambda$.*

Proof. Let us assume that $[M(\lambda) : L(\lambda')] > 0$ and $M(\lambda)$ has no singular vectors of weight λ'. Then $\lambda \neq \lambda'$, hence one can find a proper submodule $N \subset M(\lambda)$ such that $[N : L(\lambda')] > 0$. Let $S = (\mathcal{U}(\mathfrak{n})N_{\lambda'})^{\mathfrak{n}}$. By our construction $[\oplus_{\lambda''\in\Omega(S)} M(\lambda'')^{\sigma} : L(\lambda')] > 0$, hence $M(\lambda'')^{\sigma} : L(\lambda')] > 0$ and by duality $[M(\lambda'') : L(\lambda')] > 0$ for some $\lambda'' \in \Omega(S)$. Then $\lambda' < \lambda'' < \lambda$ and $[M(\lambda) : L(\lambda'')] > 0$ as $M(\lambda)$ contains a singular vector of weight λ''. □

5.1.1. **Corollary.** *If $\hat{\mathfrak{g}}$ is a Lie algebra then*

$$[M(\lambda) : L(\lambda')] > 0 \Longrightarrow \ M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda'} \neq 0.$$

Proof. If $\hat{\mathfrak{g}}$ is a Lie algebra then $M(\nu)^{\hat{\mathfrak{n}}}_{\nu'} \neq 0$ implies that $M(\nu')$ is a submodule of $M(\nu)$; the assertion follows from Proposition 5.1 by induction on $\lambda - \lambda'$. □

5.2. We will use the following lemma.

Lemma. *Assume that λ is such that $[M(\lambda) : L(\mu)] = M(\lambda)^{\hat{\mathfrak{n}}}_{\mu}$ for all $\mu \in \hat{\mathfrak{h}}^*$. Then any submodule M of $M(\lambda)$ is generated by its singular vectors.*

Proof. Let $\mathcal{Y}$ be the set of subquotients of $M(\lambda)$ and $\mathcal{Y}' := \{M \in \mathcal{Y} | \ \forall\mu\ [M : L(\mu)] = \dim M^{\hat{\mathfrak{n}}}_{\mu}\}$. Let us show that $\mathcal{Y}' = \mathcal{Y}$ and that for any $M, N \in \mathcal{Y}$ such that N is a submodule of M one has

$$\dim M^{\hat{\mathfrak{n}}}_{\mu} = \dim N^{\hat{\mathfrak{n}}}_{\mu} + \dim(M/N)^{\hat{\mathfrak{n}}}_{\mu}. \tag{2}$$

Take $M \in \mathcal{Y}'$ and let N be a submodule of M. One has

$$[M : L(\mu)] = [N : L(\mu)] + [M/N : L(\mu)] \geq \dim N^{\hat{\mathfrak{n}}}_{\mu} + \dim(M/N)^{\hat{\mathfrak{n}}}_{\mu} \geq \dim M^{\hat{\mathfrak{n}}}_{\mu}.$$

This gives $N,\ M/N \in \mathcal{Y}'$ and implies (2). Since $M(\lambda) \in \mathcal{Y}'$ we obtain $\mathcal{Y}' = \mathcal{Y}$.

Now let M be any submodule of $M(\lambda)$ and let N be the submodule generated by $M^{\hat{\mathfrak{n}}}$. By (2), $\dim(M/N)^{\hat{\mathfrak{n}}}_{\mu} = 0$ for any μ. Hence $M/N = 0$ as required. □

6. Weights of singular vectors

In this section $\hat{\mathfrak{g}} = \mathfrak{q}(n)^{(2)}$. Recall that $[\hat{\mathfrak{g}}_{2s\delta}, \hat{\mathfrak{g}}_{-2s\delta}] = 0$ and that for $\alpha \notin 2\mathbb{Z}\delta$ the pairing $\hat{\mathfrak{g}}_{\alpha} \otimes \hat{\mathfrak{g}}_{\alpha} \to \hat{\mathfrak{h}}$ given by the bracket is non-degenerate.

6.1. **Proposition.** *If $\mu \notin 2\mathbb{Z}\delta$ then the weight space $M(\lambda)_{\lambda-\mu}$ does not contain a singular vector for a generic $\lambda \in \hat{\mathfrak{h}}^*$.*

Remark. "Genericity" here means that the property holds on the complement to a hypersurface.

Proof. Assume that $M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\mu} \neq 0$ for all $\lambda \in \mathfrak{h}^*$. This means that a certain system of homogeneous linear equations with coefficients in $\mathcal{S}(\mathfrak{h})$ has a non-zero solution for each $\lambda \in \hat{\mathfrak{h}}^*$. Then we can write a "generic" solution: a non-zero element $u \in \mathcal{U}(\hat{\mathfrak{n}}_-)_{-\mu}\mathcal{S}(\hat{\mathfrak{h}})$ such that for any $\lambda \in \hat{\mathfrak{h}}^*$ the vector $u(\lambda)v_\lambda \in M(\lambda)_{\lambda-\mu}$ is singular. This means that $[\hat{\mathfrak{n}}, u] \in \mathcal{U}(\hat{\mathfrak{g}})\hat{\mathfrak{n}}$.

Set $\hat{\mathfrak{b}}_- := \hat{\mathfrak{n}}_- + \hat{\mathfrak{h}}$ and let $P_- : \mathcal{U}(\hat{\mathfrak{g}}) \to \mathcal{U}(\hat{\mathfrak{b}}_-)$ be the projection with the kernel $\mathcal{U}(\hat{\mathfrak{g}})\hat{\mathfrak{n}}$. For $e \in \hat{\mathfrak{n}}, x \in \mathcal{U}(\hat{\mathfrak{b}}_-)$ set $e.x := P_-([e,x])$. Then

$$\hat{\mathfrak{n}}.u = 0.$$

Let F be the canonical increasing filtration on $\mathcal{U}(\hat{\mathfrak{h}})h = \mathcal{S}(\hat{\mathfrak{h}})$ (by the total degree); extend F to $\mathcal{U}(\hat{\mathfrak{b}})$ by setting $F^r(\mathcal{U}(\hat{\mathfrak{b}}_-)) := \mathcal{U}(\hat{\mathfrak{n}}_-)F^r(\mathcal{S}(\hat{\mathfrak{h}}))$ and denote by $\deg_F(x)$ the degree of $x \in \mathcal{U}(\hat{\mathfrak{b}}_-)$. Write gr_F as gr and denote by gr_k the projection $\mathcal{U}(\hat{\mathfrak{b}}_-) \to \mathcal{U}(\hat{\mathfrak{n}}_-)\mathcal{S}^k(\hat{\mathfrak{h}})$.

