SHAPOVALOV DETERMINANTS OF Q-TYPE LIE SUPERALGEBRAS

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ABSTRACT. We define an analogue of Shapovalov forms for Q-type Lie superalgebras and factorize the corresponding Shapovalov determinants which are responsible for simplicity of highest weight modules. We apply the factorization to obtain a description of the centres of Q-type Lie superalgebras.

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1. Introduction

1.1. In 1972 N. Shapovalov [Sh] suggested a powerful method for studying highest weight modules of a finite dimensional simple Lie algebra. He elucidated the description of a bilinear form on the enveloping algebra of a simple finite dimensional complex Lie algebra $\mathfrak{g}$ introduced by Gelfand and Kirillov in [GK]. The kernel of this form (Shapovalov form) at a given point $\lambda \in \mathfrak{h}^*$ determines the maximal submodule $\overline{M(\lambda)}$ of a Verma module $M(\lambda)$. In particular, a Verma module $M(\lambda)$ is simple if and only if the kernel of Shapovalov form at λ is equal to zero. The Shapovalov form can be realized as a direct sum of forms S_ν; for each S_ν one can define its determinant (Shapovalov determinant). The zeroes of Shapovalov determinants determine when a Verma module is reducible. N. Shapovalov computed these determinants for the finite dimensional simple Lie algebras: he presented them as products of polynomials of degree one. As a consequence, a Verma module $M(\lambda)$ is simple if and only if λ does not belong to a union of hyperplanes.

Shapovalov's method was generalized by V. Kac, D. Kazhdan in [KK] to Kac-Moody Lie algebras with symmetrizable Cartan matrix, by V. Kac ([K2], [K4]) to Lie superalgebras with symmetrizable Cartan matrix, and by C. De Concini, V. Kac ([DK]) and A. Joseph ([J]) to quantum case. The formula for Shapovalov determinants for Lie superalgebras with symmetrizable Cartan matrix is given in [K4].

1.2. By the term "Q-type superalgebras" we mean four series of Lie superalgebras: $\mathfrak{q}(n)$ ($n \geq 2$) and its subquotients $\mathfrak{sq}(n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$ (the last one is a simple Lie superalgebra for $n \geq 3$ in the notation of [K1] it is $Q(n)$). The Q-type Lie superalgebras are rather special. First, their Cartan subalgebras are not abelian and have non-trivial odd components. Second, they possess a non-degenerate invariant bilinear form which is *odd* and they do not have quadratic Casimir elements.

1.2.1. The first peculiarity leads to the existence of two different candidates for a role of Verma module of the highest weight $\lambda \in \mathfrak{h}_{\bar{0}}^*$: a module $M(\lambda)$ which is induced from a simple $\mathfrak{h}_{\bar{0}}$-module $\mathbb{C}_\lambda$ and a module $N(\lambda)$ which is induced from a simple $\mathfrak{h}$-module. The character of $M(\lambda)$ nicely depends on λ; we call $M(\lambda)$ a *Verma module*. We call $N(\lambda)$ a *Weyl module*. Observe that each Verma module $M(\lambda)$ has a finite filtration with the factors isomorphic to $N(\lambda)$ up to a parity change. Each Weyl module has a unique simple quotient.

1.2.2. In this paper we define a Shapovalov map for Q-type superalgebras. Its kernel at a given point $\lambda \in \mathfrak{h}_{\bar{0}}^*$ determine the maximal submodule $\overline{M(\lambda)}$ which does not meet the highest weight space. The above observation implies that the Weyl module $N(\lambda)$ is simple if and only if $\overline{M(\lambda)} = 0$.

1.2.3. It turns out that the Shapovalov determinants again admit linear factorization (i.e., are the products of polynomials of degree one) and so a Weyl module $N(\lambda)$ is simple if and only if λ does not belong to a union of hyperplanes.

1.2.4. In all cases mentioned in 1.1 the calculation of Shapovalov determinants uses an explicit formula for a quadratic Casimir element which implies a linear factorization for Shapovalov determinants.

In the present work the calculation is based on an observation which allows one to deduce the linear factorizability of the Shapovalov determinants without using the quadratic Casimir elements— see 1.3.

1.2.5. *Determinants versus reduced norm.* Let $\mathfrak{g}$ be a Lie superalgebra of Q-type and $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ be its Cartan subalgebra. Each Shapovalov map S_ν is a map between two bimodules over the non-commutative algebra $R := \mathcal{U}(\mathfrak{h})$. As left and as right R-modules the source and the target of S_ν are free of the same finite rank. Viewing S_ν as an $\mathcal{S}(\mathfrak{h}_0)$-homomorphism between free $\mathcal{S}(\mathfrak{h}_0)$-modules we define $\det S_\nu \in \mathcal{S}(\mathfrak{h}_0)$. Similarly to the case of endomorphisms of modules over an Azumaya algebra, $\det S_\nu \in \mathcal{S}(\mathfrak{h}_0)$ turns out to be a power of another polynomial $\operatorname{Norm} S_\nu$ (reduced norm) which we propose as an analogue of Shapovalov determinants for Q-type superalgebras. Notice that the resulting formulas for $\operatorname{Norm} S_\nu$ look like the formulas for Shapovalov determinants for contragredient Lie superalgebras (see [K4]). We leave to Appendix a thorough definition of reduced norm which would cover our setup.

1.3. **Computation of** $\det S_\nu$**.** The computation of Shapovalov determinants in [KK],[J] has the following steps. The first one is to show that each determinant admits a linear factorization; this easily follows from the existence of a quadratic Casimir. The second step is to construct the Jantzen filtration on Verma modules which provides some information about the multiplicity of each linear factor. Finally, one computes the leading term of $\det S_\nu$ and then obtains the multiplicities.

1.3.1. *Linear factorizability.* Let $\mathfrak{g}$ be a classical Lie superalgebra which is not of type P.

Denote by W the Weyl group of $\mathfrak{g}_{\bar{0}}$ and by $\mathcal{Z}(\mathfrak{g})$ the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. A Harish-Chandra projection identifies $\mathcal{Z}(\mathfrak{g})$ with a subalgebra Z of W-invariant polynomials on $\mathfrak{h}_{\bar{0}}^*$; if $\mathfrak{g}$ is a semisimple Lie algebra one has $Z = \mathcal{S}(\mathfrak{h}_0)^W$. For any $\mathfrak{g}$ there exists a non-zero homogeneous polynomial $z_a \in Z$ such that

$$z_a \mathcal{S}(\mathfrak{h}_0)^W \subset Z. \tag{**}$$

The property (**) could be easily deduced (see [P]) from the explicit description of Z given in [S1], [K3],[S2]. One can also obtain (**) by describing the anticentre of $\mathcal{U}(\mathfrak{g})$ which is much easier to describe than Z itself (see [G1]) and then taking $z_a := T^2$ where T is an anticentral element of the minimal degree.

Now the linear factorizability of Shapovalov determinants can be achieved as follows. Let $C \in \mathcal{S}(\mathfrak{h}_0)^W$ be the standard quadratic element given by $C(\lambda) = (\lambda, \lambda)$. By (**) Z contains $z_a C$. If $\det S_\mu(\lambda) = 0$ for some μ, then a Verma module $M(\lambda)$ has a primitive vector of weight $\lambda - \nu$ for some $0 < \nu \leq \mu$ that is $(z_a C)(\lambda) = (zC)(\lambda - \nu)$ (here $\leq$ is the standard partial order on $\mathfrak{h}_0^*$). If $z_a(\lambda) \neq 0$ this implies $(\lambda, \lambda) = (\lambda - \nu, \lambda - \nu)$ that is $2(\lambda, \nu) = (\nu, \nu)$. Let Q^+ be the positive part of root lattice $Q(\pi)$. For each $\nu \in Q^+$ the last equation defines a hyperplane. Hence $\det S_\mu(\lambda) = 0$ implies $2(\lambda, \nu) = (\nu, \nu)$ for some $\nu \in Q^+$ or $z_a(\lambda) = 0$. This means that $\det S_\mu$ admits a factorization where each factor is either linear or one of irreducible factors of z_a. Notice that instead of Casimir C we could use all elements of $\mathcal{S}(\mathfrak{h}_0)^W$. This would reduce the above set of hyperplanes to those corresponding to $\nu = n\alpha$ where α is a positive root.

The computation of the leading term of $\det S_\nu$ shows that it admits a linear factorization. Hence an irreducible homogeneous polynomial of a higher degree can not be a factor of $\det S_\nu$. Since z_a is homogeneous, this implies that $\det S_\nu$ admits a linear factorization.

1.3.2. To find the multiplicities of linear factors we compute the leading term of $\det S_\nu$. Then we define a Jantzen-type filtration on a Verma module $M(\lambda)$ and prove a sum formula for the multiplicities. Comparing the leading term and the sum formula we determine the multiplicities.

The leading term can be computed by various methods. In this text we use a reduction to the minimal rank case which is $\mathfrak{sq}(2)$ for Q-type algebras.

1.4. **Applications.** The computation of Shapovalov determinants gives us immediately a criterion of irreducibility of a Weyl module.

We also obtain a description of Jantzen filtration at the points $\lambda \in \mathfrak{h}_0^*$ corresponding to a generic reducible Weyl module. This is essential for the computation of $\mathcal{Z}(\mathfrak{g})$, see below.

The Harish-Chandra projection HC provides an embedding $\mathcal{Z}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h}_0)$. Explicit knowledge of the Shapovalov determinants allows us to describe the image of this embedding following a Kac approach [K3]; we give some details in 1.4.1,1.4.2. The result is similar to the one for contragredient Lie superalgebras (see [K3]). For the case $\mathfrak{g} = \mathfrak{q}(n)$ the centre $\mathcal{Z}(\mathfrak{g})$ was described in [S2], [NS].

1.4.1. The main idea is to recover a central element by its action on Verma modules.

In Section 12 we introduce a certain completion $\hat{U}$ of $\mathcal{U}(\mathfrak{g})$; roughly speaking, $\hat{U}$ is an algebra which acts on all $\mathfrak{g}$ modules which are locally nilpotent over $\mathfrak{n}^+$. We show that the centre $\mathcal{Z}(\hat{U})$ coincides with $\mathcal{Z}(\mathfrak{g})$. This follows from the following statement suggested to the author by J. Bernstein: for any $a \in \mathcal{Z}(\hat{U})$ one has $\deg \mathrm{HC}(a) = \deg(a)$. This statement can be viewed as an analogue of Chevalley's theorem stating that for a semisimple Lie algebra the restriction of a non-zero $\mathfrak{g}$-invariant regular function on $\mathfrak{g}$ to $\mathfrak{h}$ is non-zero.

The formula $\deg \mathrm{HC}(a) = \deg(a)$ implies that $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$ if $\mathfrak{g}$ is a finite-dimensional contragredient or Q-type Lie superalgebra.

In 1.5.2 we define a $\mathfrak{g}$–$\mathfrak{h}$ bimodule $\mathcal{M}$ which plays role of a generic Verma module (so that a Verma module $M(\lambda)$ can be viewed as the evaluation of $\mathcal{M}$ at λ). The $\mathfrak{g}$-action on $\mathcal{M}$ can be extended to an action of $\hat{U}$. It turns out that $\mathcal{M}$ is a faithful $\hat{U}$-module; moreover, $z \in \hat{U}$ is central iff the action of z on $\mathcal{M}$ coincides with the right action of $\mathrm{HC}(z)$: $zv = v\,\mathrm{HC}(z)$ for all $v \in \mathcal{M}$. Using this property, we compute the centre $\mathcal{Z}(\hat{U})$, see 1.4.2 for details.

1.4.2. Knowledge of submodules of a generic reducible Weyl module gives us necessary conditions on $\mathrm{HC}(z)$ for $z \in \mathcal{Z}(\hat{U})$. Then for each $\phi \in \mathcal{S}(\mathfrak{h}_0)$ satisfying these necessary conditions we construct an element $z = \sum z_\nu \in \mathcal{Z}(\hat{U})$ with $\mathrm{HC}(z) = \phi$ by a recursive procedure introduced in [K3]. The key ingredient is that S_ν is invertible over the field of fractions of $\mathcal{S}(\mathfrak{h}_0)$ and that S_ν^{-1} has poles of order at most one at a subset of codimension two in $\mathfrak{h}_{\bar{0}}^*$. The fact that S_ν^{-1} has poles of order at most one at λ is equivalent to the statement that the Jantzen filtration of $M(\lambda)$ has length at most two; the latter holds for the *regular* and *subregular* points $\lambda \in \mathfrak{h}_{\bar{0}}^*$.

In [G2] we have checked the finiteness of the recursive procedure by an estimation of degrees and thus show that the central element $z \in \mathcal{Z}(\hat{U})$ lies in $\mathcal{Z}(\mathfrak{g})$. In the present paper we use the equality $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$ which is proven independently.

1.5. **Construction of Shapovalov maps.** The Shapovalov forms can be naturally interpreted in terms of *Shapovalov maps* which we define below. This approach was suggested to us by J. Bernstein.

In [K1] V. Kac introduced a notion of contragredient Lie superalgebra. These are Lie superalgebras which can be constructed by a standard procedure from their Cartan matrices. Let $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ be a contragredient or a Q-type Lie superalgebra. Denote by σ the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ equal to $-\mathrm{id}$ on $\mathfrak{g}$.

Let $\mathcal{C}$ be the category of $\mathfrak{h}$-modules and $\mathcal{D}$ be the category of $\mathfrak{g}$-modules graded by elements of $Q^- := -Q^+$ (the grading is consistent with the natural $Q(\pi)$-grading on $\mathcal{U}(\mathfrak{g})$). We denote by N_ν the νth homogeneous component of N. Let $\Phi_0 : \mathcal{D} \to \mathcal{C}$ be the functor given by $\Phi_0(N) = N_0$. The functor Φ_0 admits a left adjoint functor $\mathrm{Ind} : \mathcal{C} \to \mathcal{D}$ and a right adjoint functor Coind. For any $L \in \mathcal{C}$ the adjunction morphisms $L \to \Phi_0(\mathrm{Ind}(L))$ and $\Phi_0(\mathrm{Coind}(L)) \to L$ are, in fact, isomorphisms. In particular,

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{Ind}(L), \mathrm{Coind}(L)) = \mathrm{Hom}_{\mathcal{C}}(L, L).$$

Let $\Xi(L) : \mathrm{Ind}(L) \to \mathrm{Coind}(L)$ correspond to the identity map $L \to L$; in this way we obtain a morphism of functors $\Xi : \mathrm{Ind} \to \mathrm{Coind}$. The kernel of $\Xi(L)$ is the maximal graded submodule of $\mathrm{Ind}(L)$ which does not meet its zero component.

1.5.1. Set $R := \mathcal{U}(\mathfrak{h}), A := \mathcal{S}(\mathfrak{h}_0)$ ($A = R$ if $\mathfrak{g}$ is not of Q-type). View R as an object of $\mathcal{C}$. We check that the canonical morphisms $\operatorname{Ind}(E) \to \operatorname{Ind}(R) \otimes_R E$, $\operatorname{Coind}(R) \otimes_R E \to \operatorname{Coind}(E) = \operatorname{Coind}(R) \otimes_R E$ are isomorphisms; they allow to identify $\Xi(E)$ with $\Xi(R) \otimes \mathrm{id}_E$. We call $S := \Xi(R)$ a *Shapovalov map.*

Both $\mathfrak{g}$–$\mathfrak{h}$ bimodules $\operatorname{Ind}(R), \operatorname{Coind}(R)$ viewed as $\mathfrak{h}_{\bar{0}}$-bimodules are isomorphic to $\mathcal{U}(\mathfrak{b}^-)$ (where $\mathfrak{b}^- := \mathfrak{h} + \mathfrak{n}^-$). Their homogeneous components are R-bimodules which are free A-modules of the same finite rank. Thus we can decompose $S = \sum S_\nu$ where

$$S_\nu : \operatorname{Ind}(R)_{-\nu} \to \operatorname{Coind}(R)_{-\nu}$$

is an R-bimodule homomorphism. Viewing the source and the target as left A-modules we realize S_ν as an A-homomorphism between two free A-modules of the same rank. A matrix of S_ν (with entries in A) is called a *Shapovalov matrix* and its determinant is called a *Shapovalov determinant* (this is an element in $A = \mathcal{S}(\mathfrak{h}_0)$ which is defined up to an invertible scalar).

1.5.2. Define $M(\lambda)$ and $N(\lambda)$ as in 1.2.1 ($M(\lambda) = N(\lambda)$ if $\mathfrak{g}$ is not of Q-type). The family of $M(\lambda)$ can be obtained from $\mathcal{M} := \operatorname{Ind}(R)$ by evaluation. Denote by $S(\lambda)$ the evaluation of the Shapovalov map S at λ. The kernel of $S(\lambda)$ is the maximal submodule of $M(\lambda)$ which does not meet the highest weight space. We denote this submodule by $\overline{M(\lambda)}$ and define similarly $\overline{N(\lambda)}$.

1.5.3. Recall that $M(\lambda)$ (resp., $\overline{M(\lambda)}$) has a finite filtration whose factors are isomorphic to $N(\lambda)$ (resp., to $\overline{N(\lambda)}$) up to parity change. As a consequence,

$$\operatorname{Ker} S(\lambda) = 0 \iff \overline{M(\lambda)} = 0 \iff \overline{N(\lambda)} = 0.$$

Each $N(\lambda)$ has a unique simple quotient $V(\lambda)$ and $V(\lambda) = N(\lambda)/\overline{N(\lambda)}$. Hence

$$V(\lambda) = N(\lambda) \iff \det S_\nu(\lambda) \neq 0 \text{ for all } \nu.$$

1.6. **Shapovalov forms.** Historically, the Shapovalov map was introduced as a bilinear form. This can be described as follows. The module $\operatorname{Ind}(R)$ identifies with $\mathcal{U}(\mathfrak{b}^-)$ as $\mathfrak{b}^-$-R bimodule. The module $\operatorname{Coind}(R)$ can be realized (up to a parity change) as a graded dual of $\mathcal{U}(\mathfrak{b}^+)$. More precisely, $\operatorname{Coind}(R)$ identifies with the maximal graded submodule of $\operatorname{Hom}_R(\mathcal{U}(\mathfrak{b}^+), R^\sigma)$ where R^σ is an R bimodule obtained from R by the shift by σ. Using this identification, we realize the Shapovalov map as $S : \mathcal{U}(\mathfrak{b}^-) \to \operatorname{Hom}_{R_r}(\mathcal{U}(\mathfrak{b}^+), R^\sigma)$; the formula for S is

$$S(u_-)(u_+) = (-1)^{p(u_-)p(u_+)} \operatorname{HC}\big(\sigma(u_+)u_-\big)$$

where $u_\pm \in \mathcal{U}(\mathfrak{b}^\pm)$.

If $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$ (that is $R = A$) then $R^\sigma = R^* := \operatorname{Hom}_A(R, A)$.

If $\mathfrak{h} \neq \mathfrak{h}_{\bar{0}}$ (i.e., $\mathfrak{g}$ is of Q-type) the right-hand side of the above formula is an element of a non-commutative algebra R which is not very convenient. Fortunately, there exists a map $\int : R \to A$ (see 1.6.1) which induces an isomorphism $R^\sigma \to \Pi^{\dim \mathfrak{h}_{\bar{1}}}(R^*)$. In this way, $\mathrm{Coind}(R)$ identifies (up to a parity change) with the maximal graded submodule of $\mathrm{Hom}_A(\mathcal{U}(\mathfrak{b}^+), A)$. The last identification gives rise to another realization of the Shapovalov map $B : \mathcal{U}(\mathfrak{b}^-) \to \Pi^{\dim \mathfrak{h}_{\bar{1}}} \mathrm{Hom}_A(\mathcal{U}(\mathfrak{b}^+), A)$. It is given by the formula

$$B(u_-)(u_+) = (-1)^{p(u_-)p(u_+)} \int \mathrm{HC}\big(\sigma(u_+)u_-\big).$$

The map B is instrumental in the computation of the centre of $\mathcal{U}(\mathfrak{g})$ (see 1.4).