Let F_- be the canonical filtration on $\mathcal{U}(\hat{\mathfrak{n}}_-)$; extend F_- to $\mathrm{gr}\,\mathcal{U}(\hat{\mathfrak{b}}_-) \cong \mathcal{U}(\hat{\mathfrak{b}}_-)$ by setting $F^k_-(\mathcal{U}(\hat{\mathfrak{n}}_-)\mathcal{S}(\hat{\mathfrak{h}})) = F^k_-(\mathcal{U}(\hat{\mathfrak{n}}_-))\mathcal{S}(\hat{\mathfrak{h}})$. Clearly, $\mathrm{gr}_-(\mathcal{U}(\hat{\mathfrak{n}}_-)\mathcal{S}(\hat{\mathfrak{h}})) = \mathcal{S}(\hat{\mathfrak{b}}_-)$.

Set $r := \deg_F(u)$. Write

$$\mathrm{gr}(u) = \sum a_i s_i,$$

where a_i are elements of PBW basis of $\mathcal{U}(\hat{\mathfrak{n}}_-)_{-\mu}$ and s_i are polynomials of degree r in $\mathcal{S}(\hat{\mathfrak{h}})$. An element of the PBW basis is of the form $\prod f_i$, where f_i are weight elements of $\hat{\mathfrak{n}}_-$. For a weight vector $x \in \mathcal{U}(\hat{\mathfrak{g}})$ denote by $\mathrm{wt}\, x$ its weight. Denote by $\mathrm{supp} \prod f_i$ the multiset of opposites to the weights of f_i that is

$$\mathrm{supp} \prod_{i=1}^{k} f_i := \{-\mathrm{wt}\, f_i\}_{i=1}^{k}.$$

For any i one has $\mathrm{wt}\, a_i = -\mu \notin 2\mathbb{Z}\delta$ so $\mathrm{supp}\, a_i \not\subset 2\mathbb{Z}\delta$. Let β be a minimal element of the union of $\mathrm{supp}\, a_i \setminus 2\mathbb{Z}\delta$ i.e.

$$\beta := \min\big(\cup_i \mathrm{supp}\, a_i\big) \setminus 2\mathbb{Z}\delta.$$

For $e \in \hat{\mathfrak{n}}_\beta$ define a map $\hat{\mathfrak{b}}_- \to \hat{\mathfrak{h}}$ by

$$e * y = [e, y]\delta_{\beta, -\mathrm{wt}\, y}$$

and extend this action to a derivation of $\mathcal{S}(\hat{\mathfrak{b}}_-)$. We claim that for any $e \in \hat{\mathfrak{g}}_\beta$ one has

$$\mathrm{gr}_-\, \mathrm{gr}_{\deg_F(u)+1}(e.u) = e * \mathrm{gr}_-\, \mathrm{gr}(u). \tag{3}$$

Indeed, observe that $\hat{\mathfrak{n}}.\mathcal{U}(\hat{\mathfrak{n}}_-)\subset F^1(\mathcal{U}(\hat{\mathfrak{b}}_-))$ and so

$$\mathrm{gr}_{\deg_F(u)+1}(e.u)=\mathrm{gr}_{\deg_F(u)+1}(e.\,\mathrm{gr}(u))=\sum_i \mathrm{gr}_1(e.a_i)s_i.$$

For a weight element $x\in\hat{\mathfrak{n}}$ one has

$$\mathrm{gr}_1\big(x.\prod_{j\in J} f_j\big)=\sum_{J'\subset J:\mathrm{wt}\,x+\sum_{j\in J'}\mathrm{wt}\,f_j=0}\pm\prod_{j\in J\setminus J'} f_j\cdot h_{J'},\quad \text{where } h_{J'}\in\hat{\mathfrak{h}}$$

and $h_{J'}:=[\dots[[x,f_{j_1}],f_{j_2}]\dots f_{j_k}]$ for $J'=\{j_m\}_{m=1}^k$. The choice of β ensures that β can not be written as a non-trivial sum of elements from $\cup_i \operatorname{supp} a_i$. Therefore

$$\mathrm{gr}_1(e.a_i)=\mathrm{gr}_1(e.\prod_{j\in J} f_j)=\sum_{j:\mathrm{wt}\,f_j=-\beta}(-1)^{p(e)p(\prod_{m<j} f_m)}\prod_{m\neq j} f_m\cdot[e,f_j],$$

where $[e,f_j]\in\hat{\mathfrak{h}}$ and p stands for the parity on $\hat{\mathfrak{g}}$. This gives $\mathrm{gr}_-\,\mathrm{gr}_1(e.a_i)=e*a_i$ and (3) follows.

Set $W:=\hat{\mathfrak{h}}\oplus_{\alpha\neq-\beta}\hat{\mathfrak{n}}_{-,\alpha}$; then

$$\hat{\mathfrak{b}}_-=\hat{\mathfrak{g}}_{-\beta}\oplus W.$$

Let us show that

$$\{x\in\mathcal{S}(\hat{\mathfrak{b}}_-)|\ \hat{\mathfrak{g}}_\beta*x=0\}=\mathcal{S}(W). \tag{4}$$

Indeed, by the construction, for any $e\in\hat{\mathfrak{g}}_\beta$ the map $x\mapsto e*x$ is a derivation of the algebra $\mathcal{S}(\hat{\mathfrak{b}}_-)$ and $\mathcal{S}(\hat{\mathfrak{h}})$ lies in the kernel of this derivation. Extend this derivation to a derivation of the localized algebra $\mathcal{S}(\hat{\mathfrak{b}}_-)\otimes_{\mathcal{S}(\hat{\mathfrak{h}})}\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$. It is easy to see that

$$\{x\in\mathcal{S}(\hat{\mathfrak{b}}_-)\otimes_{\mathcal{S}(\hat{\mathfrak{h}})}\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})|\ \hat{\mathfrak{g}}_\beta*x=0\}=\mathcal{S}(W)\otimes_{\mathcal{S}(\hat{\mathfrak{h}})}\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}}) \tag{5}$$

which implies (4).