1.6.1. Let $\mathfrak{g}$ be a Q-type Lie superalgebra. The algebra $R = \mathcal{U}(\mathfrak{h})$ is a Clifford superalgebra over the polynomial algebra $A = \mathcal{S}(\mathfrak{h}_0)$. For each $\lambda \in \mathfrak{h}_{\bar{0}}^*$ the evaluation of R at λ is a complex Clifford superalgebra. Notice that a non-degenerate complex Clifford superalgebra is either the matrix algebra (if $\dim \mathfrak{h}_{\bar{1}}$ is even) or the algebra $Q(n)$ (this is an associative algebra whose Lie algebra is $\mathfrak{q}(n)$). In particular, it possesses a supertrace which is even if $\dim \mathfrak{h}_{\bar{1}}$ is even and odd if $\dim \mathfrak{h}_{\bar{1}}$ is odd. In both cases, there exists a map

$$\int : R \to A.$$

satisfying $\int [R, R] = 0$; the evaluation of $\int$ at λ is proportional to supertrace on the complex Clifford superalgebra if the latter is non-degenerate.

1.7. **Content of the paper.** In Section 3 we recall definitions and some properties of main objects.

In Section 4 we propose definition of Shapovalov map for Q-type Lie superalgebras which was briefly explained in 1.5, 1.6.

In Section 5 we compare Shapovalov determinants for various algebras.

In Section 6 we construct a non-graded isomorphism $\mathcal{M} \to \mathcal{M}^{\#}$.

In Section 7 we consider an example $\mathfrak{g} = \mathfrak{sq}(2)$.

In Section 8 we calculate the leading terms of Shapovalov determinants.

In Section 9 we adapt the definition of Jantzen filtration (see [Ja]) to the Q-type Lie superalgebras. As in the contragredient case, the Jantzen filtration is instrumental for computations of Shapovalov determinants.

In Section 10 we describe the anticentres of Q-type Lie superalgebras.

In Section 11 we compute Shapovalov determinants (see Theorem 11.1). We also show that the Jantzen filtration have length 2 for *subregular* (see 11.1.1) values of λ.

In Section 12 we describe a certain completion $\hat{U}$ of $\mathcal{U}(\mathfrak{g})$. We show that for any $a \in \mathcal{Z}(\hat{U})$ one has $\deg \mathrm{HC}(a) = \deg(a)$. As a consequence, $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$ if $\mathfrak{g}$ is a finite-dimensional contragredient or Q-type Lie superalgebra.

In Section 13 we describe the centre $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$. In 13.3 we prove that $\mathcal{Z}(\mathfrak{sq}(n)) = \mathcal{Z}(\mathfrak{q}(n))$ and $\mathcal{Z}(\mathfrak{psq}(n)) = \mathcal{Z}(\mathfrak{pq}(n))$.

In the appendix A we analyze the structure of $\mathcal{U}(\mathfrak{h})$ which is a Clifford algebra over the polynomial algebra $\mathcal{S}(\mathfrak{h}_0)$. We recall some basic facts on Clifford algebras and introduce the map $\int : \mathcal{U}(\mathfrak{h}) \to \mathcal{S}(\mathfrak{h}_0)$. We adapt a notion of reduced norm to $\mathcal{U}(\mathfrak{h})$.

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2. Index of notations

Symbols used frequently are given below under the section number where they are first defined. Notation used in Appendix are defined there.

3.1	$\mathbb{Z}_{\geq 0}, \mathbb{Z}_{>0}, p(u), \Pi, \deg u, V^{\oplus r}, \mathrm{gr}$		
3.2.1	σ	4.1.1	$\Phi_0, \mathrm{Ind}, \mathrm{Coind}, \Xi$
3.3.2	$h_\alpha, h_{\overline{\alpha}}, e_\alpha, f_\alpha, H_\alpha, E_\alpha, F_\alpha$	4.1.2	$\mathcal{M}, \mathcal{M}_\nu, R^\sigma, \mathcal{M}^{\#}, N^*, S, B$
3.3.3	$Q^+, \nu \geq \mu$	4.1.4	$S_\nu, B_\nu, \mathrm{Norm}\, S_\nu$
3.4	$A, R, \mathcal{C}\ell(\lambda), c(\lambda), E(\lambda)$	8.1	$\tau(\nu), \mathbf{k}, \mathcal{P}(\nu), \tau_\alpha(\nu), \lvert\mathbf{k}\rvert$
3.4.1	$M(\lambda), N(\lambda), V(\lambda)$	10.2.2	$T_{\mathfrak{p}}$
3.4.2	$\overline{N}$	10.4	$t_{\mathfrak{h}}, t_{\mathfrak{g}}$
3.5	HC	11.1.1	$\Gamma, \gamma_{h,c}$
4.1	$\mathcal{C}, \mathcal{D}, \mathcal{D}_+, N_\nu$	$A.4.2$	$\int$.

3. Preliminaries

3.1. The symbol $\mathbb{Z}_{\geq 0}$ stands for the set of non-negative integers and $\mathbb{Z}_{>0}$ for the set of positive integers. We denote by $|X|$ the number of elements in a finite set X.

Let $V = V_{\overline{0}} \oplus V_{\overline{1}}$ be a $\mathbb{Z}_2$-graded vector space. We denote by $\dim V$ the total dimension of V. For a homogeneous element $u \in V$ we denote by $p(u)$ its $\mathbb{Z}_2$-degree; in all formulae where this notation is used, u is assumed to be $\mathbb{Z}_2$-homogeneous. For a subspace $N \subset V$ we set $N_i := N \cap V_i$ for $i = 0, 1$. Let Π be the functor which switches parity, i.e. $(\Pi V)_{\overline{0}} = V_{\overline{1}}, (\Pi V)_{\overline{1}} = V_{\overline{0}}$. We denote by $V^{\oplus r}$ the direct sum of r-copies of V.

For a Lie superalgebra $\mathfrak{g}$ we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra and by $\mathcal{S}(\mathfrak{g})$ its symmetric algebra. Recall that $\mathcal{S}(\mathfrak{g}) = \mathrm{gr}\, \mathcal{U}(\mathfrak{g})$ with respect to the canonical filtration $\mathcal{F}^k(\mathfrak{g}) := \mathfrak{g}^k$. For $u \in \mathcal{U}(\mathfrak{g})$ denote by $\deg u$ the degree of $\mathrm{gr}\, u$ in $\mathcal{S}(\mathfrak{g})$.

Throughout the paper the base field is $\mathbb{C}$ and $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ denote one (unless otherwise specified, an arbitrary one) of Q-type Lie superalgebras $\mathfrak{q}(n), \mathfrak{sq}(n)$ for $n \geq 2$, $\mathfrak{pq}(n), \mathfrak{psq}(n)$ for $n \geq 3$.

Throughout the paper we are often dealing with homomorphisms between two isomorphic free R-modules where R is a commutative algebra. The determinant of such a homomorphism is defined up to an invertible element of R; in all our examples the set of invertible elements of R is $\mathbb{C}^*$.

3.2. **Q-type Lie superalgebras.** Recall that $\mathfrak{q}(n)$ consists of the matrices with the block form

$$X_{A,B} := \begin{pmatrix} A & | & B \\ -- & - & -- \\ B & | & A \end{pmatrix}$$

where A, B are arbitrary $n \times n$ matrices; $\mathfrak{q}(n)_{\bar{0}} = \{X_{A,0}\} \cong \mathfrak{gl}(n)$, $\mathfrak{q}(n)_{\bar{1}} = \{X_{0,B}\}$ and

$$[X_{A,0}, X_{A',0}] = X_{[A,A'],0}, \quad [X_{A,0}, X_{0,B}] = X_{0,[A,B]}, \quad [X_{0,B}, X_{0,B'}] = X_{0,BB'+B'B}.$$

Define $\mathrm{tr}' : \mathfrak{q}(n) \to \mathbb{C}$ by $\mathrm{tr}'(X_{A,B}) = \mathrm{tr}\, B$. In this notation,

$$\begin{aligned} \mathfrak{sq}(n) :\ &= \{x \in \mathfrak{q}(n) |\ \mathrm{tr}'\, x = 0\}, \\ \mathfrak{pq}(n) :\ &= \mathfrak{q}(n)/(\mathrm{Id}), \\ \mathfrak{psq}(n) :\ &= \mathfrak{sq}(n)/(\mathrm{Id}), \end{aligned}$$

where Id is the identity matrix.

These definitions are illustrated by the following diagram

$$\begin{tikzcd} & \mathfrak{q}(n) \arrow[dr] & \\ \mathfrak{sq}(n) \arrow[ur, hook] \arrow[dr] & & \mathfrak{pq}(n) \\ & \mathfrak{psq}(n) \arrow[ur, hook] & \end{tikzcd}$$

Clearly, the category of $\mathfrak{pq}(n)$-modules (resp., $\mathfrak{psq}(n)$-modules) is the subcategory of $\mathfrak{q}(n)$-modules (resp., of $\mathfrak{sq}(n)$-modules) which are killed by the identity matrix Id.

The map $(x, y) \mapsto \mathrm{tr}'(xy)$ gives an odd non-degenerate invariant symmetric bilinear form on $\mathfrak{q}(n)$ and on $\mathfrak{psq}(n)$.

For the quotient algebras $\mathfrak{pq}(n), \mathfrak{psq}(n)$ we denote by $X_{A,B}$ the image of the corresponding element in the appropriate algebra.

3.2.1. Recall that a linear map σ is called an *antiautomorphism* of a Lie superalgebra (resp., of an associative superalgebra) if it satisfies the rule $\sigma([x,y]) = (-1)^{p(x)p(y)}[\sigma(y),\sigma(x)]$ (resp., $\sigma(xy) = (-1)^{p(x)p(y)}\sigma(y)\sigma(x)$). A Lie superalgebra $\mathfrak{g}$ admits an antiautomorphism σ given by $\sigma(x) = -x$; we denote by σ also the induced antiautomorphisms of $\mathcal{U}(\mathfrak{g})$.

If $\mathfrak{g}$ is a classical Lie superalgebra which is not of type P, it admits a "naive antiautomorphism" $x \mapsto x^t$ satisfying the rule $[x,y]^t = [y^t, x^t]$. In an appropriate basis, this antiautomorphism is given by the matrix transposition. It preserves the elements of a Cartan subalgebra.

3.2.2. For Q-type Lie superalgebras the set of even roots ($\Delta^+_{\bar{0}}$) coincides with the set of odd roots ($\Delta^+_{\bar{1}}$). This phenomenon has two obvious consequence. The first one is that all triangular decompositions of a Q-type Lie superalgebra are conjugate with respect to inner automorphisms (this does not hold for other simple Lie superalgebras). The second one is that the element $\rho := \frac{1}{2}(\sum_{\alpha\in\Delta^+_{\bar{0}}}\alpha - \sum_{\alpha\in\Delta^+_{\bar{1}}}\alpha)$ is equal to zero.

We choose the natural triangular decomposition: $\mathfrak{q}(n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{h}_{\bar{0}}$ consists of the elements $X_{A,0}$ where A is diagonal, $\mathfrak{h}_{\bar{1}}$ consists of the elements $X_{0,B}$ where B is diagonal, and $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) consists of the elements $X_{A,B}$ where A, B are strictly upper-triangular (resp., lower-triangular). We consider the induced triangular decompositions of $\mathfrak{sq}(n), \mathfrak{pq}(n), \mathfrak{psq}(n)$.

The "naive antiautomorphism" $x \mapsto x^t$ preserves the elements of $\mathfrak{h}$ and interchanges $\mathfrak{n}^+$ with $\mathfrak{n}^-$.

3.3. In the standard notation the set of roots of $\mathfrak{gl}(n) = \mathfrak{q}(n)_{\bar{0}}$ can be written as

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j\}_{1\leq i<j\leq n}$$

and the set of simple roots as $\pi := \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n\}$. Each root space has dimension $(1|1)$.

For $\alpha \in \Delta^+$ let $s_\alpha : \mathfrak{h}^*_{\bar{0}} \to \mathfrak{h}^*_{\bar{0}}$ be the corresponding reflection: $s_{\varepsilon_i-\varepsilon_j}(\varepsilon_i) = \varepsilon_j$, $s_{\varepsilon_i-\varepsilon_j}(\varepsilon_k) = \varepsilon_k$ for $k \neq i,j$. Denote by W the Weyl group of $\mathfrak{g}_{\bar{0}}$ that is the group generated by $s_\alpha : \alpha \in \Delta^+$. Recall that W is generated by $s_\alpha : \alpha \in \pi$.

The space $\mathfrak{h}^*_{\bar{0}}$ has the standard non-degenerate W-invariant bilinear form: $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

3.3.1. Let E_{rs} be the elementary matrix: $E_{rs} = (\delta_{ir}\delta_{sj})^n_{i,j=1}$.

The elements

$$h_i := X_{E_{ii},0}$$

form the standard basis of $\mathfrak{h}_{\bar{0}}$ for $\mathfrak{g} = \mathfrak{q}(n), \mathfrak{sq}(n)$. We use the notation h_i also for the image of h_i in the quotient algebras $\mathfrak{pq}(n), \mathfrak{psq}(n)$.

The elements $H_i := X_{0,E_{ii}}$ $(i = 1, \ldots, n)$ form a convenient basis of $\mathfrak{h}_{\bar{1}} \subset \mathfrak{q}(n)$; they satisfy the relations $[H_i, H_j] = 2\delta_{ij}h_i$.

3.3.2. For each positive root $\alpha = \varepsilon_i - \varepsilon_j$ we define

$$\begin{array}{ll} h_\alpha := h_i - h_j,\ h_{\overline{\alpha}} := h_i + h_j, & H_\alpha := H_i - H_j, \\ e_\alpha := X_{E_{ij},0}, & E_\alpha := X_{0,E_{ij}}, \\ f_\alpha := X_{E_{ji},0}, & F_\alpha := X_{0,E_{ji}}. \end{array}$$

All above elements are non-zero in $\mathfrak{sq}(n), \mathfrak{pq}(n), \mathfrak{psq}(n)$ (since we excluded the cases $\mathfrak{pq}(2), \mathfrak{psq}(2)$).

The elements $h_\alpha, e_\alpha, f_\alpha$ ($\alpha \in \Delta^+$) form the standard basis of $\mathfrak{sl}(n) = [\mathfrak{gl}(n), \mathfrak{gl}(n)]$; the elements E_α (resp., F_α) form the natural basis of $\mathfrak{n}_{\overline{1}}^+$ (resp., of $\mathfrak{n}_{\overline{1}}^-$) and the elements H_α span $\mathfrak{h}_{\overline{1}} \cap \mathfrak{sq}(n)$.

For each α the elements $h_\alpha, e_\alpha, f_\alpha, h_{\overline{\alpha}}, H_\alpha, E_\alpha, F_\alpha$ span $\mathfrak{sq}(2)$ and one has

$$[e_\alpha, f_\alpha] = h_\alpha, \quad [E_\alpha, F_\alpha] = h_{\overline{\alpha}}, \quad [H_\alpha, H_\alpha] = 2h_{\overline{\alpha}}$$
$$[E_\alpha, f_\alpha] = [e_\alpha, F_\alpha] = H_\alpha.$$

3.3.3. Set

$$Q(\pi) := \sum_{\alpha \in \Delta^+} \mathbb{Z}\alpha, \quad Q^+ := \sum_{\alpha \in \Delta^+} \mathbb{Z}_{\geq 0}\alpha.$$

Define a partial order on $\mathfrak{h}_{\overline{0}}^*$ by $\nu \geq \mu$ iff $\nu - \mu \in Q^+$.

3.4. Set

$$A := \mathcal{S}(\mathfrak{h}_0), \quad R := \mathcal{U}(\mathfrak{h}).$$

Identify $\mathcal{U}(\mathfrak{h}_{\overline{0}})$ with A. The algebra R is a Clifford algebra over A: it is generated by the odd space $\mathfrak{h}_{\overline{1}}$ endowed by the A-valued symmetric bilinear form $b(H, H') = [H, H']$. We will describe some properties of this algebra in Appendix.

For $\lambda \in \mathfrak{h}_{\overline{0}}^*$ let $\mathbb{C}(\lambda)$ be the corresponding one-dimensional $\mathfrak{h}_{\overline{0}}$-module. Set

$$\mathcal{C}\ell(\lambda) := \mathcal{U}(\mathfrak{h}) \otimes_{\mathfrak{h}_{\overline{0}}} \mathbb{C}_\lambda.$$

Clearly, $\mathcal{C}\ell(\lambda)$ is isomorphic to a complex Clifford algebra generated by $\mathfrak{h}_{\overline{1}}$ endowed by the evaluated symmetric bilinear form $b_\lambda(H, H') := [H, H'](\lambda)$. Set

$$c(\lambda) := \dim \operatorname{Ker} b_\lambda.$$

For $\mathfrak{g} = \mathfrak{q}(n)$, $c(\lambda)$ is the number of zeros among $h_1(\lambda), \ldots, h_n(\lambda)$. The complex Clifford algebra $\mathcal{C}\ell(\lambda)$ is non-degenerate iff $c(\lambda) = 0$.

Denote by $E(\lambda)$ a simple $\mathcal{C}\ell(\lambda)$-module (up to a shift of grading, such a module is unique— see A.3.2). One has $\dim E(\lambda) = 2^{[\frac{\dim \mathfrak{h}_{\overline{1}} + 1 - c(\lambda)}{2}]}$.

3.4.1. Set $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+, \mathfrak{b}^- := \mathfrak{h} + \mathfrak{n}^-$. Endow $\mathcal{C}\ell(\lambda)$ with the $\mathfrak{b}$-module structure via the trivial action of $\mathfrak{n}^+$. Set

$$M(\lambda) := \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathcal{C}\ell(\lambda), \quad N(\lambda) := \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} E(\lambda).$$

Clearly, $M(\lambda)$ has a finite filtration with the factors isomorphic to $N(\lambda)$ up to parity change. We call $M(\lambda)$ a *Verma module* and $N(\lambda)$ a *Weyl module.* A Weyl module $N(\lambda)$ has a unique maximal submodule denoted by $V(\lambda)$.

As a $\mathfrak{g}_{\overline{0}}$-module $N(\lambda)$ has a filtration whose factors are $\mathfrak{g}_{\overline{0}}$-Verma modules. In particular, $N(\lambda)$ has a finite length.

3.4.2. For a diagonalizable $\mathfrak{h}_{\overline{0}}$-module N and a weight $\mu \in \mathfrak{h}_{\overline{0}}^*$ denote by N_μ the corresponding weight space. Say that a module N has the highest weight λ if $N = \sum_{\mu \leq \lambda} N_\mu$ and $N_\lambda \neq 0$. If all weight spaces N_μ are finite-dimensional we put $\operatorname{ch} N := \sum_\mu \dim N_\mu e^\mu$.

If N has a highest weight we denote by $\overline{N}$ the sum of all submodules which do not meet the highest weight space of N. One has $V(\lambda) = N(\lambda)/\overline{N}(\lambda)$.

3.5. **Harish-Chandra projection.** Denote by HC the Harish-Chandra projection $\mathrm{HC} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ along the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathcal{U}(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^-\mathcal{U}(\mathfrak{g}))$.

3.5.1. **Lemma.**

(i) *If $u \in \mathcal{U}(\mathfrak{g})$ is the product of n_- elements of $\mathfrak{n}^-$, n_+ elements of $\mathfrak{n}^+$ and n_0 elements of $\mathfrak{h}$ then* $\deg \mathrm{HC}(u) \leq \min(n_-, n_+) + n_0$.