Indeed, for (5) set $W':=\oplus_{\alpha\neq-\beta}\hat{\mathfrak{n}}_{-,\alpha}$. Then $\hat{\mathfrak{b}}_-=\hat{\mathfrak{g}}_{-\beta}\oplus\hat{\mathfrak{h}}\oplus W'$ and

$$\mathcal{S}(\hat{\mathfrak{b}}_-)\otimes_{\mathcal{S}(\hat{\mathfrak{h}})}\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})=\mathcal{S}(\hat{\mathfrak{g}}_{-\beta})\otimes\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})\otimes\mathcal{S}(W').$$

Let $e_1,\dots,e_m$ be a basis of $\hat{\mathfrak{g}}_\beta$ and $y_1,\dots,y_m$ be a dual basis of $\hat{\mathfrak{g}}_{-\beta}\otimes\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$ over $\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$ (such a basis exists since the pairing $\hat{\mathfrak{g}}_\beta\otimes\hat{\mathfrak{g}}_{-\beta}\to\hat{\mathfrak{h}}$ given by the bracket is non-degenerate). Write an element of $\mathcal{S}(\hat{\mathfrak{b}}_-)\otimes_{\mathcal{S}(\hat{\mathfrak{h}})}\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$ in the form $\sum a_ib_i$, where a_i are elements of a basis of $\mathcal{S}(W')$ and $b_i\in\mathcal{S}(\hat{\mathfrak{g}}_{-\beta})\otimes\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$. View $\mathcal{S}(\hat{\mathfrak{g}}_{-\beta})\otimes\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$ as the algebra of polynomials in $y_1,\dots,y_m$ with coefficients in $\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$. Then

$$e_j*(\sum a_ib_i)=\sum a_i(e_j*b_i)=\sum a_i\frac{\partial b_i}{\partial y_j}.$$

As a result, if $e_j*(\sum a_ib_i)=0$ for all j, then all b_i are constant polynomials that is $\sum a_ib_i\in\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})\otimes\mathcal{S}(W')=\mathcal{S}(W)\otimes_{\mathcal{S}(\hat{\mathfrak{h}})}\operatorname{Fract}\mathcal{S}(\hat{\mathfrak{h}})$. This establishes (5) and (4).

Combining (3) and (4) we get $\mathrm{gr}_-\,\mathrm{gr}\,u\in\mathcal{S}(W)$ which contradicts to the choice of β. $\square$

7. Towards the proof of Theorem 4.1 (i), (ii)

7.1. In this section we describe a suitable Zariski dense subset $\Lambda\subset\hat{\mathfrak{h}}^*$. Then we reduce Theorem 4.1 (i), (ii) to the following assertions for $\lambda\in\Lambda$:

(A) $\operatorname{ch} L(\lambda)\geq e^{\lambda}\prod_{\alpha\in\hat{\Delta}^+_{re;0}}(1-e^{-\alpha})^{-1}\prod_{\alpha\in\hat{\Delta}^+_1}(1+e^{-\alpha})$.

(B) $\mathrm{HC}_+(M(\lambda)^{\hat{\mathfrak{n}}})$ contains $\mathcal{H}_{-,\overline{0}}$.

7.2. **The set** Λ. Let Λ be the set of λs such that any singular vector of $M(\lambda)$ has weight of the form $\lambda-2s\delta$ for some $s\geq 0$:

$$\Lambda:=\bigcap_{\mu\in\hat{Q}^+,\mu\neq 2s\delta}\Lambda_\mu,\quad\text{where}\quad\Lambda_\mu:=\{\lambda\in\mathfrak{h}^*|\ M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\mu}=0\}.$$

Recall that $M(\lambda)\cong M(\lambda-2\delta)$ as $[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$-modules. As a consequence, Λ is invariant under the shift by 2δ: if $\lambda\in\Lambda$ then $\lambda-2\delta\in\Lambda$.

7.2.1. **Lemma.** *For $\lambda\in\Lambda$ any simple subquotients of $M(\lambda)$ is of the form $L(\lambda-2s\delta)$ for some $s\in\mathbb{Z}_{\geq 0}$.*

Proof. Let $\nu\in\hat{Q}^+\setminus\mathbb{N}(2\delta)$ be minimal such that $[M(\lambda):L(\lambda-\nu)]\neq 0$ for some $\lambda\in\Lambda$. By Proposition 5.1 there exists ξ such that $0<\xi<\nu$ and $[M(\lambda):L(\lambda-\xi)]$, $[M(\lambda-\xi):L(\lambda-\nu)]\neq 0$. Then $\xi\in\mathbb{N}(2\delta)$ so $\lambda-\xi\in\Lambda$. The minimality of ν gives $(\lambda-\xi)-(\lambda-\nu)\in\mathbb{N}(2\delta)$ so $\nu\in\mathbb{N}(2\delta)$. The assertion follows. □

7.2.2. **Lemma.** *The set Λ is Zariski dense in $\hat{\mathfrak{h}}^*$.*

Proof. By Proposition 6.1, Λ_μ is the complement to a hypersurface in $\hat{\mathfrak{h}}^*$ for $\mu\neq 2s\delta$. As a consequence, Λ is the complement to a union of countably many hypersurfaces in $\hat{\mathfrak{h}}^*$. This implies the assertion. □

7.3. **Algebra structure on** $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$**.** Take $\lambda\in\Lambda$. Since any singular vector in $M(\lambda)$ has weight $\lambda-2s\delta$ and $M(\lambda)\cong M(\lambda-2\delta)$ as $[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$-modules, we may (and will) identify $\mathrm{End}_{[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]}(M(\lambda))$ with $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$ via the map $\phi\mapsto\phi(v_\lambda)$. Endow $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$ with the algebra structure via this bijection.

7.3.1. **Lemma.** *Take $\lambda\in\Lambda$. The restriction $\mathrm{HC}_+:\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}\to\mathcal{U}(\mathcal{H}_-)$ is an algebra homomorphism.*

Proof. Let ι stands for the natural identification $M(\lambda)$ with $\mathcal{U}(\hat{\mathfrak{n}}_-)$. Clearly, the algebra structure on $\sum_s M(\lambda)^{\hat{\mathfrak{n}}}_{s\delta}$ is compatible with ι, that is $\iota(x)\iota(y)=\iota(xy)$. Notice that ι maps $\sum_s M(\lambda)_{s\delta}$ to $\mathcal{U}(\hat{\mathfrak{n}}_-)^{\mathfrak{h}'}$, where $\mathfrak{h}'=\hat{\mathfrak{h}}\cap[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$. Now the assertion follows from the fact that the restriction of HC_+ to $\mathcal{U}(\hat{\mathfrak{n}}_-)^{\mathfrak{h}'}$ is an algebra homomorphism. □

7.4. **Deduction of Theorem 4.1 (i),(ii) from (A) and (B).** Fix $\lambda \in \Lambda$. From Lemma 5.2 we see that the formula

$$[M(\lambda) : L(\lambda-\nu)] = \dim M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\nu} \tag{6}$$

implies that any submodule of $M(\lambda)$ is generated by singular vectors.

Let us prove (6). From Lemma 7.3.1 and the assertion (B) it follows that $\mathrm{HC}_+(M(\lambda)^{\hat{\mathfrak{n}}})$ contains $\mathcal{S}$. Then

$$[M(\lambda) : L(\lambda-\nu)] \geq \dim M(\lambda)^{\hat{\mathfrak{n}}}_{\lambda-\nu} \geq \dim \mathcal{S}_{-\nu}. \tag{7}$$

Using 2.6.2, we get

$$\operatorname{ch} M(\lambda) \geq \sum_s [M(\lambda) : L(\lambda-s\delta)] e^{-s\delta} \operatorname{ch} L(\lambda) \geq \operatorname{ch} \mathcal{S} \cdot \operatorname{ch} L(\lambda), \tag{8}$$

where the first inequality is strict if $[M(\lambda) : L(\lambda-\nu)] \neq 0$ for some $\nu \notin \mathbb{Z}\delta$ and the second inequality is strict if at least one of the inequalities of (7) is strict. The inequality (A) can be rewritten as

$$\operatorname{ch} \mathcal{S} \cdot \operatorname{ch} L(\lambda) \geq \operatorname{ch} M(\lambda). \tag{9}$$

Comparing (8) with (9) we conclude that all inequalities in (8) are in fact equalities and thus all inequalities in (7) and the inequality (A) are equalities as well. In particular, this gives (6) and hence Theorem 4.1 (ii).

Since the inequalities (7) are equalities we have $\operatorname{ch} M(\lambda)^{\hat{\mathfrak{n}}} = e^{\lambda}\mathcal{S}$. By above, $\mathrm{HC}_+(M(\lambda)^{\hat{\mathfrak{n}}})$ contains $\mathcal{S}$. As a result, the restriction of HC_+ to $M(\lambda)^{\hat{\mathfrak{n}}}$ gives a bijection: $\mathrm{HC}_+ : M(\lambda)^{\hat{\mathfrak{n}}} \xrightarrow{\sim} \mathcal{S}$; note that this is an algebra isomorphism. This proves Theorem 4.1 (i).

8. Explicit construction of singular vectors

In this section we prove (B) of 7.1 for $\lambda \in \Lambda$, where Λ is described in 7.2. More precisely, for $\lambda \in \Lambda$ in Proposition 8.2.5 we will explicitly construct a singular vector $v(h,m) \in M(\lambda)$ satisfying $\mathrm{HC}_+(v(h,m)) = h \otimes t^{-m}$ where h is any element of $\mathfrak{h}$ and m is any even positive number.

Note that Λ is dense in $\hat{\mathfrak{h}}^*$ so (B) holds for any $\lambda \in \hat{\mathfrak{h}}$.

8.1. The subalgebra $\mathfrak{q}$.

Put $\mathcal{B} := \tilde{\mathcal{N}}^+ \oplus \mathcal{H}$, $\mathfrak{q} = \mathcal{B} \cap \hat{\mathfrak{n}}$.

8.1.1. **Lemma.** *Take $\lambda \in \Lambda$. A vector $v \in M(\lambda)_{\lambda-2s\delta}$ is singular iff $\mathfrak{q}v = 0$.*

Proof. Take $v \in M(\lambda)_{\lambda-s\delta}$ satisfying $\mathfrak{q}v = 0$. The subspace $\mathcal{U}(\hat{\mathfrak{n}})v$ contains a singular vector. Write $\hat{\mathfrak{n}} = \mathfrak{q} \oplus \mathfrak{s}$ where $\mathfrak{s} := \sum_{i>0} \mathfrak{n}_- \otimes t^i$. Notice that $\mathfrak{s}$ is a subalgebra of $\hat{\mathfrak{n}}$ and $\mathcal{U}(\hat{\mathfrak{n}})v = \mathcal{U}(\mathfrak{s})v$ because $\mathfrak{q}v = 0$. Weight vectors in $\mathcal{U}(\mathfrak{s})v$ which are not proportional to v have weights of the form $\lambda - \mu$ where $\mu \in \hat{Q}^+,\ \mu \notin \mathbb{N}\delta$; such vectors are not singular. Hence v is singular. □

8.1.2. Set

$$\mathcal{N}^+ = \sum_{r\geq 0} \mathfrak{n}\otimes t^r = \tilde{\mathcal{N}}^+ \cap \hat{\mathfrak{n}}.$$

Lemma. *Take $\lambda\in\Lambda,\ \mu\in\Omega(\mathcal{N}^+)$. One has*

$$L(\lambda)_{\lambda-\mu}\subset\sigma(\mathfrak{q})L(\lambda),\quad M'(\lambda)_{\lambda-\mu}\subset\sigma(\mathfrak{q})M'(\lambda).$$

Proof. Writing $\hat{\mathfrak{n}}_- = \sigma(\mathfrak{q})\oplus\sigma(\mathfrak{s})$ where $\mathfrak{s}$ is introduced in the proof of Lemma 8.1.1, we get $\mathcal{U}(\hat{\mathfrak{n}}_-) = \mathcal{U}(\sigma(\mathfrak{s})) + \sigma(\mathfrak{q})\mathcal{U}(\hat{\mathfrak{n}}_-)$.