(ii) *For $i = 1, \dots, k$ let x_i be an element of weight α_i and y_i be an element of weight $-\beta_i$, where $\alpha_i, \beta_i \in \Delta^+$. If $\mathrm{HC}(\prod_{i=1}^k x_i \prod_{i=1}^k y_i)$ has degree k then the multisets $\{\alpha_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ are equal.*

Proof. The proof of (i) is an easy induction on $n_- + n_+ + n_0$. Indeed, write $u = u'x$ or $u = u'xy_1 \dots y_r$ with $x \in (\mathfrak{h} + \mathfrak{n}^+), y_1 \dots y_r \in \mathfrak{n}^-, u' \in \mathcal{U}(\mathfrak{g})$. If $u = u'x$ then $\mathrm{HC}(u) = \mathrm{HC}(u') \mathrm{HC}(x)$ and the assertion follows from by induction. In the case $u = u'xy_1 \dots y_r$, write

$$u = \pm u'y_1 \dots y_r x + \sum_{i=1}^r \pm u'y_1 \dots y_{i-1}((\operatorname{ad} x)y_i)y_{i+1} \dots y_r.$$

As we have already checked $\deg \mathrm{HC}(u'y_1 \dots y_r x) \leq \min(n_-, n_+) + n_0$. The remaining summands are the products of n'_- elements of $\mathfrak{n}^-$, n'_+ elements of $\mathfrak{n}^+$ and n'_0 elements of $\mathfrak{h}$ where $n'_- + n'_+ + n'_0 = n_- + n_+ + n_0 - 1$. In all cases $\min(n'_-, n'_+) + n'_0 \leq \min(n_-, n_+) + n_0$ which implies (i); (ii) easily follows from (i). $\square$

4. Shapovalov map

In this section we construct a Shapovalov map and define an analogue of Shapovalov determinants. In 4.1 we define the main objects and formulate the results of this section. The proofs are given in 4.2–4.4.

In this section $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a classical Lie superalgebra. If $\mathfrak{g}$ is a Q-type superalgebra we keep the above notation. Otherwise $\mathfrak{h}_{\bar{1}} = 0$ and we set $M(\lambda) := N(\lambda)$ for $\lambda \in \mathfrak{h}^*$.

4.1. **Brief description of the main results.** View $\mathcal{U}(\mathfrak{g})$ as a $Q(\pi)$-graded algebra via the adjoint action of $\mathfrak{h}_{\bar{0}}$. Let $\tilde{D}$ be the category of left $Q(\pi)$-graded $\mathfrak{g}$-modules. Let Q^- be the set of weights of $\mathcal{U}(\mathfrak{n}^-)$ that is $Q^- = -Q^+$. Let $\mathcal{D}$ (resp., $\mathcal{D}_+$) be the subcategory of $\tilde{D}$ where the objects are graded $\mathfrak{g}$-modules for which the graded components outside Q^- (resp., outside Q^+) vanish. Let $\mathcal{C}$ be the category of left $\mathfrak{h}$-modules. For $K, L \in \mathcal{D}$ let $\mathrm{Hom}_{\mathcal{D}}(K, L)$ be the space of degree zero homomorphisms. For $N \in \mathcal{D}$ we write $N = \sum_{\nu \in Q^-} N_\nu$. View $M(\lambda)$ as an object of $\mathcal{D}$: the Q^--grading is defined by assigning degree zero to the highest weight vectors.

4.1.1. Let $\Phi_0 : \mathcal{D} \to \mathcal{C}$ be the functor given by $\Phi_0(N) = N_0$. The functor Φ_0 admits a left adjoint functor $\mathrm{Ind} : \mathcal{C} \to \mathcal{D}$ given by $\mathrm{Ind}(L) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} L$ where the action of $\mathfrak{n}^+$ on L is supposed to be trivial. It turns out that Φ_0 also admits a right adjoint functor which we denote by Coind. The adjunction morphism $L \to \mathrm{Ind}(L)_0$ is an isomorphism for any $L \in \mathcal{C}$; thus

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{Ind}(L), \mathrm{Coind}(L)) = \mathrm{Hom}_{\mathcal{C}}(\mathrm{Ind}(L)_0, L) = \mathrm{Hom}_{\mathcal{C}}(L, L).$$

Let $\Xi(L) : \mathrm{Ind}(L) \to \mathrm{Coind}(L)$ be the morphism corresponding to the identity map id_L; in this way we obtain a morphism of functors $\Xi : \mathrm{Ind} \to \mathrm{Coind}$. The following claim is proven in 4.2.2 below.

Claim. $\mathrm{Ker}\,\Xi(L)$ is the maximal graded submodule of $\mathrm{Ind}(L)$ which does not meet the zero component.

View $R = \mathcal{U}(\mathfrak{h})$ as an $\mathfrak{h}$-bimodule. As a left module R belongs to $\mathcal{C}$; both $\mathrm{Ind}(R), \mathrm{Coind}(R)$ inherit the right action of $\mathfrak{h}$ and $\Xi(R) : \mathrm{Ind}(R) \to \mathrm{Coind}(R)$ is a $\mathfrak{g}$-$\mathfrak{h}$ bimodule map. We call $S := \Xi(R)$ the *Shapovalov map.* The following claim is proven in 4.2.1 below.

Claim. One has canonical isomorphisms $\mathrm{Ind}(E) \xrightarrow{\sim} \mathrm{Ind}(R) \otimes_R E$, $\mathrm{Coind}(R) \otimes_R E \xrightarrow{\sim} \mathrm{Coind}(E)$ identifying $\Xi(E)$ with $\Xi(R) \otimes_R \mathrm{id}_E$.

4.1.2. Notice that $\mathrm{Ind}(R)$ has a nice structure: it can be identified with $\mathcal{U}(\mathfrak{b}^-)$ as $\mathfrak{b}^-$-R bimodule. Set

$$\mathcal{M} := \mathrm{Ind}(R).$$

The νth homogeneous component with respect to Q^--grading takes form

$$\mathcal{M}_\nu = \{v \in \mathcal{M} \mid hv - vh = -\mu(h)v \text{ for all } h \in \mathfrak{h}_{\overline{0}}\}.$$

We shall now present convenient realizations of $\mathrm{Coind}(R)$. The proofs are given in 4.2.1 below.

Denote by $\mathrm{Hom}_{R_r}(-,-)$ the set of homomorphisms of *right* R-modules. Define on R a new bimodule structure R^σ via $v.r := (-1)^{p(r)p(v)}\sigma(r)v, r.v := (-1)^{p(r)p(v)}v\sigma(r)$ where the dot stands for the new action, σ is the antiautomorphism introduced in 3.2.1, r is an element of the algebra R and $v \in R^\sigma$. It turns out that R^σ is isomorphic to $R^* := \mathrm{Hom}_A(R, A)$ up to a change of parity. In A.5 we exhibit various connections between R^σ and $\mathrm{Hom}_{R_r}(R, -)$.

Define the functor $\mathrm{Ind}_+ : \mathcal{C} \to \mathcal{D}_+$ similarly to Ind. Let $\mathcal{M}^\#$ be the maximal graded subspace of $\mathrm{Hom}_{R_r}(\mathrm{Ind}_+(R), R^\sigma)$ that is

$$\mathcal{M}^\# := \oplus_{\nu \in Q^+} \mathcal{M}^\#_{-\nu}, \quad \text{where } \mathcal{M}^\#_{-\nu} := \mathrm{Hom}_{R_r}(\mathrm{Ind}_+(R)_\nu, R^\sigma).$$

Convert the natural structure of R-$\mathfrak{g}$-bimodule on $\mathcal{M}^\#$ to a $\mathfrak{g}$-R-bimodule structure via the antiautomorphism σ. Then $\mathcal{M}^\# \in \mathcal{D}$ and, moreover, $\mathcal{M}^\# \cong \mathrm{Coind}(R)$ as $\mathfrak{g}$-R-bimodules.

For a $\mathfrak{g}$-R bimodule N define a dual module N^* as $\mathrm{Hom}_A(N, A)$ (where $A = \mathcal{S}(\mathfrak{h}_0)$) endowed by the $\mathfrak{g}$-R bimodule structure via σ. In A.4.2 we describe an A-homomorphism $\int : R \to A$ whose parity is equal to the parity of $\dim \mathfrak{h}_{\overline{1}}$. For N being a free R-module, the map $\psi \mapsto \int \psi$ provides a map $\mathrm{Hom}_{R_r}(N, R^\sigma) \xrightarrow{\sim} \Pi^{\dim \mathfrak{h}_{\overline{1}}}(N^*)$. Both maps are isomorphism of $\mathfrak{g}$-R bimodules (see A.5). Putting $N := \mathrm{Ind}_+(R)$ we obtain $\mathcal{M}^\# \xrightarrow{\sim} \mathrm{Ind}_+(R)^*$ if $\dim \mathfrak{h}_{\overline{1}}$ is even and $\mathcal{M}^\# \xrightarrow{\sim} \Pi(\mathrm{Ind}_+(R)^*)$ if $\dim \mathfrak{h}_{\overline{1}}$ is odd. Thus we obtain two realizations of $\mathrm{Coind}(R)$: $\mathcal{M}^\#$ and $\Pi^{\dim \mathfrak{h}_{\overline{1}}}(\mathrm{Ind}_+(R)^*)$.

4.1.3. The above realizations of $\mathrm{Coind}(R)$ give the following realizations of the Shapovalov map: $S : \mathrm{Ind}(R) \to \mathrm{Hom}_{R_r}(\mathrm{Ind}_+(R), R^\sigma)$ and $B : \mathrm{Ind}(R) \to \Pi^{\dim \mathfrak{h}_{\overline{1}}}(\mathrm{Ind}_+(R)^*)$. Using the natural identification $\mathrm{Ind}(R) = \mathcal{U}(\mathfrak{b}^-), \mathrm{Ind}_+(R) = \mathcal{U}(\mathfrak{b}^+)$ we obtain the following formulas:

$$S(u_-)(u_+) = (-1)^{p(u_-)p(u_+)}\,\mathrm{HC}\big(\sigma(u_+)u_-\big). \tag{1}$$

$$B(u_-)(u_+) = (-1)^{p(u_-)p(u_+)} \int \mathrm{HC}\big(\sigma(u_+)u_-\big). \tag{2}$$

4.1.4. Recall that S is homogeneous of degree zero and write $S = \sum_{\nu \in Q^+} S_\nu, B = \sum_{\nu \in Q^+} B_\nu$ where $S_\nu : \mathcal{M}_{-\nu} \to \mathcal{M}^\#_{-\nu}$ is the restriction of S and B_ν is defined similarly. Observe that S_ν is an A-homomorphism between two free A-modules of the same finite rank.

Thus $\det S_\nu \in A = \mathcal{S}(\mathfrak{h}_0)$ is defined up to an invertible scalar. If $\mathfrak{g}$ is a contragredient Lie superalgebra, $\det S_\nu$ is called a *Shapovalov determinant.*

In 4.3.1 we show that for a Q-type Lie superalgebras

$$\det S_\nu = (\operatorname{Norm} S_\nu)^{2^{\dim \mathfrak{h}_{\bar 1}}}$$

where $\operatorname{Norm} S_\nu \in A$ is a *reduced norm* of the operator S_ν. We suggest $\operatorname{Norm} S_\nu$ as an analogue of Shapovalov determinant for Q-type superalgebras.

Since B is the composition of S and a $\mathfrak{g}$-R isomorphism, one has

$$\det S_\nu = \det B_\nu.$$

The map B is more convenient than S because the matrices of B_ν have entries in the polynomial algebra A. In particular, $\det B_\nu$ is equal to the determinant of the corresponding matrix which we also denote as B_ν.

4.1.5. Observe that $M(\lambda) = \operatorname{Ind}(\mathcal{C}\ell(\lambda))$. The map $\Xi(\mathcal{C}\ell(\lambda))$ is obtained from $\Xi(R)$ by the evaluation at λ. Its kernel is $\overline{M(\lambda)}$ (see 3.4.2 for the notation) and it coincides with $\operatorname{Ker} B(\lambda)$. This gives

$$\overline{M(\lambda)} = \{uv_\lambda |\ \Big(\int \operatorname{HC}(u'u)\Big)(\lambda) = 0 \ \text{ for all } u \in \mathcal{U}(\mathfrak{b}^+)\} \tag{3}$$

where v_λ is the canonical generator of $M(\lambda)$ (i.e., the image of $1 \in R$).

Recall that $M(\lambda)$ has a filtration with factors isomorphic to $N(\lambda)$ up to the change of parity. Thus $\overline{M(\lambda)} = 0$ if and only if $\overline{N(\lambda)} = 0$. Hence $\overline{N(\lambda)} = 0$ if and only if $S(\lambda)$ is injective.

Corollary. $N(\lambda)$ is simple iff $\operatorname{Norm} S_\nu(\lambda) \neq 0$ for all $\nu \in Q^+$.

Notice that the matrices of the evaluated maps $B_\nu(\lambda)$ have complex entries. In particular, the dimension of the kernel of the evaluated map $B_\nu(\lambda)$ is equal to the corank of the corresponding matrix.

Corollary. $\dim \overline{M(\lambda)}_\nu = \operatorname{corank} B_\nu(\lambda)$.

4.1.6. Let $\mathfrak{g}$ be a Q-type Lie superalgebra. As we show later $\cap_{\lambda\in\mathfrak{h}_{\bar 0}^*} \operatorname{Ann} N(\lambda) = 0$. Since $N(\lambda)$ is a subquotient of $\mathcal{M}$ one has $\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})} \mathcal{M} = 0$. We will use the module $\mathcal{M}$ for the calculation of the centre of $\mathcal{U}(\mathfrak{g})$ in Sect. 13.

4.2. **The proofs of the claims 4.1.1.** Retain the notation of 4.1. It is easy to check that Ind is left adjoint to Φ_0 and one has a canonical isomorphism

$$\operatorname{Ind}(L) \xrightarrow{\sim} \operatorname{Ind}(R) \otimes_R L.$$

Recall that $\mathcal{M}^{\#} \in \mathcal{D}$ is the maximal graded subspace of $\mathrm{Hom}_{R_r}(\mathrm{Ind}_+(R), R^{\sigma})$. We will identify R^{σ} with R and use the dot to indicate the R-bimodule structure on R^{σ}. Then $\mathfrak{g}$-R-bimodule structure is given by the following formulas

$$\begin{array}{l}\psi(xr) = \psi(x).r = (-1)^{p(r)p(\psi(x))}\sigma(r)\psi(x),\\ (g\psi)(x) = (-1)^{p(g)p(\psi)}\psi(\sigma(g)x),\\ (\psi r)(x) = (-1)^{p(r)p(\psi)}\sigma(r).\psi(x) = (-1)^{p(r)p(x)}\psi(x)r.\end{array} \tag{4}$$

where $g \in \mathcal{U}(\mathfrak{g}), r \in R, \psi \in \mathcal{M}^{\#}, x \in \mathrm{Ind}_+(R)$.

4.2.1. **Proposition.**

(i) *The functor Φ_0 admits a right adjoint functor* Coind *which is exact.*
(ii) *For any $K \in \mathcal{C}$ the adjunction map* $\mathrm{Coind}(K)_0 \to K$ *is an isomorphism.*
(iii) *One has a $\mathfrak{g}$-R-bimodule isomorphism $\beta : \mathcal{M}^{\#} \xrightarrow{\sim} \mathrm{Coind}(R)$ satisfying $\beta^{-1}(x)(y) = (-1)^{p(x)p(y)}\sigma(y)x$ for each $x \in \mathrm{Coind}(R)_0 = R$ and $y \in \mathrm{Ind}_+(R)_0 = R$.*
(iv) *One has a canonical isomorphism* $\mathrm{Coind}(R) \otimes_R L \to \mathrm{Coind}(L)$.
(v) *Under the above identifications $\Xi(L)$ identifies with $\Xi(R) \otimes_R \mathrm{id}_L$.*

Proof. (i) The functor Φ_0 is exact. By Freyd Theorem (see [ML] Ch. V), Φ_0 admits a right adjoint functor. The functor Coind is left exact as it admits left adjoint. It remains to check that an epimorphism $L \to L'$ induces an epimorphism $\mathrm{Coind}(L) \to \mathrm{Coind}(L')$.

Below we construct a family of modules $X(\nu) \in \mathcal{D}$ ($\nu \in Q^+$) with the following properties:

- $X(\nu)_0$ is a free R-module and
- one has a natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(X(\nu), N) \xrightarrow{\sim} N_{-\nu}$ for all $N \in \mathcal{D}$.

To check the surjectivity of $\mathrm{Coind}(L) \to \mathrm{Coind}(L')$ it is enough to check it on the homogeneous components that is to verify the surjectivity of the map

$$\mathrm{Hom}_{\mathcal{D}}(X(\nu), \mathrm{Coind}(L)) \to \mathrm{Hom}_{\mathcal{D}}(X(\nu), \mathrm{Coind}(L')).$$

The latter amounts to the surjectivity of $\mathrm{Hom}_{\mathcal{C}}(X(\nu)_0, L) \to \mathrm{Hom}_{\mathcal{C}}(X(\nu)_0, L')$ which follows from the freeness of $X(\nu)_0$. We construct $X(\nu) \in \mathcal{D}$ as follows: fix a $Q(\pi)$-grading on $\mathcal{U}(\mathfrak{g})$ by assigning to 1 degree $-\nu$ and then take the maximal quotient belonging to $\mathcal{D}$; in other words, $X(\nu) := \mathcal{U}(\mathfrak{g})/\sum_{\lambda \not\leq \nu} \mathcal{U}(\mathfrak{g})\mathcal{U}(\mathfrak{g})_{\lambda}$ with the grading shifted by ν. One can easily see from PBW theorem that $X(\nu)_0$ can be identified with $\mathcal{U}(\mathfrak{b}^+)_{\nu}$. Hence $X(\nu)_0$ is a free left R-module; the second property is clear. This proves (i).

Observe that $X(\nu)$ is a cyclic $\mathfrak{g}$-module and that $X(\nu)_0$ is a finitely generated R-module.

(ii) Identify $\mathrm{Ind}(R)_0$ with R (see 4.2). One has

$$K = \mathrm{Hom}_{\mathcal{C}}(R, K) = \mathrm{Hom}_{\mathcal{D}}(\mathrm{Ind}(R), \mathrm{Coind}(K)) = \mathrm{Hom}_{\mathcal{C}}(R, \mathrm{Coind}(K)_0).$$

Hence $K = \mathrm{Hom}_{\mathfrak{h}}(R, \mathrm{Coind}(K)_0) = \mathrm{Coind}(K)_0$ as required.

(iii) One has an R-bimodule isomorphism $R \xrightarrow{\sim} \mathcal{M}_0^{\#} = \mathrm{Hom}_{R_r}(R, R^{\sigma})$ given by $x \mapsto r_x$ where $r_x(y) = \sigma(x)y$. Observe that the evaluated map $\mathcal{C}\ell(\lambda) \to \mathrm{Hom}_{R_r}(R, \mathcal{C}\ell(\lambda))$ is bijective for any λ. The inverse map $\beta' : \mathcal{M}_0^{\#} \to R$ induces an $\mathfrak{g}$-R homomorphism $\beta : \mathcal{M}^{\#} \to \mathrm{Coind}(R)$ satisfying $\beta_0 = \beta'$. We show below that $\beta_{-\nu}\, \mathcal{M}_{-\nu}^{\#} \to \mathrm{Coind}(R)_{-\nu}$ is a morphism of free A-modules of the same finite rank and that the evaluated maps $\beta(\lambda)$ are injective for all λ. Thus $\det \beta_{-\nu}$ has nonzero evaluations at all points and hence is invertible. Therefore $\beta_{-\nu}$ is bijective for all ν and so β is bijective as well.