One has $L(\lambda) = \mathcal{U}(\hat{\mathfrak{n}}_-)v_\lambda$. The condition $\mu\in\Omega(\mathcal{N}^+)$ ensures that $\mathcal{U}(\sigma(\mathfrak{s}))v_\lambda$ does not meet $L(\lambda)_{\lambda-\mu}$. Thus $L(\lambda)_{\lambda-\mu}\subset\sigma(\mathfrak{q})L(\lambda)$.

For the second inclusion, observe that $\lambda\in\Lambda$ forces $M'(\lambda) = \mathcal{U}(\hat{\mathfrak{n}}_-)M''$ where $M'' := \sum_{s>0} M'(\lambda)_{\lambda-2s\delta}$. The assumption $\mu\in\Omega(\mathcal{N}^+)$ ensures that $\mathcal{U}(\sigma(\mathfrak{s}))M''$ does not meet $M(\lambda)_{\lambda-\mu}$. Thus $M'(\lambda)_{\lambda-\mu}\subset\sigma(\mathfrak{q})M''$ and the second inclusion follows. □

8.1.3. **Proposition.** *For any $\lambda\in\Lambda,\ \mu\in\Omega(\mathcal{N}^+)$ one has*

$$H^r(\mathfrak{q}, L(\lambda))_{\lambda-\mu} = 0 \ \text{ and } \ H^r(\mathfrak{q}, M(\lambda))_{\lambda-\mu} = 0 \ \text{ for } r = 0,1.$$

Proof. The first formula follows from Lemma 10.1 and Lemma 8.1.2. The second formula is an easy consequence of the first one. Indeed, $M(\lambda)$ has a local series at $\lambda-\mu$ with simple quotients $L(\lambda_i)$ where, by Lemma 7.2.1, $\lambda_i = \lambda - 2s_i\delta$ for some $s_i\geq 1$. By the long exact sequence of Lie algebra cohomology, it is enough to show that for $r = 0,1$ one has $H^r(\mathfrak{q}, L(\lambda_i))_{\lambda-\mu} = 0$ for all indexes i. The last follows from the first formula and the fact that $\lambda_i\in\Lambda$ since Λ is invariant under the shift by 2δ. □

8.2. Let m be an even positive number and $h\in\mathfrak{h}$. Set $h(-m) = h\otimes t^{-m}\in\mathcal{H}_-$. In this subsection we construct for each $\lambda\in\Lambda$ a singular vector $v(h,m)\in M(\lambda)$ satisfying

$$\mathrm{HC}_+(v(h,m)) = h(-m).$$

8.2.1. Let $T_{2k} : \mathcal{N}^+ + \mathbb{C}h \to \mathcal{B}$ be a linear map given by

$$T_{2k}(u\otimes t^s) = u\otimes t^{s-2k}.$$

Observe that T_{2k} is $\mathcal{N}^+$-invariant: $T_{2k}([a,b]) = [a, T_{2k}(b)]$ if $a\in\mathcal{N}^+, b\in\mathcal{N}^+ + \mathbb{C}h$.

8.2.2. *Notation.* View $\mathcal{B}=\mathcal{H}\oplus\mathcal{N}^+$ as a $\mathfrak{q}$-module via the adjoint action. Consider the natural grading on $\hat{\mathfrak{g}}$: $\hat{\mathfrak{g}}_0=\mathfrak{gl}(n), \hat{\mathfrak{g}}_s=\mathfrak{sl}(n)\otimes t^s$ for $s\neq 0$ and for a homogeneous subspace $X\subset\hat{\mathfrak{g}}$ set $X_s:=X\cap\hat{\mathfrak{g}}_s$. Set

$$N:=\mathbb{C}h+\sum_{s\geq 0}\mathfrak{n}\otimes t^s,\quad N':=\sum_{s\geq m}\mathfrak{n}\otimes t^s,\quad V:=\mathbb{C}h+\sum_{0\leq s<m}\mathfrak{n}\otimes t^s;$$

note that N,N' are $\mathfrak{q}$-submodule of $\mathcal{B}$ and V is an $\hat{\mathfrak{n}}_0$-submodule of $\mathcal{B}$.

Let V^* be the orthogonal complement of N' in N^* that is

$$V^*:=\{f\in\operatorname{Hom}(N,\mathbb{C})|\ f(N')=0\}.$$

Notice that V^* viewed as $\hat{\mathfrak{n}}_0$-module is dual to V.

Both N,N' are $(\mathfrak{q}+\hat{\mathfrak{h}})$-modules. View N^* and V^* as $(\mathfrak{q}+\hat{\mathfrak{h}})$-modules via the antiautomorphism $-\operatorname{id}$. Let $h^*\in V^*$ be the "dual to" h that is $h^*(h)=1, h^*(a)=0$ for all $a\in\sum_{s\geq 0}\mathfrak{n}^+\otimes t^s$.

8.2.3. **Lemma.** *For any $\lambda\in\Lambda$ there exists a unique $\mathfrak{q}$-homomorphism $\psi: V^*\to M(\lambda)$ such that $\psi(h^*)=v_\lambda$.*

Proof. Let T^ν be a one-dimensional $(\hat{\mathfrak{h}}+\mathfrak{q})$-module of weight ν (i.e. $\mathfrak{q}T^\nu=0,\ h|_{T^\nu}=\nu(h)\operatorname{id}$). Recall that $\Omega(\mathfrak{q})\subset\hat{Q}^+$ and so the action of $\mathfrak{q}$ raises the weight. As a result, V^* admits an increasing $(\hat{\mathfrak{h}}+\mathfrak{q})$-filtration $\{W^k\}_{k\geq 0}$ with one-dimensional factors and $W^0=\mathbb{C}h^*\cong T^0$, $W^k/W^{k+1}\cong T^\nu$ for $\nu\in-\Omega(\mathcal{N}^+)$.