For each $\lambda \in \mathfrak{h}_{\bar{0}}^*$ the evaluated map $\beta(\lambda)$ has the source $\mathcal{M}^{\#}(\lambda) := \mathrm{Hom}_{R_r}(\mathrm{Ind}_+(R), \mathcal{C}\ell(\lambda)^{\sigma})$. It is easy to see that any non-zero submodule of $\mathcal{M}^{\#}(\lambda)$ meets $\mathcal{M}^{\#}(\lambda)_0$. Since $\beta(\lambda)_0 = \beta'(\lambda)$ is injective, the map $\beta(\lambda)$ is injective as well.

Using (i), one obtains the natural isomorphisms of right R-modules

$$\mathrm{Coind}(R)_{-\nu} = \mathrm{Hom}_{\mathcal{D}}(X(\nu), \mathrm{Coind}(R)) = \mathrm{Hom}_{\mathcal{C}}(X(\nu)_0, R) = \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{b}^+)_{\nu}, R) = \mathcal{M}_{-\nu}^{\#}$$

where the right action of R on $\phi \in \mathrm{Hom}(X, Y)$ is induced by the right action of R on Y. Hence the source and the target of $\beta_{-\nu}$ are free A-modules of the same finite rank. This establishes (iii).

(iv) Define a canonical map

$$\mathrm{Coind}(R) \otimes_R L \longrightarrow \mathrm{Coind}(L) \tag{5}$$

as the one adjoint to the map

$$(\mathrm{Coind}(R) \otimes_R L)_0 \longrightarrow L$$

obtained from the identification $\mathrm{Coind}(R)_0 = R$ (see (ii)).

Recall that Coind is an exact functor (see (i)). It is easy to show that the map in (5) is an isomorphism for each L iff the functor Coind commutes with infinite direct sums. To verify the latter observe that

$$\mathrm{Hom}_{\mathcal{D}}(X(\nu), \mathrm{Coind}(\oplus Y_i)) = \mathrm{Hom}_{\mathcal{C}}(X(\nu)_0, \oplus Y_i) = \oplus\, \mathrm{Hom}_{\mathcal{C}}(X(\nu)_0, Y_i)$$

because $X(\nu)_0$ is a finitely generated R-module. On the other hand, since $X(\nu)$ is finitely generated $\mathfrak{g}$-module one has

$$\mathrm{Hom}_{\mathcal{D}}(X(\nu), \oplus\, \mathrm{Coind}(Y_i)) = \oplus\, \mathrm{Hom}_{\mathcal{D}}(X(\nu), \mathrm{Coind}(Y_i)) = \oplus\, \mathrm{Hom}_{\mathcal{C}}(X(\nu)_0, Y_i).$$

This finally yields a bijection of the νth graded components of $\mathrm{Coind}(\oplus Y_i)$ and $\oplus\, \mathrm{Coind}(Y_i)$ (see the proof of (i)). Now (iv) follows.

(v) The last claim amounts to checking the diagram below is commutative for each L.

$$\begin{tikzcd}
\mathrm{Ind}(L) \arrow[r, "\Xi(L)"] & \mathrm{Coind}(L) \\
\mathrm{Ind}(R)\otimes_R L \arrow[u] \arrow[r, "\Xi(R)\otimes \mathrm{id}"] & \mathrm{Coind}(R)\otimes_R L \arrow[u]
\end{tikzcd}$$

Since the functors Coind and $X \mapsto X_0$ are adjoint, this follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Ind}(L)_0 \arrow[r] & L \\
(\mathrm{Ind}(R)\otimes_R L)_0 \arrow[u] \arrow[r] & (\mathrm{Coind}(R)\otimes_R L)_0 \arrow[u]
\end{tikzcd}$$

which is obvious. $\square$

4.2.2. **Proposition.** $\mathrm{Ker}\,\Xi(L)$ *is the maximal graded submodule of* $\mathrm{Ind}(L)$ *which does not meet* L*.*

Proof. A composition

$$X \longrightarrow \mathrm{Ind}(L) \xrightarrow{\Xi(L)} \mathrm{Coind}(L)$$

is zero if and only if the adjoint composition

$$X_0 \longrightarrow \mathrm{Ind}(L)_0 = L \longrightarrow \mathrm{Coind}(L)$$

is zero. This implies the statement. $\square$

4.2.3. Let us identify $\mathrm{Coind}(R)$ with $\mathcal{M}^{\#} \subset \mathrm{Hom}_{R_r}(\mathrm{Ind}_+(R), R^{\sigma})$ via β (see Proposition 4.2.1). By Proposition 4.2.1, the restriction of $S : \mathcal{M} \to \mathcal{M}^{\#}$ to $\mathcal{M}_0 = R$ is given by $x \mapsto r_x$ where $r_x(y) = (-1)p(x)p(y)\sigma(y)x$ if $y \in \mathrm{Ind}_+(R)_0 = R$ and $r_x(y) = 0$ if $y \in \mathrm{Ind}_+(R)_\nu$ for $\nu \neq 0$.

Viewing 1 as an element of $\mathrm{Ind}(R)$ (resp., $\mathrm{Ind}_+(R)$) via the identification $R = \mathrm{Ind}(R)_0$ (resp., $R = \mathrm{Ind}_+(R)_0$) write an element of $\mathrm{Ind}(R)$ as $u1$ and an element of $\mathrm{Ind}_+(R)$ as $u'1$ $(u, u' \in \mathcal{U}(\mathfrak{g}))$.

4.2.4. **Claim.**

$$S(u1)(u'1) = (-1)^{p(u)p(u')}\,\mathrm{HC}(\sigma(u')u)1.$$

Proof. Since S is an even $\mathfrak{g}$-homomorphism one has

$$\begin{aligned} S(u1)(u'1) &= (-1)^{p(u)p(u')}(\sigma(u')S)(u1)(1) = (-1)^{p(u)p(u')}S(\sigma(u')u_{-}u1)(1) \\ &= (-1)^{p(u)p(u')}S(\mathrm{HC}(\sigma(u')u)1)(1) = (-1)^{p(u)p(u')}\,\mathrm{HC}(\sigma(u')u)1. \end{aligned}$$

□

Identifying $\mathcal{M}$ with $\mathcal{U}(\mathfrak{b}^-)$ and $\mathcal{M}^\#$ with $\mathcal{U}(\mathfrak{b}^+)$ we obtain the formulas (1), (2).

4.3. **Shapovalov determinants.** Let $S_\nu : \mathcal{M}_{-\nu} \to \mathcal{M}^\#_{-\nu}$ be the restriction of S to $\mathcal{M}_{-\nu}$. Viewed as left A-modules $\mathcal{M}_{-\nu}$ and $\mathcal{M}^\#_{-\nu}$ are free of the same finite rank. Thus $\det S_\nu$ is an element of A defined up to the multiplication by an invertible element, i.e. by an element of $\mathbb{C}^*$.

If $\mathfrak{g}$ is a contragredient Lie superalgebra, $\det S_\nu$ is called a *Shapovalov determinant.*

4.3.1. *Reduced norms.* Let $\mathfrak{g}$ be a Q-type Lie superalgebra.

Set $\tilde{R} := R \otimes R$ where the tensor product means that of the graded algebras. View $\tilde{R}$ as a non-graded algebra. Notice that $\tilde{R}$ is a Clifford A-algebra whose evaluation at a generic point is isomorphic to the matrix algebra $\mathrm{Mat}(k, \mathbb{C})$ where $k = 2^{\dim \mathfrak{h}_{\bar{1}}}$. As we will explain in A.6, for any $\tilde{R}$-module L which is free over A there exists a unique map $\mathrm{Norm} : \mathrm{End}_{\tilde{R}}(L) \to A$ which satisfies the properties

$$\mathrm{Norm}(\mathrm{id}) = 1, \quad \mathrm{Norm}(\psi\psi') = \mathrm{Norm}\,\psi\,\mathrm{Norm}\,\psi', \quad \det\psi = (\mathrm{Norm}\,\psi)^k.$$

Convert R-bimodules $\mathcal{M}_{-\nu}$ and $\mathcal{M}^\#_{-\nu}$ to left (non-graded) $\tilde{R}$-modules via the antiautomorphism σ; denote these modules by X and Y respectively. In Sect. 6 we will construct an isomorphism $\Psi : X \to Y$. This allows us to define a map $\mathrm{Norm} : \mathrm{Hom}_{\tilde{R}}(X, Y) \to A$ by setting $\mathrm{Norm}\,\psi := \mathrm{Norm}(\Psi^{-1}\psi)$. One has

$$\det\psi = (\mathrm{Norm}\,\psi)^{2^{\dim \mathfrak{h}_{\bar{1}}}}.$$

Since S_ν is an even homomorphism of R-bimodules, it can be viewed as an element of $\mathrm{Hom}_{\tilde{R}}(X, Y)$. We call $\mathrm{Norm}\,S_\nu$ a Shapovalov determinant.

4.3.2. The results presented in 4.1.5 immediately follow from the above.

4.4. **Applications to Verma modules $M(\lambda)$.** Recall that $\mathcal{C}\ell(\lambda)$ has a filtration with the factors of the form $E(\lambda), \Pi(E(\lambda))$. This filtration induces a filtration on $M(\lambda)$ which has the same number of factors and each factor is either $N(\lambda)$ or $\Pi(N(\lambda))$. In its turn, $\overline{M(\lambda)}$ admits a filtration with the factors of the form $\overline{N(\lambda)}$ or $\Pi(\overline{N(\lambda)})$ and the number of factors is not greater than one for the previous filtrations. Therefore

$$\mathrm{ch}\,\overline{M(\lambda)} = m\,\mathrm{ch}\,\overline{N(\lambda)}, \text{ for some } 1 \le m \le \frac{2^{\dim \mathfrak{h}_{\bar{1}}}}{\dim E(\lambda)}.$$

The example $\mathfrak{g}=\mathfrak{sq}(2)$, $\lambda=0$ (see 7.3.4) illustrates that the numbers of factors in the above filtrations can be not equal. Indeed, the module $N(0)$ has the filtration of length two: $N(0)/M(0)\cong\Pi(M(0))$, but $\overline{N(0)}=\overline{M}(0)$. We have $\dim\mathfrak{h}_{\bar 1}=1,\dim E(\lambda)=1$ and $m=1$.

4.4.1. If $c(\lambda)=0$ (see 3.4 for the notation), we conclude from A.3.2 below that for $\dim\mathfrak{h}_{\bar 1}$ being even

$$\begin{array}{l} M(\lambda)\cong N(\lambda)^{\oplus s}\oplus\Pi(N(\lambda))^{\oplus s},\\ \overline{M}^{\oplus}(\lambda)\cong\overline{M}(\lambda)^{\oplus s}\oplus\Pi(\overline{M}(\lambda))^{\oplus s},\end{array}$$

and for $\dim\mathfrak{h}_{\bar 1}$ being odd

$$\begin{array}{l} N(\lambda)\cong\Pi(N(\lambda)),\\ M(\lambda)\cong N(\lambda)^{\oplus s},\\ \overline{M(\lambda)}\cong\overline{N(\lambda)}^{\oplus s}\end{array}$$

where $s:=2^{[\frac{\dim\mathfrak{h}_{\bar 1}-1}{2}]}$.

4.4.2. Combining 4.1.5 and the above analysis of filtrations we conclude

$$\begin{array}{l} \dim\overline{M(\lambda)}_{\lambda-\nu}=\operatorname{corank}B_\nu(\lambda),\\ \operatorname{corank}B_\nu(\lambda)/r\leq\dim\overline{N(\lambda)}_{\lambda-\nu}\leq\operatorname{corank}B_\nu(\lambda),\\ \dim V(\lambda)_{\lambda-\nu}=\operatorname{rank}B_\nu(\lambda)/r\ \text{if}\ c(\lambda)=0\end{array}\tag{6}$$

where $r=2^{\dim\mathfrak{h}_{\bar 1}}/\dim E(\lambda)$. Recall that the condition $c(\lambda)=0$ simply means that $\mathcal{C}\ell(\lambda)$ is a non-degenerate Clifford algebra; the example 7.3.4 shows that this condition is essential in the last formula.

4.4.3. **Corollary.**

(i) $\overline{M(\lambda)}_{\lambda-\nu}=0\iff\overline{N(\lambda)}_{\lambda-\nu}=0\iff\operatorname{Norm}S_\nu\neq 0$,
(ii) $N(\lambda)$ *is simple iff* $\operatorname{Norm}S_\nu\neq 0$ *for all* $\nu\in Q^+$.

4.4.4. *Example:* $\nu=0$. In this case $Y=R$ and the map $S_0:R\to\operatorname{Hom}_A(R,A)$ coincides with α. From the formula (36) we see that the matrix B_0 written with respect to an appropriate basis have zero entries everywhere except the secondary diagonal; the entries of this diagonal are equal to ±1. Thus $\det B_0=\pm1$ and $\operatorname{Norm}S_0=1$.

5. On Shapovalov determinants for various algebras

5.1. **Assumptions.** The construction described in Sect. 4 is applicable to a large class of Lie superalgebras admitting "nice triangular decompositions". The main assumption is that $[\mathfrak{h}_{\bar 0},\mathfrak{h}]=0$.

Let $\mathfrak{g}=\mathfrak{n}^-+\mathfrak{h}+\mathfrak{n}^+,\mathfrak{g}'=(\mathfrak{n}^-)'+\mathfrak{h}'+(\mathfrak{n}^+)'$ be Lie superalgebras satisfying the assumption. Let $\psi:\mathfrak{g}\to\mathfrak{g}'$ be a homomorphism such that the restriction of ψ gives bijections $\mathfrak{n}^\pm\xrightarrow{\sim}$

$(\mathfrak{n}^{\pm})'$. In this section we show that $\operatorname{Norm} S'_\nu = \psi(\operatorname{Norm} S_\nu)$; notice that $\det S'_\nu \neq \psi(\det S_\nu)$ if $\dim \mathfrak{h}_{\bar 1} \neq \dim \mathfrak{h}'_{\bar 1}$.

5.1.1. Retain notations of Sect. 4; set $R' := \mathcal{U}(\mathfrak{h}')$, $\mathcal{M}' := \operatorname{Ind}(R')$ and so on. Extend ψ to the homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}')$. The module $\mathcal{M}'$ inherits a structure of $\mathfrak{g}$-R bimodule and it lies in the category $\mathcal{D}$. The restriction of ψ to R induces $\mathfrak{g}$-R-maps $\mathcal{M} \to \mathcal{M}'$ and $(\mathcal{M}^{\#})' \to \mathcal{M}^{\#}$. The following diagram is commutative

$$\begin{tikzcd} \mathcal{M} \arrow[r, "S"] \arrow[d] & \mathcal{M}^{\#} \\ \mathcal{M}' \arrow[r, "S'"] & (\mathcal{M}^{\#})'. \arrow[u] \end{tikzcd}$$

In particular, $\mathcal{M}', (\mathcal{M}^{\#})'$ inherit Q^--gradings and $S' = \sum_{\nu \in Q^+} S'_\nu$ ($Q^+, Q^- \subset Q$ where Q stands for the root lattice corresponding to $\mathfrak{g}$).

5.2. **Theorem.** *One has* $\operatorname{Norm} S'_\nu = \psi(\operatorname{Norm} S_\nu)$ *for each* ν.

Proof. Let $H_1, \dots, H_n$ (resp., $H'_1, \dots, H'_m$) be a basis of $\mathfrak{h}_{\bar 1}$ (resp., of $\mathfrak{h}'_{\bar 1}$) and $\psi(H_i) = H'_i$ for $i = 1, \dots, s$, $\psi(H_i) = 0$ for $= s+1, n$. Normalize $\int, \int'$ is such a way that $\int H_1 \dots H_n = \int' H'_1 \dots H'_m = 1$. For $J \subset \{1, \dots, n\}$ set $H_J := \prod_{j \in J} H_j$ ($H_\emptyset = 1$) and define H'_J similarly. Clearly, H_J form a free basis of $\mathcal{U}(\mathfrak{h})$ over $\mathcal{S}(\mathfrak{h}_0)$. Fix bases $\{x_1, \dots, x_r\}$ in $\mathcal{U}(\mathfrak{n}^-)_{-\nu}$ and $\{y_1, \dots, y_r\}$ in $\mathcal{U}(\mathfrak{n}^+)_\nu$. Then the products $x_i H_J$ form a basis in $\mathcal{U}(\mathfrak{b}^-)$ and the products $y_i H_J$ form a basis in $\mathcal{U}(\mathfrak{b}^+)$. Consider Shapovalov matrices B_ν, B'_ν corresponding to this choice of bases. More precisely one has

$$B_\nu = \big(b_{(i,I;j,J)}\big), \quad b_{(i,I;j,J)} = (-1)^{p(x_i H_I) p(y_j H_J)} \int \mathrm{HC}\big(\sigma(x_i H_I) y_j H_J\big)$$

and the similar formulas for B'_ν.

Consider three cases:

(i) the restrictions of ψ gives bijection $\mathfrak{h}_{\bar 1} \xrightarrow{\sim} \mathfrak{h}'_{\bar 1}$;

(ii) the restrictions of ψ gives a monomorphism $\mathfrak{h}_{\bar 1} \to \mathfrak{h}'_{\bar 1}$ and $\dim \mathfrak{h}'_{\bar 1} - \dim \mathfrak{h}_{\bar 1} = 1$;

(iii) the restrictions of ψ gives an epimorphism $\mathfrak{h}_{\bar 1} \to \mathfrak{h}'_{\bar 1}$ and $\dim \mathfrak{h}_{\bar 1} - \dim \mathfrak{h}'_{\bar 1} = 1$.

In the first case $\operatorname{Ker} \psi \subset \mathfrak{h}_{\bar 0}$ and so the matrix B'_ν is the evaluation of the matrix B_ν. In particular, $\det B'_\nu = \psi(\det B_\nu)$ and so $\operatorname{Norm} S_\nu = \psi(\operatorname{Norm} S'_\nu)$.

In the second case $n = s = m - 1$. One has $\int' \psi(a) H_m = \int a$ for all $a \in \mathcal{U}(\mathfrak{h})$. Then $\int' \psi(a)\psi(b) H_m = \int ab = \pm \int' H_m \psi(a)\psi(b))$ and $\int' \psi(a)\psi(b) = \int' H_m \psi(a)\psi(b) H_m = 0$ for all $a, b \in \mathcal{U}(\mathfrak{g})$. Therefore

$$B'_\nu = \begin{pmatrix} 0 & B_\nu \\ \pm B_\nu & 0 \end{pmatrix}$$

Thus $\det B'_\nu = \pm(\det B_\nu)^2$ and so $\operatorname{Norm} S_\nu = \operatorname{Norm} S'_\nu$.

The third case $m = s = n-1$ is similar to the second one.

Finally, remark that general ψ can be presented as the composition of maps of the form (i)–(iii). Indeed, by the assumption on $\mathfrak{g}$ one has $[\mathfrak{h}_{\bar{0}}, \mathfrak{h}] = 0$. Since $\operatorname{Ker}\psi$ is a subspace of $\mathfrak{h}$ and an ideal, $\operatorname{Ker}\psi \cap \mathfrak{h}_{\bar{0}}$ is an ideal as well. Thus $\psi = \psi_2 \circ \psi_1$ where ψ_1 is of the form (i) and the restriction of ψ_2 gives a bijection $\mathfrak{h}_{\bar{0}} \xrightarrow{\sim} \mathfrak{h}'_{\bar{0}}$. Therefore in is enough to show that ψ satisfying $\operatorname{Ker}\psi \cap \mathfrak{h}_{\bar{0}} = 0$ an be presented as the composition of maps of the form (ii), (iii). It suffices to construct a chain of ideals $I_1 \subset I_2 \subset \ldots \subset I_k = \operatorname{Ker}\psi \subset \mathfrak{g}$ satisfying $\dim I_j = j$ and a chain of subalgebras $\operatorname{Im}\psi = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_t = \mathfrak{g}'$ satisfying $\dim \mathfrak{p}_i = \dim \mathfrak{p}_{i-1} + 1$. Recall that $\operatorname{Ker}\psi \subset \mathfrak{h}_{\bar{1}}$ and so all elements of $\operatorname{Ker}\psi$ are central and so any subspace of $\operatorname{Ker}\psi$ is an ideal. Similarly, the assumption $[\mathfrak{h}'_{\bar{0}}, \mathfrak{h}'] = 0$ implies that the sum $\operatorname{Im}\psi + X$ is a subalgebra for any $X \subset \mathfrak{h}_{\bar{1}}$. The statement follows. $\square$

5.3. **Shapovalov determinants for various Q-type algebras.** Recall that $\mathfrak{pq}(n), \mathfrak{psq}(n)$ are quotients of $\mathfrak{q}(n)$ and $\mathfrak{sq}(n)$ respectively, by a one-dimensional centre $\mathbb{C}h_0$. As a consequence, the Shapovalov determinants for $\mathfrak{pq}(n), \mathfrak{psq}(n)$ are evaluations at $h_0 = 0$ of the corresponding Shapovalov determinants for $\mathfrak{q}(n), \mathfrak{sq}(n)$.

Let S_ν be the Shapovalov map for $\mathfrak{q}(n)$ and S'_ν be the Shapovalov map for $\mathfrak{sq}(n)$. The natural embedding $\mathfrak{sq}(n) \to \mathfrak{q}(n)$ gives $\operatorname{Norm} S_\nu = \operatorname{Norm} S'_\nu$.

6. Comparing $\mathcal{M}$ to $\mathcal{M}^\#$.

The modules $\mathcal{M}$ and $\mathcal{M}^\#$ are not isomorphic as R-bimodules. In this section we show that they become (non-canonically) isomorphic when considered as non-graded $R \otimes R$-modules. This fact is used in the definition of reduced norms in 4.3.1.

6.1. **$\mathcal{M}$ and $\mathcal{M}^\#$ as R-bimodules.** Set $\tilde{R} := R \otimes R$ where the tensor product sign stands for the tensor product of graded algebras.

Any R-bimodule V can be viewed as a left $\tilde{R}$ module via the antiautomorphism σ:

$$(u \otimes 1)v := uv, \quad (1 \otimes u)v := (-1)^{p(v)p(u)} v\sigma(u).$$

View $\tilde{R}$ as a non-graded algebra; we will prove that $\mathcal{M} \cong \mathcal{M}^\#$ as (non-graded) $\tilde{R}$-modules.

6.1.1. Recall that as R-bimodules $\mathcal{M}=\mathcal{U}(\mathfrak{b}^-)$ and $\mathcal{M}^{\#}=\sum_{\nu\in Q^+}\operatorname{Hom}_{R_r}(\mathcal{U}(\mathfrak{b}^+)_\nu,R^\sigma)$. As modules over $1\otimes R$ both $\mathcal{M},\mathcal{M}^{\#}$ can be decomposed as

$$\mathcal{M}=(1\otimes R)\otimes\mathcal{U}(\mathfrak{n}^-),\quad \mathcal{M}^{\#}=(1\otimes R)\otimes\mathcal{U}(\mathfrak{n}^+)^{\#}$$

where $\mathcal{U}(\mathfrak{n}^+)^{\#}$ is the image of a natural embedding $\operatorname{Hom}(\sum_{\nu\in Q^+}\mathcal{U}(\mathfrak{n}^+)_\nu,\mathbb{C})\to\mathcal{M}^{\#}$. It is easy to see that both $\mathcal{U}(\mathfrak{n}^-)$ and $\mathcal{U}(\mathfrak{n}^+)^{\#}$ are $\operatorname{ad}\mathfrak{h}$-stable.

We construct in 6.2 below an isomorphism $\Psi':\mathcal{U}(\mathfrak{n}^-)\xrightarrow{\sim}\mathcal{U}(\mathfrak{n}^+)^{\#}$ of non-graded $\operatorname{ad}\mathfrak{h}$-modules.

Finally, the following Lemma 6.1.2 implies that an isomorphism of $\operatorname{ad}\mathfrak{h}$ modules $\mathcal{U}(\mathfrak{n}^-)$ and $\mathcal{U}(\mathfrak{n}^+)^{\#}$ extends canonically to an $\tilde{R}$-isomorphism $\mathcal{M}\xrightarrow{\sim}\mathcal{M}^{\#}$.

6.1.2. **Lemma.** *Let $\mathfrak{p}$ be a Lie superalgebra, $U:=\mathcal{U}(\mathfrak{p})$ and let $\Delta:U\to U\otimes U$ be the comultiplication. Let X,Y be non-graded modules over $U\otimes U$ which, as modules over $1\otimes U\subset U\otimes U$, have the form*

$$X=(1\otimes U)\otimes X',\quad Y=(1\otimes U)\otimes Y'$$

where X',Y' are $\Delta(\mathfrak{p})$-stable subspaces.

Then any homomorphism of $\Delta(\mathfrak{p})$-modules $\Psi':X'\to Y'$ can be uniquely extended to a $U\otimes U$-homomorphism $\Psi:X\to Y$.

Proof is straightforward.

6.1.3. Let us rewrite the above lemma in terms of bimodules.

Let $\tilde{X},\tilde{Y}$ be U-bimodules and X,Y are corresponding $U\otimes U$-modules (defined via the antiautomorphism σ) viewed as non-graded modules. Notice that the action of $\Delta(\mathfrak{p})$ on X,Y corresponds to the action of $\operatorname{ad}\mathfrak{p}$ on $\tilde{X},\tilde{Y}$.

Assume that $\tilde{X}=X'\otimes U$, $\tilde{Y}=Y'\otimes U$ as right modules where X',Y' are $\operatorname{ad}\mathfrak{p}$-stable. Lemma 6.1.2 says that any non-graded $\operatorname{ad}\mathfrak{p}$-homomorphism $\Psi':X'\to Y'$ can be uniquely extended to a $U\otimes U$-homomorphism $\Psi:X\to Y$.

Thus, we have proven the following

6.1.4. **Corollary.** *$\mathcal{M}$ and $\mathcal{M}^{\#}$ are isomorphic as non-graded $\tilde{R}$-modules.*

6.2. **An isomorphism $\mathcal{U}(\mathfrak{n}^-)\to\mathcal{U}(\mathfrak{n}^+)^{\#}$.** In this subsection we will construct a non-graded $\mathfrak{h}$-isomorphism $\mathcal{U}(\mathfrak{n}^-)\xrightarrow{\sim}\mathcal{U}(\mathfrak{n}^+)^{\#}$. All $\mathfrak{h}$-module structures in this subsection are induced by the adjoint action.

6.2.1. Denote by $\mathcal{U}(\mathfrak{g}_\alpha)^\#$ the maximal $\mathfrak{h}$-invariant submodule of $\mathrm{Hom}(\mathcal{U}(\mathfrak{g}_\alpha),\mathbb{C})$. We claim that $\mathcal{U}(\mathfrak{g}_{-\alpha})$ and $\mathcal{U}(\mathfrak{g}_\alpha)^\#$ are isomorphic as non-graded $\mathfrak{h}$-modules. Indeed, $\mathcal{U}(\mathfrak{g}_{-\alpha})$ and $\mathcal{U}(\mathfrak{g}_\alpha)^\#$ are isomorphic as $\mathfrak{h}_{\overline{0}}$-modules and their weights are of the form $-r\alpha$; their zero weight spaces are even trivial $\mathfrak{h}$-modules and for $r>0$ the corresponding subspaces are simple two-dimensional $\mathfrak{h}$-modules. The algebra $\mathfrak{h}$ has at most two non-isomorphic simple modules of a given $\mathfrak{h}_{\overline{0}}$-weight, these modules are $E(-r\alpha)$ and $\Pi(E(-r\alpha))$ (see A.3.2). Thus as non-graded $\mathfrak{h}$-modules $\mathcal{U}(\mathfrak{g}_{-\alpha})$ and $\mathcal{U}(\mathfrak{g}_\alpha)^\#$ are isomorphic (see Remark 6.2.3 for more details).

6.2.2. The product in $\mathcal{U}(\mathfrak{g})$ gives the following isomorphisms of graded $\mathfrak{h}$-modules:

$$\mathcal{U}(\mathfrak{n}^-)\cong\otimes_{\alpha\in\Delta^+}\mathcal{U}(\mathfrak{g}_{-\alpha}),\qquad \mathcal{U}(\mathfrak{n}^+)\cong\otimes_{\alpha\in\Delta^+}\mathcal{U}(\mathfrak{g}_{\alpha})$$

where $\otimes$ stands for the graded tensor product. Then $\mathcal{U}(\mathfrak{n}^+)^\#$ is isomorphic to the maximal $\mathfrak{h}$-invariant submodule of the graded tensor product $\mathrm{Hom}(\otimes_{\alpha\in\Delta^+}\mathcal{U}(\mathfrak{g}_\alpha),\mathbb{C})=\otimes_{\alpha\in\Delta^+}\mathrm{Hom}(\mathcal{U}(\mathfrak{g}_\alpha),\mathbb{C})$ that is

$$\mathcal{U}(\mathfrak{n}^+)^\#\cong\otimes_{\alpha\in\Delta^+}\mathcal{U}(\mathfrak{g}_\alpha)^\#.$$

Now the existence of a non-graded isomorphism $\mathcal{U}(\mathfrak{n}^+)\cong\mathcal{U}(\mathfrak{n}^+)^\#$ follows from 6.2.1.

6.2.3. *Remark.* Let us explain why an isomorphism between $\mathcal{U}(\mathfrak{g}_{-\alpha})$ and $\mathcal{U}(\mathfrak{g}_\alpha)^\#$ is non-graded. As $\mathfrak{h}$-modules, $\mathcal{U}(\mathfrak{g}_{-\alpha})\cong\sum_{r=0}^\infty\mathcal{S}^r(\mathfrak{g}_{-\alpha})$ and $\mathcal{U}(\mathfrak{g}_\alpha)^\#=\sum_{r=0}^\infty\mathcal{S}^r(\mathfrak{g}_\alpha^*)$ where $\mathcal{S}^r$ stands for the rth symmetric power. It is easy to check that there exists an isomorphism $\psi_1:\mathfrak{g}_{-\alpha}\xrightarrow{\sim}\Pi(\mathfrak{g}_\alpha^*)$. This induces an isomorphism $\psi_r:\mathcal{S}^r(\mathfrak{g}_{-\alpha})\xrightarrow{\sim}\Pi\big(\mathcal{S}^r(\mathfrak{g}_\alpha^*)\big)$ for all $r>0$. On the other hand, $\mathcal{S}^0(\mathfrak{g}_{-\alpha})\not\cong\Pi\big(\mathcal{S}^0(\mathfrak{g}_\alpha^*)\big)$ since $\mathcal{S}^0(\mathfrak{g}_{-\alpha}),\mathcal{S}^0(\mathfrak{g}_\alpha^*)$ are trivial even $\mathfrak{h}$-modules. Therefore $\sum_{r>0}\psi_r:\sum_{r=1}^\infty\mathcal{S}^r(\mathfrak{g}_{-\alpha})\xrightarrow{\sim}\Pi\big(\sum_{r=1}^\infty\mathcal{S}^r(\mathfrak{g}_\alpha^*)\big)$ can not be extended to an isomorphism $\mathcal{U}(\mathfrak{g}_{-\alpha})\to\Pi\big(\mathcal{U}(\mathfrak{g}_\alpha)^\#\big)$.

7. The case $\mathfrak{g}=\mathfrak{sq}(2)$

In this section we write down the Shapovalov matrices for the simplest case $\mathfrak{g}:=\mathfrak{sq}(2)$.

The simple modules $V(\lambda)=N(\lambda)/\overline{N(\lambda)}$ for $\mathfrak{sq}(2)$ were described in [P]. We explicitly calculate the kernels of Shapovalov maps which gives both $\overline{M(\lambda)}$ and $\overline{N(\lambda)}$.

The even part $\mathfrak{g}_{\overline{0}}=\mathfrak{gl}(2)$ has the only root α; the elements $e_\alpha,f_\alpha,h_\alpha$ form the standards basis of $\mathfrak{sl}(2)\subset\mathfrak{gl}(2)$ and $h':=h_{\overline{\alpha}}$ is a central element in $\mathfrak{gl}(2)$. The space $\mathfrak{h}_{\overline{1}}$ is spanned by an element H satisfying $H^2=h'$.

We will omit the lower index α writing f instead of f_α and etc.

7.1. **Verma and Weyl modules.** Observe that $\mathcal{U}(\mathfrak{h})$ has rank two over $\mathcal{S}(\mathfrak{h}_0)$ and $c(\lambda)=1$ if $h'(\lambda)=0$, $c(\lambda)=0$ if $h'(\lambda)\neq 0$. The superalgebra $\mathcal{C}\ell(\lambda)$ is a simple module over itself iff $c(\lambda)=0$. Therefore $M(\lambda)=N(\lambda)\cong\Pi(N(\lambda))$ if $h'(\lambda)\neq0$. If λ is such that $h'(\lambda)=0$ then $M(\lambda)$ has a submodule $N(\lambda)$ (having an odd highest weight vector) whose quotient is $\Pi(N(\lambda))$. Notice that for $n>0$ the weight space $\mathcal{U}(\mathfrak{n}^-)_{n\alpha}$ is two-dimensional. This gives the following formulae for non-graded characters

$$\operatorname{ch} M(\lambda)=e^{\lambda}(2+4(e^{-\alpha}+e^{-2\alpha}+\ldots))$$

and

$$\operatorname{ch} N(\lambda)=\begin{cases} e^{\lambda}(1+2(e^{-\alpha}+e^{-2\alpha}+\ldots)), & h'(\lambda)=0,\\ e^{\lambda}(2+4(e^{-\alpha}+e^{-2\alpha}+\ldots)) & h'(\lambda)\neq 0.\end{cases}$$

7.2. **Shapovalov matrices.** Set $R:=\mathcal{U}(\mathfrak{h}), A:=\mathcal{S}(\mathfrak{h}_0)$. The elements $1,H$ form a free A-basis of R; one has $\int H=1, \int 1=0$.

The weight space $\mathcal{U}(\mathfrak{b}^-)_0$ coincides with R. The matrix B_0 written with respect to the basis $1,H$ takes form

$$B_0=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

For $k\geq 0$ the elements $f^{k+1}, f^kFH, f^{k+1}H, f^kF$ form a free A-basis of $\mathcal{U}(\mathfrak{b}^-)_{(k+1)\alpha}$. Using the formulas of 7.5 we see that the matrix $B_{(k+1)\alpha}$ written in this basis takes form

$$B_{(k+1)\alpha}=\begin{pmatrix} 0 & 0 & (-1)^{k+1}(k+1)hh_k & (-1)^{k+1}(k+1)h_k\\ 0 & 0 & (-1)^k(k+1)h'h_k & (-1)^k h'h_k\\ (-1)^k(k+1)hh_k & (-1)^k(k+1)h'h_k & 0 & 0\\ (-1)^{k+1}(k+1)h_k & (-1)^{k+1}h'h_k & 0 & 0\end{pmatrix}$$

where

$$h_0:=1,\quad h_k:=k!(h-1)\ldots(h-k) \text{ for } k>0.$$

Therefore $\det B_{(k+1)\alpha}=(h'h_kh_{k+1})^2$ and so $\operatorname{Norm} S_{(k+1)\alpha}=\mathbb{C}^*h'h_kh_{k+1}$ that is

$$\operatorname{Norm} S_{(k+1)\alpha}=h_{\overline{\alpha}}(h_\alpha-1)^2\ldots(h_\alpha-k)^2(h_\alpha-(k+1)).$$

7.3. **Structure of $M(\lambda)$ as $\mathfrak{g}_{\overline{0}}$-module.** Writing $\lambda=(x,a)$ where $x:=h(\lambda), a:=h'(\lambda)$, we see that $N(\lambda)$ is not simple iff $x\in\mathbb{Z}_{>0}$ or $a=0$. Let us describe $V(\lambda)=N(\lambda)/\overline{N(\lambda)}$ as an $\mathfrak{sl}(2)$-module. Denote by $M_{\mathfrak{sl}(2)}(x)$ (resp., $V_{\mathfrak{sl}(2)}(x)$) a Verma (resp., simple) $\mathfrak{sl}(2)$-module of the highest weight x.

Recall that the evaluated Shapovalov matrices $B_\nu(\lambda)$ correspond to the restriction of Shapovalov map $S(\lambda)$ to $M(\lambda)_{\lambda-\nu}$. Let $E(\lambda)$ be a simple $\mathcal{C}\ell(\lambda)$-module. By Claim 4.2.2, $\overline{N(\lambda)}$ coincides with the kernel of Shapovalov map $\Xi(E(\lambda))$. Since $E(\lambda)$ is a submodule of $\mathcal{C}\ell(\lambda)$, $N(\lambda)$ is a submodule of $M(\lambda)$ and $\Xi(E(\lambda))$ is the restriction of $S(\lambda)$ to $N(\lambda)$.

Identify $M(\lambda)$ with $\mathcal{U}(\mathfrak{b}^-)$. The submodule $N(\lambda)$ is given by the formula

$$N((x,a))=\begin{cases}\mathcal{U}(\mathfrak{b}^-), & a\neq 0,\\ \mathcal{U}(\mathfrak{n}^-)H, & a=0.\end{cases}$$

7.3.1. Take $\lambda=(m,a)$ such that $a\neq 0$ and $m\in\mathbb{Z}_{>0}$. The Shapovalov matrix $B_{k\alpha}$ has rank 4 if $0<k<m$, $B_{m\alpha}$ has rank 2 and $B_{k\alpha}$ has rank 0 for $k>m$. Therefore $V(1,a)=V_{\mathfrak{sl}(2)}(1)\oplus V_{\mathfrak{sl}(2)}(1)$ and $V(m,a)=V_{\mathfrak{sl}(2)}(m)^{\oplus 2}\oplus V_{\mathfrak{sl}(2)}(m-2)^{\oplus 2}$ for $m>1$.

7.3.2. Observe that

$$N((x,0))\cong M_{\mathfrak{sl}(2)}(x)\oplus M_{\mathfrak{sl}(2)}(x-2)\ \text{ for } x\neq 0.$$

Take $\lambda=(x,0)$ where $x\not\in\mathbb{Z}_{\geq 0}$. The Shapovalov matrices $B_{(k+1)\alpha}$ $(k\geq 0)$ have rank two and $\operatorname{Ker} B_{(k+1)\alpha}$ is spanned by $f^kFH, f^k(Fh-fH)$. Therefore $\operatorname{Ker} B_{(k+1)\alpha}\cap\mathcal{U}(\mathfrak{n}^-)H$ is spanned by f^kFH and thus

$$\overline{M}((x,0))\cong M_{\mathfrak{sl}(2)}(x-2),\quad V((x,0))\cong M_{\mathfrak{sl}(2)}(x)\cong V_{\mathfrak{sl}(2)}(x)\quad \text{ for } x\not\in\mathbb{Z}_{\geq 0}.$$

7.3.3. Take $\lambda=(m,0)$ where $m\in\mathbb{Z}_{>0}$. For $0\leq k<m$ the Shapovalov matrix $B_{(k+1)\alpha}$ has rank two and $\operatorname{Ker} B_{(k+1)\alpha}$ is spanned by $f^kFH, f^k(Fh-fH)$; thus $\operatorname{Ker} B_{(k+1)\alpha}\cap\mathcal{U}(\mathfrak{n}^-)H$ is spanned by f^kFH. The Shapovalov matrices $B_{(k+1)\alpha}$ $(k\geq m)$ are equal to zero. Therefore

$$\overline{M}((m,0))\cong M_{\mathfrak{sl}(2)}(m-2)\oplus M_{\mathfrak{sl}(2)}(-m-2),\quad V((m,0))\cong V_{\mathfrak{sl}(2)}(m).$$

7.3.4. Finally, take $\lambda=(0,0)$. The module $V(0)$ is one-dimensional. The matrices $B_{(k+1)\alpha}$ $(k\geq 0)$ have rank 2: the kernels are spanned by $f^kFH, f^{k+1}H$. Thus $\overline{N(0)}=\overline{M(0)}$.