Define a twisted $\hat{\mathfrak{h}}$-action on $M(\lambda)$ by $h.v=(h-(\lambda)(h))v$ for $h\in\hat{\mathfrak{h}}, v\in M(\lambda)$. The twisted action is compatible with the action of $\mathfrak{q}$; view $M(\lambda)$ as $(\hat{\mathfrak{h}}+\mathfrak{q})$-module with respect to this action and notice that ψ is a $(\hat{\mathfrak{h}}+\mathfrak{q})$-homomorphism . In the formulas below we use this twisted $(\hat{\mathfrak{h}}+\mathfrak{q})$-module structure on $M(\lambda)$; Hom stands for $\operatorname{Hom}_{(\hat{\mathfrak{h}}+\mathfrak{q})}$.

The short exact sequence

$$0\to W^{k-1}\to W^k\to T^\nu\to 0$$

gives

$$\begin{aligned}0\to\operatorname{Hom}(T^\nu,M(\lambda))=H^0(\mathfrak{q},M(\lambda))_{\nu'}\to\operatorname{Hom}(W^k,M(\lambda))\to\\ \to\operatorname{Hom}(W^{k-1},M(\lambda))\to\operatorname{Ext}^1(T^\nu,M(\lambda))=H^1(\mathfrak{q},M(\lambda))_{\lambda+\nu}.\end{aligned}$$

For $k>0$ one has $\nu\in-\Omega(\mathcal{N}^+)$ so

$$\operatorname{Hom}(T^\nu,M(\lambda))=\operatorname{Ext}^1(T^\nu,M(\lambda))=0,$$

by Proposition 8.1.3. Therefore $\operatorname{Hom}(W^k,M(\lambda))=\operatorname{Hom}(W^{k-1},M(\lambda))$. As a result, $\operatorname{Hom}(W^k,M(\lambda))=\operatorname{Hom}(W^0,M(\lambda))=\operatorname{Hom}(\mathbb{C}h^*,M(\lambda))$. The assertion follows. $\square$

8.2.4. Note that ψ shifts weights by λ that is $\psi(V^*_{-\mu})\subset M(\lambda)_{\lambda-\mu}$.

8.2.5. Retain notation of 8.2.1, 8.2.2.

Proposition. *Let m be an even positive number. Fix $\lambda \in \Lambda$ and $h \in \mathfrak{h}$. Let $\psi : V^* \to M(\lambda)$ be a $\mathfrak{q}$-homomorphism constructed in Lemma 8.2.3, let $\gamma : \hat{\mathfrak{g}} \otimes M(\lambda) \to M(\lambda)$ be the natural map $\gamma(u \otimes v) = uv$, and let $\mathrm{id}' \in V \otimes V^*$ corresponds to the identity map $V \to V$. Then*

$$v(h,m) = \gamma\big((T_m \otimes \psi)\,\mathrm{id}'\big)$$

satisfies $\mathrm{HC}_+(v(h,m)) = h(-m)$ and $v(h,m)$ is singular.

Proof. Let B be a weight basis of $V \cap \mathcal{N}^+$; then

$$B' := \{h\} \cup B$$

is a weight basis of V. For $b \in B$ denote by b^* the element of the dual basis $\{h^*\} \cup B^*$ of V^*. One has

$$v(h,m) = h(-m)v_\lambda + \sum_{b \in B} T_m(b)\psi(b^*).$$

Since $\mathcal{N}^+$ is T_m-invarinat, $T_m(b) \in \mathcal{N}^+$ and so $\mathrm{HC}_+(v(h,m)) = h(-m)$.

Let us check that $v(h,m)$ is singular. In the light of Lemma 8.1.1 it is enough to verify that $\mathfrak{q}v(h,m) = 0$.

It is easy to see that

$$T_m([g,x]) = [g, T_m(x)] \text{ for all } g \in \mathfrak{q}, x \in V. \tag{10}$$

Take $u \in \mathfrak{q}$. Using (10) and the fact that ψ is a $\mathfrak{q}$-homomorphism we have

$$\begin{aligned} uv(h,m) &= \textstyle\sum_{b \in B'} uT_m(b)\psi(b^*) \\ &= \textstyle\sum_{b \in B'} [u, T_m(b)]\psi(b^*) + (-1)^{p(u)p(b)} T_m(b)u\psi(b^*) \\ &= \textstyle\sum_{b \in B'} T_m([u,b])\psi(b^*) + (-1)^{p(u)p(b)} T_m(b)u\psi(b^*). \end{aligned} \tag{11}$$

For $b_s \in B'$ write

$$[u, b_s] = \sum c_{sj} b_j + w_s \quad \text{where } w_s \in \sum_{t \geq m} \mathcal{N}_t^+.$$

Then

$$ub_s^* = (-1)^{p(u)p(b_s^*)+1} \sum c_{js} b_j^*$$

and (11) gives

$$uv(h,m) = \sum_s T_m(w_s)\psi(b_s^*) = \psi\big(\sum_s T_m(w_s)b_s^*\big),$$

where the last equality follows from the fact that ψ is $\mathfrak{q}$-invariant and $T_m(\sum_{t \geq m} \mathcal{N}_t^+) \subset \mathfrak{q}$.

Denote by $\mathrm{wt}\, a$ the weight of a. One has $\mathrm{wt}\, w_s = \mathrm{wt}\, u + \mathrm{wt}\, b_s$ and $\mathrm{wt}\, b_s^* = -\mathrm{wt}\, b_s$ so $\mathrm{wt}\, T_m(w_s)b_s^* = m\delta + \mathrm{wt}\, u \in \Omega(\mathcal{B})$ because $\mathrm{wt}\, u \in \Omega(\mathcal{N}^+)$. However $T_m(w_s)b_s^* \in V^*$ and $\Omega(V^*) = -\Omega(V) \subset -\Omega(\mathcal{N}^+) \cup \{0\}$. Thus $T_m(w_s)b_s^* = 0$ or $T_m(w_s)b_s^*$ has zero weight. If $\mathrm{wt}\, u \neq m\delta$ we obtain $T_m(w_s)b_s^* = 0$ so $uv(h,m) = 0$ as required.

It remains to check the case when u has weight $m\delta$ that is $u=h'(m)$ for some $h'\in\mathfrak{h}$. Let us show that

$$\sum_s T_m(w_s)b_s^*=0.$$

As we have shown the left-hand side has zero weight so is proportional to h^*. Thus we need to verify that $\sum_s(T_m(w_s)b_s^*)(h)=0$. Observe that for any $x\in V$ one has $[h'(m),x]\in\sum_{t\geq m}\mathcal{N}_t^+$ and so $w_s=[h'(m),b_s]$. Therefore $T_m(w_s)=T_m([h'(m),b_s])=h'(\mathrm{wt}\,b_s)b_s$ because $h'(m)$ is even. Then

$$\sum_s(T_m(w_s)b_s^*)(h)=\sum_s h'(\mathrm{wt}\,b_s)(b_sb_s^*)(h)=\sum_s(-1)^{p(b_s)+1}h'(\mathrm{wt}\,b_s)h(\mathrm{wt}\,b_s)=\mathrm{str}_V(\mathrm{ad}\,h)(\mathrm{ad}\,h').$$

Recall that $V=\mathbb{C}h+\sum_{j=0}^{m-1}\mathfrak{n}\otimes t^j$. One has

$$\mathrm{str}_{\mathfrak{n}\otimes t^j}(\mathrm{ad}\,h)(\mathrm{ad}\,h')=(-1)^j\,\mathrm{tr}_{\mathfrak{n}}(\mathrm{ad}\,h)(\mathrm{ad}\,h');$$

since m is even we obtain $\mathrm{str}_V(\mathrm{ad}\,h)(\mathrm{ad}\,h')=0$ and this completes the proof. □

9. Proof of 7.1 (A) and of Theorem 4.1 (iii)

9.1. **Proof of 7.1 (A).** Set

$$N_+:=\sum_{r>0}\mathfrak{n}\otimes t^{-r},\ \ N_-:=\sum_{r\geq 0}\mathfrak{n}_-\otimes t^{-r},\ \ H_1:=\sum_{r>0}\mathfrak{h}\otimes t^{-2r+1}=\mathcal{H}_{-,\overline{1}}.$$

Notice that N_+,N_-,H_1 are Lie subalgebras of $\hat{\mathfrak{n}}_-$. In a Shapovalov matrix S_ν consider the minor corresponding to the space $\mathcal{U}(N_+)\mathcal{U}(N_-)\mathcal{U}(H_1)$. The leading term of this minor takes form

$$\prod_{\alpha\in\hat{\Delta}^+\setminus\hat{\Delta}^+_{\mathrm{Im},\overline{0}}} D_\alpha^{r_\alpha(\nu)},$$

for some $r_\alpha(\nu)\geq 0$, see [GS]. From 2.4.2 it follows that this minor is a non-zero polynomial. Therefore for a generic λ the space $\mathcal{U}(N_+)\mathcal{U}(N_-)\mathcal{U}(H_1)v_\lambda$ does not meet $M'(\lambda)$. This proves (A) of 7.1.

9.2. **Proof of Theorem 4.1 (iii).** We need to show that for a generic λ the submodule generated by a singular weight vector in $M(\lambda)$ is a Verma module.

Take λ such that for any $s\geq 0$ the assertion of Theorem 4.1 (i) holds for $\lambda':=\lambda-2s\delta$. Let $v\in M(\lambda)_{\lambda-2s\delta}$ be a singular vector. The submodule M generated by v is a quotient of $M(\lambda-2s\delta)$. If $M\neq M(\lambda-2s\delta)$ then $M=M(\lambda-2s\delta)/M''$ where M'' contains a singular vector v'. Write $v=uv_\lambda, v'=u'v_{\lambda-2s\delta}$, $(u,u'\in\mathcal{U}(\hat{\mathfrak{n}}_-))$. Then $uu'=0$. By Theorem 4.1 (i) $\mathrm{HC}_+(u),\mathrm{HC}_+(u')\neq 0$. Hence $\mathrm{HC}_+(uu')=\mathrm{HC}_+(u')\,\mathrm{HC}_+(u)\neq 0$. and this contradicts to $u'u=0$. This proves Theorem 4.1 (iii).

9.3. *Remark.* Now we can precisely formulate the condition on λ in Theorem 4.1.

From 9.1 we see that Theorem 4.1 (i), (ii) hold if the minors corresponding to $\big(\mathcal{U}(N_+)\mathcal{U}(N_-)\mathcal{U}(H_1)\big)_\nu$ in the Shapovalov matrices S_ν are non-zero at λ for each $\nu \in \hat{Q}^+$. Since $\hat{Q}^+$ is countable, this condition excludes countably many hypersurfaces.

From 9.2 we see that for Theorem 4.1 (iii) it is enough if Theorem 4.1 (i), (ii) hold for $\lambda - 2s\delta$ for each $s \geq 0$. This again excludes countably many hypersurfaces.

Using a Shapovalov technique it is easy to see that the leading term of the above minors is not divisible by K; thus these minors are not identically equal to zero at any hyperplane $\{\lambda : K(\lambda) = k\}$ ($k \in \mathbb{C}$). Hence Theorem 4.1 holds for a generic weight at each level.

10. A vanishing lemma

If $\mathfrak{p}$ is a Lie algebra, N is a $\mathfrak{p}$-module and N' is a subspace of N, denote by $\mathfrak{p}N'$ the vector space spanned by xv where $x \in \mathfrak{p}$, $v \in N'$.