7.3.5. Summarizing we obtain

$$V((x,a))\cong\begin{cases} V_{\mathfrak{sl}(2)}(1)\oplus V_{\mathfrak{sl}(2)}(1), & x=1, \quad a\neq 0,\\ V_{\mathfrak{sl}(2)}(x)^{\oplus 2}\oplus V_{\mathfrak{sl}(2)}(x-2)^{\oplus 2}, & x\in\mathbb{Z}_{>0}, x\neq 1,\quad a\neq 0,\\ V_{\mathfrak{sl}(2)}(x), & a=0\end{cases}\tag{7}$$

and $V((x,a))=N((x,a))$ for $x\not\in\mathbb{Z}_{>0}$ and $a\neq 0$.

7.4. **Corollary.** *Let $\mathfrak{g}$ be a Q-type Lie superalgebra and α_i be a simple root.*

(i) *If $\lambda\in\mathfrak{h}_{\overline{0}}^*$ is such that $m:=\lambda(h_{\alpha_i})\in\mathbb{Z}_{>0}$ then $N(\lambda)$ has a primitive vector of the weight $s_\alpha(\lambda)=\lambda-m\alpha_i$.*
(ii) *If $\lambda\in\mathfrak{h}_{\overline{0}}^*$ is such that $\lambda(h_{\overline{\alpha}_i})=0$ then $N(\lambda)$ has a primitive vector of the weight $\lambda-\alpha_i$.*

Proof. Fix a simple root $\alpha:=\alpha_i$. The superalgebra $\mathfrak{p}:=\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}+[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sq}(2)$. Let $U^-(\alpha)$ be a subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by $f_\alpha, F_\alpha, H_\alpha$. Let $v\in N(\lambda)$ be a highest weight vector. Clearly, $N':=U^-(\alpha)v$ is a Weyl $\mathfrak{p}$-module. For λ satisfying the assumption (i) (resp., (ii)) the formula (7) shows the existence of a non-zero vector $v'\in N'_{-m\alpha}$ (resp., $v'\in N'_{-\alpha}$) such that $E_\alpha v'=e_\alpha v'=0$. For any simple root $\alpha_j\neq\alpha$ one has $E_{\alpha_j}N'=e_{\alpha_j}N'=0$ because all vectors in N' have weights of the form $\lambda-r\alpha_i$ and

$N(\lambda)$ does not have vectors of weight $\lambda - r\alpha_i + \alpha_j$. Since $\{E_\beta, e_\beta : \beta \in \pi\}$ generate $\mathfrak{n}^+$, we conclude that v' is primitive. $\square$

7.5. **Useful formulas.** In 7.2 we used the following formulas.

Lemma.

$$\begin{array}{ll} (i) & \mathrm{HC}(e^k f^k) = k!h(h-1)\dots(h-(k-1)),\\ (ii) & \mathrm{HC}(Ee^k f^k F) = k!h'(h-1)(h-2)\dots(h-k),\\ (iii) & \mathrm{HC}(Ee^k f^{k+1}) = \mathrm{HC}(e^{k+1} f^k F) = (k+1)!(h-1)\dots(h-k)H, \end{array}$$

Proof. By Lemma 3.5.1, $\mathrm{HC}(e^k f^k)$ has degree k. Note that the term $e^k f^k$ annihilates $V_{\mathfrak{sl}(2)}(m)$ for $m = 0, 1, \dots, k-1$. Thus, up to a constant, $\mathrm{HC}(e^k f^k) = h(h-1)\dots(h-(k-1))$. Let v_1 be a highest weight vector of $V_{\mathfrak{sl}(2)}(1)$ and $v_1^{\otimes k}$ be the corresponding vector in $V_{\mathfrak{sl}(2)}(1)^{\otimes k}$. It is easy to check that $e^k f^k(v_1^{\otimes k}) = k!^2(v_1^{\otimes k})$. Since $v_1^{\otimes k}$ is a primitive vector of weight k this gives $\mathrm{HC}(e^k f^k)(k) = k!^2$ and (i) follows.

For (ii) set $a_k := \mathrm{HC}(e^k f^k)$. Modulo the left ideal $\mathcal{U}(\mathfrak{g})e^2$ one can write $e^k f^k = a_k + f b_k e$ where $b_k \in \mathbb{C}[h]$ ($b_1 = 1$). Then $a_{k+1} = \mathrm{HC}(e^{k+1} f^{k+1}) = e a_k f + e f b_k e f$ that is $h^2 b_k = a_{k+1} - h\zeta(a_k)$ where $\zeta : \mathbb{C}[h] \to \mathbb{C}[h]$ is an algebra homomorphism given by $h \mapsto h-2$. Therefore $b_k = k!k(h-2)\dots(h-k)$. Now $\mathrm{HC}(Ee^k f^k F) = \mathrm{HC}(E(a_k + f b_k e)F) = h'\zeta(a_k) + H b_k H = h'\zeta(a_k) + h' b_k$ which gives (ii). Finally, for (iii) one has $\mathrm{HC}(e^{k+1} f^k F) = \mathrm{HC}(e(a_k + f b_k e)F) = H\zeta(a_k) + h b_k H = (\zeta(a_k) + h b_k)H$ and similarly $\mathrm{HC}(Ee^k f^{k+1}) = (\zeta(a_k) + h b_k)H$. $\square$

8. The leading term of Shapovalov determinant

Recall that $\det S_\nu = \det B_\nu \in \mathcal{S}(\mathfrak{h}_0)$ for each $\nu \in Q^+$. In this section we calculate the leading term of the polynomial $\det S_\nu$.

8.1. **The main result of the section.** The Kostant partition function $\tau :\ Q \to \mathbb{Z}_{\geq 0}$ is defined by the formula

$$\mathrm{ch}\,\mathcal{U}(\mathfrak{n}^-) = \prod_{\alpha\in\Delta_1^+} (1+e^{-\alpha}) \prod_{\alpha\in\Delta_0^+} (1-e^{-\alpha})^{-1} =: \sum_{\eta\in Q} \tau(\eta) e^{-\eta}.$$

Note that $\tau(Q \setminus Q^+) = 0$.

8.1.1. **Definition.** A vector $\mathbf{k} = \{k_\alpha, k_{\overline{\alpha}}\}_{\alpha\in\Delta^+}$ is called a *partition of* $\nu \in Q^+$ if

$$\nu = \sum_{\alpha\in\Delta^+} (k_\alpha + k_{\overline{\alpha}})\alpha; \quad k_\alpha \in \mathbb{Z}_{\geq 0},\ k_{\overline{\alpha}} \in \{0,1\} \text{ for all } \alpha \in \Delta^+.$$

Denote by $\mathcal{P}(\nu)$ the set of all partitions of ν. Clearly, $|\mathcal{P}(\nu)| = \tau(\nu)$.

For each $\alpha \in \Delta^+$ set

$$\tau_\alpha(\nu) := |\{\mathbf{k} \in \mathcal{P}(\nu)|\ k_{\overline{\alpha}} = 0\}|.$$

In this section we will show that for suitable bases one has the

8.1.2. **Claim.** *The leading term of the polynomial* $\operatorname{Norm} S_\nu$ *is equal to*

$$\prod_{\alpha\in\Delta^+} h_\alpha^{\sum_{m=1}^\infty \tau(\nu-m\alpha)} h_{\overline{\alpha}}^{\tau_\alpha(\nu-\alpha)}$$

In particular, $\det S_\nu \neq 0$ for all $\nu \in Q^+$. By Corollary 4.4.3, this is equivalent to the existence of a simple Weyl module.

Claim 8.1.2 is proven in 8.2–8.6.5 below.

Using Lemma 8.6.5 we obtain the following useful formula

$$\deg \operatorname{Norm} S_\nu = \sum_{\mathbf{m}\in\mathcal{P}(\nu)} |\mathbf{m}|. \tag{8}$$

where $|\mathbf{m}| := \sum_{\alpha\in\Delta^+} m_\alpha + m_{\overline{\alpha}}$.

8.2. **Outline of the proof.** We reduce a computation of leading term of Shapovalov determinants to the case $\mathfrak{sq}(2)$. This is done in several steps described below.

As it was explained in 5.3, Shapovalov determinants for various Q-type algebras can be expressed via Shapovalov determinants for $\mathfrak{sq}(n)$. In particular, it is enough to prove Claim 8.1.2 for $\mathfrak{g} := \mathfrak{sq}(n)$. The proof for $\mathfrak{sq}(n)$ goes as follows. Define a filtration F on $\mathfrak{g}$ by setting

$$F^0(\mathfrak{g}) = 0,\ F^1(\mathfrak{g}) = \mathfrak{n}^- + \mathfrak{n}^+ + \mathfrak{h}_{\overline{1}},\ F^i(\mathfrak{g}) = \mathfrak{g} \text{ for } i > 1,\quad \dot{\mathfrak{g}} := \operatorname{gr}_F \mathfrak{g}.$$

Denote by S (resp., $\dot{S}$) the Shapovalov map for $\mathfrak{g}$ (resp., for $\dot{\mathfrak{g}}$). As we will show later (8.3.1), $\operatorname{Norm} \dot{S}_\nu$ is either zero or equal to the leading term of $\operatorname{Norm} S_\nu$.

Now let

$$\mathfrak{g}^\sqcap = \prod_{\alpha\in\Delta^+} \dot{\mathfrak{s}}^{(\alpha)}$$

where $\dot{\mathfrak{s}}^{(\alpha)} = \operatorname{gr}_F \mathfrak{sq}(2)$. Observe that $[\dot{\mathfrak{g}}, \dot{\mathfrak{g}}] = \dot{\mathfrak{h}}$ where $\dot{\mathfrak{h}} := \operatorname{gr}_F \mathfrak{h}$ and that $\dot{\mathfrak{h}} \cong \mathfrak{h}$ as Lie algebras. As a consequence, $\dot{\mathfrak{g}}$ is the quotient of $\mathfrak{g}^\sqcap \to \dot{\mathfrak{g}}$ by an ideal lying in $\mathfrak{h}^\sqcap$. By Theorem 5.2, $\operatorname{Norm} \dot{S}_\nu = \psi(\operatorname{Norm} S_\nu^\sqcap)$ where $S_\nu^\sqcap$ is the Shapovalov map for $\mathfrak{g}^\sqcap$. Hence the leading term of $\operatorname{Norm} S_\nu$ is equal to $\psi(\operatorname{Norm} S_\nu^\sqcap)$ provided the latter is non-zero. Since $\mathfrak{g}^\sqcap$ is the direct product of copies of $\dot{\mathfrak{sq}}(2)$, a computation of $\operatorname{Norm} S_\nu^\sqcap$ is reduced to the case $\dot{\mathfrak{sq}}(2)$— see 8.5.

8.3. **The algebras $\dot{\mathfrak{g}}$.** Retain notation of 8.2 and extend the filtration F to $\mathcal{U}(\mathfrak{g})$. Then $\mathcal{U}(\dot{\mathfrak{g}})$ identifies with the associated graded of $\mathcal{U}(\mathfrak{g})$; denote by $\dot{x}$ the image of $x \in \mathcal{U}(\mathfrak{g})$ or $x \subset \mathcal{U}(\mathfrak{g})$ in $\mathcal{U}(\dot{\mathfrak{g}})$.

The algebra $\dot{\mathfrak{g}} = \operatorname{gr}_F \mathfrak{g}$ admits a decomposition $\dot{\mathfrak{g}} = \dot{\mathfrak{n}}^- + \dot{\mathfrak{h}} + \dot{\mathfrak{n}}^+$. Construct a Shapovalov map $\dot{S}$ via the above decomposition. The algebra $\mathcal{U}(\dot{\mathfrak{n}}^+)$ inherit the gradings by Q^+ and this grading leads to a decomposition $\dot{S}=\sum\dot{S}_\nu$. In this way we obtain the Shapovalov matrices $\dot{B}_\nu$ and the determinants $\det\dot{B}_\nu=(\operatorname{Norm}\dot{S}_\nu)^{2^{\dim\mathfrak{h}_{\bar 1}}}$.

Let us explain why $\operatorname{Norm}S'_\nu$ is either zero or equal to the leading term of $\operatorname{Norm}S_\nu$.

8.3.1. Recall that the Rees algebra $\tilde{\mathfrak{g}}:=\oplus_{k=0}^\infty\varepsilon^kF^k(\mathfrak{g})$ is a Lie superalgebra over $\mathbb{C}[\varepsilon]$ with the bracket induced by the bracket in $\mathfrak{g}$; as a Lie superalgebra over $\mathbb{C}$ $\tilde{\mathfrak{g}}$ is graded with kth homogeneous component $\varepsilon^kF^k(\mathfrak{g})$.

The evaluation at $\varepsilon=0$ is canonically isomorphic to $\operatorname{gr}_F\mathfrak{g}$ and the evaluation at $\varepsilon=1$ is canonically isomorphic to $\mathfrak{g}$: $\tilde{\mathfrak{g}}/\varepsilon\tilde{\mathfrak{g}}\cong\operatorname{gr}_F\mathfrak{g}=\dot{\mathfrak{g}}$, $\tilde{\mathfrak{g}}/(\varepsilon-1)\tilde{\mathfrak{g}}\cong\mathfrak{g}$.

The algebra $\tilde{\mathfrak{g}}$ inherits a triangular decomposition where $\tilde{\mathfrak{n}}^\pm:=\oplus_{k=0}^\infty\varepsilon^k(F^k(\mathfrak{g})\cap\mathfrak{n}^\pm)$, $\tilde{\mathfrak{h}}:=\oplus_{k=0}^\infty\varepsilon^k(F^k(\mathfrak{g})\cap\mathfrak{h})$. Define the Harish-Chandra projection $\widetilde{\mathrm{HC}}:\mathcal{U}(\tilde{\mathfrak{g}})\to\mathcal{U}(\tilde{\mathfrak{h}})$ and the Shapovalov map $\tilde{S}$. The algebras $\mathcal{U}(\tilde{\mathfrak{n}}^\pm)$ inherit the gradings by $Q^\pm$ and these gradings leads to a decomposition $\tilde{S}=\sum\tilde{S}_\nu$. In this way we obtain the Shapovalov matrices $\tilde{B}_\nu$ and the determinants $\det\tilde{B}_\nu=(\operatorname{Norm}\tilde{S}_\nu)^{2^{\dim\mathfrak{h}_{\bar 1}}}$.

The evaluation of $\tilde{B}_\nu$ at $\varepsilon=0$ (resp., at $\varepsilon=1$) is the Shapovalov matrix $\dot{B}_\nu$ for $\dot{\mathfrak{g}}$ (resp., B_ν for $\mathfrak{g}$).

8.3.2. For each $h\in\mathfrak{h}_{\bar 0}$ one has $\tilde{h}:=\varepsilon^2h\in\tilde{\mathfrak{h}}$; the reduction modulo ε identifies $\tilde{h}$ with $\dot{h}\in\dot{\mathfrak{h}}$ and the reduction modulo $\varepsilon-1$ identifies $\tilde{h}$ with h. Observe that the entries of Shapovalov matrices are polynomials homogeneous in ε. As a consequence, the leading term of B_ν is equal to $\det\dot{B}_\nu$ providing that the latter is non-zero via the obvious identification $h\mapsto\dot{h}$ of $\mathfrak{h}_{\bar 0}$ with $\dot{\mathfrak{h}}_{\bar 0}$.

For instance, for $\mathfrak{g}=\mathfrak{sl}(2)$, one has $\tilde{B}_{2\alpha}=2(\varepsilon^2h)^2-2\varepsilon^4h=2\tilde{h}^2-2\varepsilon^2\tilde{h}$; the identification $\tilde{\mathfrak{g}}/\varepsilon\tilde{\mathfrak{g}}$ with $\mathfrak{g}$ gives $B_{2\alpha}=2h^2-2h$ and the identification $\tilde{\mathfrak{g}}/(\varepsilon-1)\tilde{\mathfrak{g}}$ with $\dot{\mathfrak{g}}$ gives $\dot{B}_{2\alpha}=2\dot{h}^2$.

8.4. **The algebra $\mathfrak{g}^\sqcap$.** For each α in Δ^+ let $\mathfrak{s}^{(\alpha)}$ be a subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{\pm\alpha}$ and let $\dot{\mathfrak{s}}^{(\alpha)}$ be its image in $\dot{\mathfrak{g}}$. Clearly, $\mathfrak{s}^{(\alpha)}\cong\mathfrak{sq}(2)$.

Consider a Lie algebra $\mathfrak{g}^\sqcap:=\prod\limits_{\alpha\in\Delta^+}\dot{\mathfrak{s}}^{(\alpha)}$ (the direct product of Lie superalgebras). Set $\mathfrak{h}^\sqcap:=\prod_{\alpha\in\Delta^+}\dot{\mathfrak{h}}^{(\alpha)}$ where $\dot{\mathfrak{h}}^{(\alpha)}\subset\dot{\mathfrak{s}}^{(\alpha)}$ is the image of $\mathfrak{h}\cap\mathfrak{s}^{(\alpha)}$.

8.4.1. Consider the triangular-type decomposition $\mathfrak{g}^\sqcap=(\mathfrak{n}^\sqcap)^-+\mathfrak{h}^\sqcap+(\mathfrak{n}^\sqcap)^+$ where $(\mathfrak{n}^\sqcap)^\pm:=\prod_{\alpha\in\Delta^+}\mathfrak{g}_{\pm\alpha}$. Let $S^\sqcap$ be the Shapovalov map; write $S^\sqcap=\sum_{\nu\in Q^+}S^\sqcap_\nu$ as above. In 8.5 we will prove the following formula

$$\operatorname{Norm}S^\sqcap_\nu=\prod_{\alpha\in\Delta^+}\dot{h}_\alpha^{\sum_{m=1}^\infty\tau(\nu-m\alpha)}\dot{h}_{\overline{\alpha}}^{\tau_\alpha(\nu-\alpha)}\tag{9}$$

where $\dot{h}_\alpha,\dot{h}_{\overline{\alpha}}$ stand for the images of these elements in $\dot{\mathfrak{s}}^{(\alpha)}\subset\mathfrak{g}^\sqcap$.

8.4.2. Recall that $\mathfrak{s}^{(\alpha)} \cong \mathfrak{sq}(2)$ has a $(2|1)$-dimensional Cartan algebra with a basis $h_\alpha, h_{\overline{\alpha}}, H_\alpha$. Observe that $(\mathfrak{n}^{\sqcap})^{\pm} \cong \dot{\mathfrak{n}}^{\pm}$ since they are commutative Lie superalgebras. This implies $\dot{\mathfrak{g}} \cong \mathfrak{g}^{\sqcap}/I$ where

$$I := \mathrm{span}\{\dot{h}_{\alpha+\beta} - \dot{h}_\alpha - \dot{h}_\beta,\ \dot{h}_{\overline{\alpha+\beta}} - \dot{h}_{\overline{\alpha}} - \dot{h}_{\overline{\beta}},\ \dot{H}_{\alpha+\beta} - \dot{H}_\alpha - \dot{H}_\beta |\ \alpha, \beta, \alpha+\beta \in \Delta^+\}.$$

Now using Theorem 5.2, we obtain Claim 8.1.2 from the formula (9).