10.1. **Lemma.** *Let $\mathfrak{m}$ be a subalgebra of $\hat{\mathfrak{n}}$. Assume that $\lambda, \mu \in \hat{\mathfrak{h}}^*$ are such that*

$$L(\lambda)_\mu \subset \sigma(\mathfrak{m})L(\lambda), \quad M'(\lambda)_\mu \subset \sigma(\mathfrak{m})M'(\lambda)$$

then

$$H^r(\mathfrak{m}, L(\lambda))_\mu = 0 \quad \textit{for } r = 0, 1.$$

Proof. Let $\mathfrak{l} = \mathfrak{m} + \hat{\mathfrak{h}}$. Obviously,

$$\mathrm{Ext}^r_{\mathfrak{l}}(V, W) = \mathrm{Ext}^r_{\sigma(\mathfrak{l})}(W^\sigma, V^\sigma).$$

In what follows we consider only extensions which are semi-simple over $\hat{\mathfrak{h}}$. If T^ν is the one-dimensional $\mathfrak{l}$ module of weight ν, we have to show that

$$\mathrm{Ext}^r_{\mathfrak{l}}(T^{\lambda-\nu}, L(\lambda)) = 0$$

for $r = 0, 1$, which is equivalent to proving

$$\mathrm{Ext}^r_{\sigma(\mathfrak{l})}(L(\lambda), T^{\lambda-\nu}) = 0,$$

since $(T^\mu)^\sigma \cong T^\mu$ and $L(\lambda)^\sigma \cong L(\lambda)$.

First we note that that $V_\mu \subset \sigma(\mathfrak{m})V$ implies $\mathrm{Hom}_{\sigma(\mathfrak{l})}(V, T^\mu) = 0$ for any $\sigma(\mathfrak{l})$-module V. Indeed, let $f \in \mathrm{Hom}_{\sigma(\mathfrak{l})}(V, T^\mu)$, then $f(v) \neq 0$ for some $v \in V_\mu$. But $v = Xw$ for some $X \in \sigma(\mathfrak{m})$, $w \in V$ and $f(w) = 0$ because the weight of w is greater than μ. Contradiction. Hence $\mathrm{Hom}_{\sigma(\mathfrak{l})}(L(\lambda), T^{\lambda-\nu}) = 0$ and $\mathrm{Hom}_{\sigma(\mathfrak{l})}(M'(\lambda), T^{\lambda-\nu}) = 0$. Now we use the exact sequence

$$0 \to M'(\lambda) \to M(\lambda) \to L(\lambda) \to 0.$$

The Verma module $M(\lambda)$ is free over $\sigma(\mathfrak{m})$ and hence projective over $\sigma(\mathfrak{l})$. Therefore

$$\mathrm{Ext}^1_{\sigma(\mathfrak{l})}(M(\lambda), T^{\lambda-\nu}) = 0.$$

Now applying the long exact sequence for Ext and using

$$\mathrm{Hom}_{\sigma(\mathfrak{l})}(M'(\lambda), T^{\lambda-\nu}) = 0$$

one immediately obtains

$$\mathrm{Ext}^1_{\sigma(\mathfrak{l})}(L(\lambda), T^{\lambda-\nu}) = 0$$

as required. □

References

[Ar] T. Arakawa, *A new proof of the Kac-Kazhdan conjecture*, IMRN, **vol. 2006**, Art. Id 27091, 5 pp, (2006).

[DGK] V. V. Deodhar, O. Gabber, V. G. Kac, *Structure of some categories of representations of infinite dimensional Lie algebras*, Adv. in Math., **45**, (1982), p. 92–116.

[GS] M. Gorelik, V. Serganova, *Shapovalov forms for Poisson Lie superalgebras*, Contemprorary mathematics **391**, (2005).

[G] M. Gorelik *On the generic Verma module at the critical level over affine Lie superalgebras*, arXiv:math/0504253, to appear in IMRN.

[GW] R. Goodman, N. Wallach, *Higher order Sugawara operators for affine Kac-Moody algebras*, Trans. AMS, **315**, (1989), 1–66.

[H] T. Hayashi, *Sugawara operators and Kac-Kazhdan conjecture*, Inv. Math., **94** (1988), 13–52.

[HS] C. Hoyt, V. Serganova *Kac-Moody Lie superalgebras of finite growth* , to appear in Communications in Algebra.

[FF] B. Feigin, E. Frenkel, *The family of representations of affine Lie algebras*, Russian Mth. Survey, **43**, (1988), 221–222.

[F] E. Frenkel, *Wakimoto modules, opers and the center at the critical level*, Adv. in Math., **195**, (2005) no. 2, 297–404.

[K1] V. G. Kac, *Lie superalgebras*, Adv. in Math., **26**, (1977), 8–96.

[K2] V. G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 3d edition, 1990.

[KK] V. G. Kac, D. A. Kazhdan, *Structure of representations with highest weight of infinite dimensional Lie algebra*, Adv. in Math., **34**, (1979), p. 97–108.

[Ku] J.-M. Ku, *Structure of the Verma module $M(-\rho)$ over Euclidean Lie algebras*, J. of Algebra, **124**, (1989), p.367–387.

[M] O. Mathieu, *On some modular representations of affine Kac-Moody algebras at the critical level*, Compositio Math., **102**, (1996), 305–312.

[S] V. Serganova, *Automorphisms of simple Lie superalgebras*, Math. USSR Izv., **24** (1985), 539–551.

[Sz] M. Szczesny, *Wakimoto modules for twisted affine Lie algebras*, Math. Res. Lett., **9**, (2002), 433-448.

[vdL] J. W. van de Leur, *A classification of contragredient Lie superalgebras of finite growth*, Comm. in Algebra, **17**, (1989), 1815–1841.

[Wk1] M. Wakimoto, *Fock reprsentations of the affine Lie algebra $A_1^{(1)}$*, Comm. Math. Phys., **104**, (1986), 604–609.

[Wk2] M. Wakimoto, *Infinite Dimensional Lie Algebras*, Translation of Mathematical Monographs **195** (2001).

[Wl] N. Wallach, *A class of non-standard modules for affine Lie algebras*, Math. Z. **196**, (1987) No. 3, 303–313.

Dept. of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: maria.gorelik@weizmann.ac.il

Dept. of Mathematics, Univ. of California; Berkeley, CA 94720-3840, USA
E-mail address: serganov@math.berkeley.edu