8.5. **Proof of (9).** Define the equivalence relation on $\mathcal{P}(\nu)$ by setting

$$\mathbf{m} \approx \mathbf{m}' \iff \forall \alpha \in \Delta^+ \ \ m_\alpha + m_{\overline{\alpha}} = m'_\alpha + m'_{\overline{\alpha}}.$$

Denote by $\overline{\mathcal{P}}(\nu)$ the set of of equivalence classes in $\mathcal{P}(\nu)$. For $\mathbf{k} \in \overline{\mathcal{P}}(\nu)$ set $k_\alpha = m_\alpha + m_{\overline{\alpha}}$ where $\mathbf{m} \in \mathcal{P}(\nu)$ belongs to the class $\mathbf{k}$. For $\mathbf{m} \in \mathcal{P}(\nu)$ set

$$\mathrm{supp}\, \mathbf{m} := \{\alpha \in \Delta^+ :\ m_\alpha + m_{\overline{\alpha}} \neq 0\}.$$

Then supp is well-defined for $\mathbf{k} \in \overline{\mathcal{P}}(\nu)$ and $\mathrm{supp}\, \mathbf{k} := \{\alpha \in \Delta^+ :\ k_\alpha \neq 0\}$.

8.5.1. Recall that $\mathfrak{g}^{\sqcap} := \prod_{\alpha\in\Delta^+} \dot{\mathfrak{s}}^{(\alpha)}$. Let $S^{(\alpha)} : \mathcal{M}^{(\alpha)} \to (\mathcal{M}^{\#})^{(\alpha)}$ be the Shapovalov map for $\dot{\mathfrak{s}}^{(\alpha)}$. It is easy to see that $\mathcal{M}^{\sqcap} = \otimes_{\alpha\in\Delta^+} \mathcal{M}^{(\alpha)}$, $(\mathcal{M}^{\#})^{\sqcap} = \otimes_{\alpha\in\Delta^+} (\mathcal{M}^{\#})^{(\alpha)}$ and $S^{\sqcap} = \otimes_{\alpha\in\Delta^+} S^{(\alpha)}$. Choose the integral $\int$ on $\mathfrak{g}^{\sqcap}$ to be $\otimes_{\alpha\in\Delta^+} \int^{(\alpha)}$ where $\int^{(\alpha)}$ is the integral for $\dot{\mathfrak{s}}^{(\alpha)}$. Recall that B is the composition of S and a map induced by $\int$. We obtain $B^{\sqcap} = \otimes_{\alpha\in\Delta^+} B^{(\alpha)}$ where $B^{\sqcap}$ (resp., $B^{(\alpha)}$) is the map B for $\mathfrak{g}^{\sqcap}$ (resp., for $\dot{\mathfrak{s}}^{(\alpha)}$).

Recall that $\mathcal{M}^{\sqcap}$ (resp., $\mathcal{M}^{(\alpha)}$) is a free module over a polynomial algebra $\mathcal{U}(\mathfrak{h}^{\sqcap}_{\overline{0}})$ (resp., over $\mathcal{U}(\dot{\mathfrak{h}}^{(\alpha)})$). If L is a free submodule of $\mathcal{M}^{\sqcap}$ (resp., of $\mathcal{M}^{(\alpha)}$) we will use the notation "$\mathrm{rank}\, L$" for the rank over this polynomial algebra.

8.5.2. One has

$$\mathcal{M}^{\sqcap}_{-\nu} = \oplus_{\mathbf{k}\in\overline{\mathcal{P}}(\nu)} \mathcal{M}^{\sqcap}_{\mathbf{k}}, \ \text{ where } \ \mathcal{M}^{\sqcap}_{\mathbf{k}} := \otimes_{\alpha\in\Delta^+} \mathcal{M}^{(\alpha)}_{-k_\alpha\alpha}$$

with a similar formula for $(\mathcal{M}^{\#})^{\sqcap}_{-\nu}$. Let $B^{\sqcap}_{\mathbf{k}}$ be the restriction of $B^{\sqcap}$ to $\mathcal{M}^{\sqcap}_{\mathbf{k}}$. One has

$$B^{\sqcap}_{\mathbf{k}} = \otimes_{\alpha\in\Delta^+} B^{(\alpha)}_{-k_\alpha\alpha} \tag{10}$$

where $B^{(\alpha)}_{r\alpha}$ is the restriction of $B^{(\alpha)}$ to $\mathcal{M}^{(\alpha)}_{-r\alpha}$. Since $B^{(\alpha)}$ maps $\mathcal{M}^{(\alpha)}_{-r\alpha}$ to $(\mathcal{M}^{\#})^{(\alpha)}_{-r\alpha}$, $B^{\sqcap}_{\mathbf{k}}$ maps $\mathcal{M}^{\sqcap}_{\mathbf{k}}$ to $\mathcal{M}^{\sqcap\#}_{\mathbf{k}}$. As a consequence,

$$\det B^{\sqcap}_{\nu} = \prod_{\mathbf{k}\in\overline{\mathcal{P}}(\nu)} \det B^{\sqcap}_{\mathbf{k}}. \tag{11}$$

8.5.3. Fix $\mathbf{k}\in\overline{\mathcal{P}}(\nu)$ and let us compute $\det B^{\sqcap}_{\mathbf{k}}$ using the decomposition (10). Recall that for $\psi_i\in\mathrm{End}(V_i)$ one has $\det(\otimes\psi_i)=\prod(\det\psi_i)^{n/n_i}$ where $n_i:=\dim V_i, n:=\prod n_i=\dim\otimes V_i$. The module $\mathcal{M}^{(\alpha)}_{-r\alpha}$ has rank 2 for $r=0$ and rank 4 for $r>0$ (see 7.2). Therefore

$$\begin{array}{rl}\det B^{\sqcap}_{\mathbf{k}} & =\prod_{\alpha\in\operatorname{supp}\mathbf{k}}(\det B^{(\alpha)}_{k_\alpha\alpha})^{r(\mathbf{k})/4},\\ & \text{where } r(\mathbf{k}):=\operatorname{rank}\mathcal{M}^{\sqcap}_{\mathbf{k}}.\end{array}$$

Recall that $\dot{\mathfrak{s}}^{(\alpha)}=\mathrm{gr}_F\,\mathfrak{sq}(2)$. By 8.3, the entries of $B^{(\alpha)}_{(k+1)\alpha}$ are the leading terms of the entries of Shapovalov matrix $B_{k\alpha}$ for $\mathfrak{sq}(2)$. Shapovalov matrices $B_{k\alpha}$ were computed in 7.2; using the formulas there we get

$$\det B^{(\alpha)}_0=1,\quad \det B^{(\alpha)}_{k_\alpha\alpha}=(k_\alpha-1)!^2\dot{h}^2_{\overline{\alpha}}\dot{h}^{4k_\alpha-2}_{\alpha}\ \text{for } k_\alpha>0.$$

Substituting our formulas in (11) we obtain, up to a non-zero scalar,

$$\begin{array}{rl}\det B^{\sqcap}_{\nu} & =\prod\limits_{\mathbf{k}\in\overline{\mathcal{P}}(\nu)}\prod\limits_{\alpha\in\Delta^+}\dot{h}^{d(\overline{\alpha})}_{\overline{\alpha}}\dot{h}^{d(\alpha)}_{\alpha},\\ & \text{where } d(\overline{\alpha}):=\frac{1}{2}\sum_{\mathbf{k}\in\overline{\mathcal{P}}(\nu):\alpha\in\operatorname{supp}\mathbf{k}}\operatorname{rank}\mathcal{M}^{\sqcap}_{\mathbf{k}},\\ & \quad d(\alpha):=\sum_{\mathbf{k}\in\overline{\mathcal{P}}(\nu)::\alpha\in\operatorname{supp}\mathbf{k}}\frac{4k_\alpha-2}{4}\operatorname{rank}\mathcal{M}^{\sqcap}_{\mathbf{k}}.\end{array}\tag{12}$$

We will simplify the expressions for $d(\overline{\alpha}),d(\alpha)$ in 8.6 below.

8.6. **The multiplicities** $d(\alpha),d(\overline{\alpha})$**.**

8.6.1. For a partition $\mathbf{m}\in\mathcal{P}(\nu)$ view $\mathbf{f}^{\mathbf{m}}:=\prod f_\alpha^{m_\alpha}F_\alpha^{m_{\overline{\alpha}}}$ as an element of $\mathfrak{g}^{\sqcap}$. Identify $\mathcal{M}^{\sqcap}$ with $\mathcal{U}(\mathfrak{h}^{\sqcap}+(\mathfrak{n}^{\sqcap})^-)$ and for each $\mathbf{m}\in\mathcal{P}(\nu)$ denote by $\mathcal{M}^{\sqcap}_{\mathbf{m}}$ the space $\mathcal{U}(\mathfrak{h}^{\sqcap})\mathbf{f}^{\mathbf{m}}\subset\mathcal{M}^{\sqcap}$. Recall that $\overline{\mathcal{P}}(\nu)=\mathcal{P}(\nu)/\approx$. For any $\mathbf{k}\in\overline{\mathcal{P}}(\nu)$ one has

$$\mathcal{M}^{\sqcap}_{\mathbf{k}}=\bigoplus_{\mathbf{m}\in\mathcal{P}(\nu),\ \mathbf{m}\in\mathbf{k}}\mathcal{M}^{\sqcap}_{\mathbf{m}}.$$

Recall that $\mathcal{U}(\mathfrak{h}^{\sqcap})$ is a Clifford algebra over the polynomial algebra $\mathcal{U}(\mathfrak{h}^{\sqcap})_{\overline{0}})$ and so $\mathcal{M}^{\sqcap}_{\mathbf{m}}$ is a free module of rank 2^N over this algebra where $N:=\dim\mathfrak{h}^{\sqcap}_{\overline{1}}=|\Delta^+|$.

8.6.2. Let us simplify the expression $d(\overline{\alpha})=\frac{1}{2}\operatorname{rank}\sum_{\mathbf{k}\in\overline{\mathcal{P}}(\nu):\alpha\in\operatorname{supp}\mathbf{k}}\mathcal{M}^{\sqcap}_{\mathbf{k}}$ obtained in (12). One has

$$d(\overline{\alpha})=2^{N-1}|\{\mathbf{m}\in\mathcal{P}(\nu): m_\alpha+m_{\overline{\alpha}}\neq 0\}|$$

because $\sum_{\mathbf{k}\in\overline{\mathcal{P}}(\nu):\alpha\in\operatorname{supp}\mathbf{k}}\mathcal{M}^{\sqcap}_{\mathbf{k}}=\sum_{\mathbf{m}\in\mathcal{P}(\nu):\alpha\in\operatorname{supp}\mathbf{m}}\mathcal{M}^{\sqcap}_{\mathbf{m}}$. Observe that

$$|\{\mathbf{m}\in\mathcal{P}(\nu): m_\alpha+m_{\overline{\alpha}}\neq 0\}|=2|\{\mathbf{m}\in\mathcal{P}(\nu): m_{\overline{\alpha}}\neq 0\}|=2\sum_{\mathbf{m}\in\mathcal{P}(\nu)}m_{\overline{\alpha}}.$$

Using Lemma 8.6.5 (ii) we obtain

$$d(\overline{\alpha})=2^N\sum_{\mathbf{m}\in\mathcal{P}(\nu)}m_{\overline{\alpha}}=2^N\tau_\alpha(\nu-\alpha).\tag{13}$$

8.6.3. Let us simplify the term $d(\alpha)$. One can rewrite (12) as

$$d(\alpha)+d(\overline{\alpha})=\sum_{\mathbf{k}\in\overline{\mathcal{P}}(\nu)} k_\alpha \operatorname{rank}\mathcal{M}^{\sqcap}_{\mathbf{k}}=\sum_{\mathbf{m}\in\mathcal{P}(\nu)}(m_\alpha+m_{\overline{\alpha}})\operatorname{rank}\mathcal{M}^{\sqcap}_{\mathbf{m}}=2^N\sum_{\mathbf{m}\in\mathcal{P}(\nu)}(m_\alpha+m_{\overline{\alpha}}).$$

Using Lemma 8.6.5 (i) we get

$$d(\alpha)=2^N\sum_{r=1}^{\infty}\tau(\nu-r\alpha). \tag{14}$$

8.6.4. Finally, recalling that $N=\dim\mathfrak{h}^{\sqcap}_{\overline{1}}$ and substituting (13),14) into (12) we obtain the formula (9).

8.6.5. Retain notation of 8.1.

Lemma. *For all $\alpha\in\Delta^+$ one has*

(i) $$\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_\alpha=\sum_{m=1}^{\infty}\tau(\nu-m\alpha);$$

(ii) $$\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_{\overline{\alpha}}=\tau(\nu)-\tau_\alpha(\nu)=\tau_\alpha(\nu-\alpha)$$

Proof. For $\nu\not\geq\alpha$ the assertions obviously hold since both sides of both equations are equal to zero. Fix $\nu\geq\alpha$ and assume that (i) holds for all $\mu<\nu$. The map $\mathbf{k}\mapsto(\mathbf{k}-\alpha)$ gives a bijection from the set $\{\mathbf{k}\in\mathcal{P}(\nu)|\ k_\alpha\neq 0\}$ onto $\mathcal{P}(\nu-\alpha)$. Therefore

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_\alpha &= \sum_{\mathbf{k}\in\mathcal{P}(\nu-\alpha)}(k_\alpha+1)=\sum_{m=1}^{\infty}\tau(\nu-\alpha-m\alpha)+|\mathcal{P}(\nu-\alpha)|\\ &=\sum_{m=2}^{\infty}\tau(\nu-m\alpha)+\tau(\nu-\alpha)=\sum_{m=1}^{\infty}\tau(\nu-m\alpha)\end{aligned}$$

and (i) follows. The map $\mathbf{k}\mapsto(\mathbf{k}-\alpha)$ gives a bijection

$$\{\mathbf{k}\in\mathcal{P}(\nu)|\ k_{\overline{\alpha}}=1\}\xrightarrow{\sim}\{\mathbf{k}\in\mathcal{P}(\nu-\alpha)|\ k_{\overline{\alpha}}=0\}.$$

This gives (ii). □

9. Jantzen filtration

The Jantzen filtration and sum formula were described by Jantzen in [Ja] for a Verma module over a semisimple Lie algebra. In this section we adapt this construction for Q-type superalgebras.

Throughout this section we assume that all Shapovalov determinants are non-zero polynomials that is $\det B_\nu\neq 0$ for all ν.

We define the Jantzen filtration in 9.2.

9.1. **Main properties of Jantzen filtration.** The construction of Jantzen filtration depends on a "generic vector" $\rho' \in \mathfrak{h}_{\bar{0}}^*$ satisfying the following property

(J1) the hypersurfaces $\det B_\nu = 0$ *do not contain straight lines parallel to* ρ'.

In other words, $\forall \lambda \in \mathfrak{h}_{\bar{0}}^*\ \ \exists c \in \mathbb{C}$ such that $\det B_\nu(\lambda + c\rho') \neq 0$. It is not hard to show that the condition $\det B_\nu \neq 0$ for all ν ensures the existence of ρ' satisfying (a). For semisimple Lie algebras one can take $\rho' := \rho$.

For each $\lambda \in \mathfrak{h}_{\bar{0}}^*$ the Jantzen filtration $\{F^r(M(\lambda))\}$ is a finite decreasing filtration by $\mathfrak{g}$-submodule; one has

$$F^0(M(\lambda)) = M(\lambda), \quad F^1(M(\lambda)) = \overline{M(\lambda)}.$$

9.1.1. Define the order l of zero of a polynomial $q \in \mathcal{S}(\mathfrak{h}_0)$ at $\mu \in \mathfrak{h}_{\bar{0}}^*$ as follows: $l = 0$ if $q(\mu) \neq 0$; $l = 1$ if $q(\mu) = 0$ and there exists a non-zero partial derivative $\frac{\partial q}{\partial x}(\mu) \neq 0$ and so on.

Denote by $m_\nu(\lambda, \rho')$ the order of zero of $\det B_\nu(\lambda + x\rho') \in \mathbb{C}[x]$ at $x = 0$ and by $m_\nu(\lambda)$ the order of zero of $\det B_\nu$. Let Γ be the set of irreducible components of hypersurfaces $\det B_\nu = 0$ and $d_\gamma(\nu)$ be the multiplicity of γ in $\det B_\nu = 0$ (thus $d_\gamma(\nu)$ is a non-negative integer). Then, if all $\gamma \in \Gamma$ are smooth at λ one has

$$m_\nu(\lambda) = \sum_{\gamma \in \Gamma : \lambda \in \gamma} d_\gamma(\nu).$$

Note that the formula remains valid if $\lambda \notin \gamma$ for $\gamma \in \Gamma$.

Clearly,

$$\operatorname{corank} B_\nu(\lambda) \leq m_\nu(\lambda) \leq m_\nu(\lambda, \rho').$$

Moreover

$$m_\nu(\lambda, \rho') = m_\nu(\lambda)$$

if the vector ρ' is transversal to the hypersurface $\det B_\nu = 0$ at point λ i.e., it is transversal to all irreducible components passing through λ.

9.1.2. The following property is proven in [Ja]

$$m_\nu(\lambda, \rho') = \sum_{r=1}^{\infty} \dim F^r(M(\lambda))_{\lambda - \nu}. \tag{15}$$

Assume that λ is such that

(J2) the vector ρ' is transversal to the hypersurfaces $\det B_\nu = 0$ *at point* λ.

Then the following "sum formula" holds

$$\sum_{\gamma \in \Gamma : \lambda \in \gamma} d_\gamma(\nu) = \sum_{r=1}^{\infty} \dim F^r(M(\lambda))_{\lambda - \nu}. \tag{16}$$

It is not always possible to find ρ' such that (J2) holds for all λ: for instance, for $\det B_\nu = h_1^2 + h_2^2 - 1$ there is no ρ' with this property. However, one can always choose ρ' transversal to the hypersurface $\det B_\nu = 0$ at all points $\lambda \in \mathfrak{h}_{\bar 0}^* \setminus X$ where X is a set of codimension two. Thus the sum formula holds for a generic point of each hypersurface $\det B_\nu = 0$.

9.1.3. *Remark.* Since $F^1(M(\lambda)) = \overline{M(\lambda)}$ one has $\operatorname{corank} B_\nu(\lambda) = \dim F^1(M(\lambda))_{\lambda-\nu}$. Using (15) we get

$$F^2(M(\lambda)) = 0 \iff m_\nu(\lambda, \rho) = \operatorname{corank} B_\nu(\lambda) \ \ \forall \nu \in Q^+.$$

9.1.4. Later on we show that the Shapovalov determinants for Q-type Lie superalgebras admit linear factorizations. In other words, the union of the hypersurfaces $\det B_\nu = 0$ is the union of countably many hyperplanes. We choose ρ' which is not parallel to these hyperplanes. Clearly, ρ' satisfies the condition (J2) for all λ. Hence the sum formula (16) holds for all $\lambda \in \mathfrak{h}_{\bar 0}^*$.

9.1.5. Interesting questions are whether the Jantzen filtration depends on ρ' and whether for given ρ' the filtration induces a unique filtration of $N(\lambda)$. For semisimple Lie algebras the Jantzen filtration does not depend on a "generic vector", see [BB], 5.3.1.

9.2. **A construction of Jantzen filtration.** Let x be an indeterminate, L be the local ring $\mathbb{C}[x]_{(x)}$, and F be its field of fractions. Endow L and F with the trivial $\mathbb{Z}_2$-grading: $L_{\bar 1} = F_{\bar 1} = 0$.

We shall extend the scalars of "our favorite objects" from $\mathbb{C}$ to L and to F. For a $\mathbb{C}$-vector superspace V denote by V_L the L-module $L \otimes V$ and by V_F the F-vector superspace $F \otimes V$. We identify $\mathcal{U}(\mathfrak{g}_L)$ with the superalgebra $\mathcal{U}(\mathfrak{g})_L$. Retain notation of 4.1 and define the Shapovalov map Ξ_L. Clearly, $\mathcal{M}_L = \operatorname{Ind}(R_L)$. For any $\mu \in (\mathfrak{h}_{\bar 0}^*)_L$ define a $\mathfrak{g}_L$-$\mathfrak{h}_L$ bimodules $M_L(\mu) := \operatorname{Ind}(\mathcal{C}\ell(\mu))$ and $\operatorname{Coind}(\mathcal{C}\ell(\mu))$ Observe that a Shapovalov matrix for $\mathfrak{g}_L$ written with respect to a bases lying in $\mathfrak{g}$ coincides with the Shapovalov matrix for $\mathfrak{g}$ written with respect to the same bases. Consequently, the Shapovalov determinants $\det B_\nu \in \mathcal{S}(\mathfrak{h}_0)$ viewed as elements of the algebra $\mathcal{S}(\mathfrak{h}_0)_L$ coincide with the Shapovalov determinants $\det B_\nu$ constructed for $\mathcal{U}(\mathfrak{g}_L)$.

The vector space $\mathfrak{g}_F$ admits the natural structure of a Lie superalgebra and $\mathcal{U}(\mathfrak{g}_F) := \mathcal{U}(\mathfrak{g})_F$ is its enveloping superalgebra. For any $\mu \in (\mathfrak{h}_{\bar 0}^*)_L$ the localized module $M_L(\mu) \otimes_L F$ is naturally isomorphic to the $\mathfrak{g}_F$-module $M_F(\mu)$ where μ is viewed as an element of $(\mathfrak{h}_{\bar 0}^*)_F$ via the natural embedding $(\mathfrak{h}_{\bar 0}^*)_L \hookrightarrow (\mathfrak{h}_{\bar 0}^*)_F$.

9.2.1. Fix $\lambda \in \mathfrak{h}_{\bar{0}}^*$ and set

$$M := M_L(\lambda + x\rho'), \quad M' := \mathrm{Coind}(\mathcal{C}\ell(\lambda + x\rho')), \quad \varphi := S(\lambda + x\rho').$$

Thus $\phi : M \to M'$ is the evaluation of the Shapovalov map at $\lambda + x\rho'$.

Define a decreasing $\mathbb{Z}$-filtration on M by setting $F^r(M) = M$ for $r \leq 0$ and

$$F^r(M) := \{v|\varphi(v) \in (x^r)M'\} \quad \text{for } r > 0.$$

Notice that each term $F^r(M)$ is a $\mathfrak{g}_L$-$\mathfrak{h}_L$ bisubmodule because φ is a $\mathfrak{g}_L$-$\mathfrak{h}_L$ homomorphism by 4.4. Due to the condition (J1) of 9.1 $\det B_\nu(\lambda+x\rho') \neq 0$ for all $\nu \in Q^+$. Thus $\mathrm{Ker}\,\varphi = 0$ and so

$$\bigcap_{r=0}^{\infty} F^r(M) = 0.$$

9.2.2. Observe that $M(\lambda) = M/(xM)$ and $\varphi/(x\varphi) = S(\lambda) : M(\lambda) \to \mathrm{Coind}(\mathcal{C}\ell(\lambda))$. Let $F^r(M(\lambda))$ be the image of $F^r(M)$. We get a decreasing $\mathfrak{g}$-$\mathfrak{h}$ filtration on $M(\lambda)$ with the property

$$\bigcap_{r=0}^{\infty} F^r(M(\lambda)) = 0.$$

The filtration is finite since $M(\lambda)$ has a finite length.

One has

$$\begin{aligned} F^0(M(\lambda)) &= M(\lambda), \\ F^1(M(\lambda)) &= \mathrm{Ker}\, S(\lambda) = \overline{M(\lambda)}. \end{aligned}$$

The filtration $F^r(M(\lambda))$ is an analogue of the Jantzen filtration for the module $M(\lambda)$.

9.3. **Example: The case $\mathfrak{g} = \mathfrak{sq}(2)$.** Retain notation of 7.3. In 7.3 we described $\overline{N(\lambda)}$ and $\overline{M(\lambda)} = F^1(M(\lambda))$. Below we describe the Jantzen filtration on $M(\lambda)$.

For $\lambda = (x, a)$ where $x \notin \mathbb{Z}_{>0}$ and $a \neq 0$ the module $M(\lambda) = N(\lambda)$ is simple.

For $\lambda = (m, a)$ where $m \in \mathbb{Z}_{>0}$ and $a \neq 0$ the module $M(\lambda) = N(\lambda)$ has length two and its Jantzen filtration has length two: $F^2(M(\lambda)) = 0$. More precisely, $M(\lambda)$ has a unique non-trivial submodule $N(s_\alpha\lambda) = M(s_\alpha\lambda) = F^1(M(\lambda))$.

For $\lambda = (x, 0)$ where $x \notin \mathbb{Z}_{>0}$ one has $F^2(M(\lambda)) = 0$. The module $\overline{F^1(M(\lambda))}$ has a basis $f^iFH, f^i(Fh - fH)$ with $i \geq 0$ and so $F^1(M(\lambda)) \cong M(\lambda - \alpha)/\overline{M(\lambda - \alpha)}$. The module $N(\lambda)$ has length two: its unique non-trivial submodule is $\overline{N(\lambda)} \cong V(\lambda - \alpha)$. The module $M(\lambda)$ has length four because it admits a submodule isomorphic to $N(\lambda)$ with the quotient isomorphic to $\Pi(N(\lambda))$.

For $\lambda = (m, 0)$ where $m \in \mathbb{Z}_{>0}$ one has $F^3(M(\lambda)) = 0$. The term $F^1(M(\lambda))$ contains the weight spaces $M(\lambda)_\mu$ for $\mu < s_\alpha\lambda$ and the vectors $f^iFH, f^i(Fh - fH)$. The term $F^2(M(\lambda))$ is spanned by $f^{m+k}FH$ $(k \geq 0)$ and thus $F^2(M(\lambda)) \cong V(s_\alpha\lambda)$.

9.4. **The sum formulas.** The formula(15) is an immediate consequence of the following fact proven in [Ja]. Let L be the local ring $\mathbb{C}[x]_{(x)}$ and let N, N' be free L-modules of a finite rank r. Let $\varphi : N \to N'$ be an injective linear map. Define a decreasing filtration on N by setting

$$F^j(N) := \{v \in N |\ \varphi(v) \in (x^j)N'\}.$$

The claim is that the sum $\sum_{j=1}^{\infty} \dim\big(F^j(N)/(F^j(N) \cap xN)\big)$ is equal to the order of zero of $\det D$ at the origin where $D \in \mathrm{Mat}_r(L)$ is a matrix corresponding to φ. Observe that for different choice of free bases in N, N' the determinants of corresponding matrices differ by the multiplication on an invertible scalar and so have equal orders of zero at the origin. It is easy to see that we can choose free bases $v_1, \dots, v_r$ in N and $v'_1, \dots, v'_r$ in N' in such a way that $\varphi(v_i) = t^{s_i} v'_i$ for some $s_i \in \mathbb{Z}_{\geq 0}$. Let D be the matrix of φ with respect to these bases. The order of zero of $\det D$ at the origin is $\sum_{i=1}^{r} s_i$ and $\dim F^j(N)/(F^j(N) \cap xN) = |\{i : s_i \geq j\}|$. Since $\sum_i s_i = \sum_{j=1}^{\infty} |\{i : s_i \geq j\}|$, the claim results.

10. The anticentre

For a finite dimensional Lie superalgebra $\mathfrak{p}$ the anticentre $\mathcal{A}(\mathfrak{p})$ can be defined as the set of invariants of $\mathcal{U}(\mathfrak{p})$ with respect to a twisted adjoint action: $\mathcal{A}(\mathfrak{p}) := \mathcal{U}(\mathfrak{p})^{\mathrm{ad}' \mathfrak{p}}$ where ad' is given by the formula

$$(\mathrm{ad}'\, g)u = gu - (-1)^{p(g)(p(u)+1)} ug.$$

We see that the odd elements of the anticentre $\mathcal{A}(\mathfrak{p})$ commute with all elements of $\mathcal{U}(\mathfrak{p})$ and the even elements of $\mathcal{A}(\mathfrak{p})$ commute with the even elements of $\mathcal{U}(\mathfrak{p})$ and anticommute with the odd ones. Clearly, the product of two anticentral elements is central.

In this section we describe $\mathcal{A}(\mathfrak{g})$, see Theorem 10.4. This provides us a bunch of central elements (see Corollary 10.4.2). As it was indicated in 1.3.1, the central elements are useful for the proof of linear factorizability of Shapovalov determinants, see Sect. 11 below.

10.1. **Schur's lemma.** Recall that Schur's lemma for Lie superalgebras takes the following form: for a simple $\mathfrak{p}$-module $V = V_{\bar{0}} \oplus V_{\bar{1}}$ either $\mathrm{End}(V)^{\mathrm{ad}\,\mathfrak{p}} = \mathbb{C}\,\mathrm{id}$ or $\mathrm{End}(V)^{\mathrm{ad}\,\mathfrak{p}} = \mathbb{C}\,\mathrm{id} \oplus \mathbb{C}\upsilon$ where υ is odd and satisfies $\upsilon^2 = \mathrm{id}$.

Let $\theta : V \to V$ be the map $v \mapsto (-1)^{p(v)} v$. It is easy to see that the action of z on a simple $\mathfrak{p}$-module V is proportional to

$$\begin{array}{ll} \mathrm{id}, & \text{if } z \in \mathcal{Z}(\mathfrak{p}) \text{ and } z \text{ is even},\\ 0, & \text{if } z \in \mathcal{Z}(\mathfrak{p}) \text{ and } z \text{ is odd},\\ \theta, & \text{if } z \in \mathcal{A}(\mathfrak{p}) \text{ and } z \text{ is even},\\ \upsilon\theta, & \text{if } z \in \mathcal{A}(\mathfrak{p}) \text{ and } z \text{ is odd}. \end{array} \tag{17}$$

10.1.1. For $\mathfrak{p}$ being a basic classical or Q-type Lie superalgebra the formula (17) holds for Weyl modules as well. This follows from the fact that the action of z on a Weyl module is determined by its action on the highest weight space which is a simple $\mathfrak{h}$-module.

10.2. Assume that $\mathfrak{p}$ satisfies the following condition

$$\bigwedge^{\mathrm{top}}\mathfrak{p}_{\bar{1}} \text{ is a trivial } \mathfrak{p}_{\bar{0}}\text{-module.} \qquad (*)$$

Then the anticentre $\mathcal{A}(\mathfrak{p})$ admits the following description (see [G1], Sect.3).

Let $\mathbb{C}v$ be the even trivial $\mathfrak{p}_{\bar{0}}$-module. The induced module $\mathrm{Ind}_{\mathfrak{p}_{\bar{0}}}^{\mathfrak{p}}\mathbb{C}v$ contains a unique trivial $\mathfrak{p}$-submodule; let $u_0\in\mathcal{U}(\mathfrak{p})$ be such that u_0v generates this submodule. The map

$$\vartheta: z\mapsto (\mathrm{ad}'\,u_0)(z)$$

is a linear isomorphism from $\mathcal{Z}(\mathfrak{p}_{\bar{0}})$ to $\mathcal{A}(\mathfrak{p})$. The map ϑ is defined up to a multiplicative scalar: if u_0,u_0' are such that $\mathfrak{p}(u_0v)=\mathfrak{p}(u_0'v)=0$ and ϑ,ϑ' are the corresponding isomorphisms then $\vartheta=c\vartheta'$ for some $c\in\mathbb{C}^*$. One has $\mathrm{gr}(\vartheta(z))\in\bigwedge^{\mathrm{top}}\mathfrak{p}_{\bar{1}}\mathrm{gr}\,z$. In particular, the anticentre is pure even if $\dim\mathfrak{p}_{\bar{1}}$ is even and is pure odd otherwise.

10.2.1. Define a filtration $\mathcal{F}_{1/2}$ of $\mathcal{U}(\mathfrak{p})$ by letting the odd elements of $\mathfrak{p}$ have degree $1/2$ and the even elements of $\mathfrak{p}$ have degree 1. Denote by $\deg_{1/2}u$ the degree of $u\in\mathcal{U}(\mathfrak{p})$ with respect to $\mathcal{F}_{1/2}$. Observe that $\deg_{1/2}u=\deg u$ for $u\in\mathcal{U}(\mathfrak{p}_{\bar{0}})$.

We claim that

$$\deg_{1/2}\vartheta(z)\leq\frac{\dim\mathfrak{p}_{\bar{1}}}{2}+\deg z. \qquad (18)$$

Indeed, let X be a basis of $\mathfrak{p}_{\bar{1}}$. The module $\mathrm{Ind}_{\mathfrak{p}_{\bar{0}}}^{\mathfrak{p}}\mathbb{C}v$ has a basis of the form $\{u_kv\}$ $(k=1,\dots,2^{\dim\mathfrak{p}_{\bar{1}}})$ where each u_k is a product of distinct elements of X and so $\deg_{1/2}u_k\leq\frac{\dim\mathfrak{p}_{\bar{1}}}{2}$. One can choose u_0 to be a linear combination of u_k. Then $\deg_{1/2}u_0\leq\frac{\dim\mathfrak{p}_{\bar{1}}}{2}$ and hence (18).

10.2.2. *Element* $T_{\mathfrak{p}}$. By 10.2, $\mathcal{A}(\mathfrak{p})$ contains a unique (up to a scalar) element $T_{\mathfrak{p}}$ such that $\mathrm{gr}\,T_{\mathfrak{p}}\in\bigwedge^{\mathrm{top}}\mathfrak{p}_{\bar{1}}$. One has $T_{\mathfrak{p}}=(\mathrm{ad}'\,u_0)(1)$ and so $\deg_{1/2}T_{\mathfrak{p}}\leq\frac{\dim\mathfrak{p}_{\bar{1}}}{2}$.

10.3. Let $\mathfrak{g}$ be a Q-type Lie superalgebra. The algebras $\mathfrak{p}=\mathfrak{g},\mathfrak{h}$ satisfy the condition (*). In A.2 we show that $\mathcal{A}(\mathfrak{h})=\mathcal{S}(\mathfrak{h}_0)T_{\mathfrak{h}}$ where $T_{\mathfrak{h}}$ is given by the formula (35). We describe $T_{\mathfrak{g}}$ and $\mathcal{A}(\mathfrak{g})$ in Theorem 10.4 below.

10.3.1. The following proposition is proven in [LM] (Cor. D):

Proposition. *For any Zariski dense subset* Ω *of* $\mathfrak{h}_{\bar{0}}^*$ *one has*

$$\bigcap_{\lambda\in\Omega}\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}N(\lambda)=0.$$

The proof in [LM] is based on the similar assertion for semisimple Lie algebras. The assertion for semisimple Lie algebras can be deduced (see, for instance, [J], 7.1.9) from the fact that the determinants of all Shapovalov forms are not equal to zero. The same reasoning as in [J], 7.1.9 works in our case: Proposition 10.3.1 can be easily deduced from the inequalities $\det B_\nu \neq 0$ (for all $\nu \in Q^+$) obtained in 8.1.2.

10.3.2. **Lemma.**

(i) *The restriction of* HC *to* $\mathcal{Z}(\mathfrak{g})$ *is injective and its image lies in* $\mathcal{S}(\mathfrak{h}_0)^W$. *In particular,* $\mathcal{Z}(\mathfrak{g})$ *is pure even.*
(ii) *The restriction of* HC *to* $\mathcal{A}(\mathfrak{g})$ *is injective and its image lies in* $\mathcal{A}(\mathfrak{h})$.

Proof. Obviously, $\mathrm{HC}(\mathcal{Z}(\mathfrak{g})) \subset \mathcal{Z}(\mathcal{U}(\mathfrak{h})), \mathrm{HC}(\mathcal{A}(\mathfrak{g})) \subset \mathcal{A}(\mathfrak{h})$. It is easy to see that $\mathcal{Z}(\mathcal{U}(\mathfrak{h})) = \mathcal{S}(\mathfrak{h}_0)$. Observe that $z \in \mathcal{Z}(\mathfrak{g}) \cup \mathcal{A}(\mathfrak{g})$ kills a Weyl module $N(\lambda)$ iff z kills its highest weight space $N(\lambda)_\lambda$. For $v \in N(\lambda)_\lambda$ one has $zv = \mathrm{HC}(z)v$. Combining the above observation with Proposition 10.3.1 we get $\mathrm{HC}(z) \neq 0$ for $z \neq 0$. Hence (ii). The inclusion $\mathrm{HC}(\mathcal{Z}(\mathfrak{g})) \subset \mathcal{S}(\mathfrak{h}_0)^W$ follows from Corollary 7.4 (i) and the fact that $zv = \mathrm{HC}(z)(\lambda)v$ for v being a primitive vector of weight λ. □

10.4. Recall that $T_\mathfrak{g}$ is defined up to an invertible scalar.

Theorem.

(i) *We can choose* $T_\mathfrak{g}$ *such that*

$$\mathrm{HC}(T_\mathfrak{g}) = T_\mathfrak{h} \prod_{\alpha \in \Delta^+_{\overline{0}}} h_{\overline{\alpha}}.$$

(ii) *The restriction of* HC *to* $\mathcal{A}(\mathfrak{g})$ *is a linear isomorphism* $\mathcal{A}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h}_0)^W \,\mathrm{HC}(T_\mathfrak{g})$.

Proof. Let $\vartheta : \mathcal{Z}(\mathfrak{g}_{\overline{0}}) \xrightarrow{\sim} \mathcal{A}(\mathfrak{g})$ be the map described in 10.2. Recall that $\mathrm{HC}(\mathcal{A}(\mathfrak{g})) \subset \mathcal{A}(\mathfrak{h}) = \mathcal{S}(\mathfrak{h}_0)T_\mathfrak{h}$. This gives an injective linear map $\mathcal{Z}(\mathfrak{g}_{\overline{0}}) \to \mathcal{S}(\mathfrak{h}_0)$

$$z \mapsto u_z \quad \text{such that } \mathrm{HC}(\vartheta(z)) = T_\mathfrak{h} u_z.$$

Retain notation of 3.3.1. In A.2.2 we obtain the formula

$$t_\mathfrak{h} := T_\mathfrak{h}^2 = \begin{cases} \pm h_1 \dots h_n & \text{for } \mathfrak{g} = \mathfrak{q}(n), \mathfrak{pq}(n) \\ \pm \sum h_1 \dots \hat{h}_i \dots h_n & \text{for } \mathfrak{g} = \mathfrak{sq}(n), \mathfrak{psq}(n). \end{cases} \tag{19}$$

Let us estimate $\deg u_z$. Observe that $\deg_{1/2} u_z = \deg u_z$ because $u_z \in \mathcal{S}(\mathfrak{h}_0)$. Taking into account that $\mathrm{gr}_{1/2}\mathcal{U}(\mathfrak{h}) \cong \mathcal{U}(\mathfrak{h})$ and that $u_z \in \mathcal{S}(\mathfrak{h}_0)$ is a non-zero divisor in $\mathcal{U}(\mathfrak{h})$, we conclude that $\mathrm{gr}_{1/2} u_z$ is a non-zero divisor in $\mathrm{gr}_{1/2}\mathcal{U}(\mathfrak{h})$ and so $\deg_{1/2}(u_z T_\mathfrak{h}) = \deg_{1/2} u_z + \deg_{1/2} T_\mathfrak{h}$. By (18), $\deg_{1/2} T_\mathfrak{h} \leq \frac{\dim \mathfrak{h}_{\overline{1}}}{2}$. Since $t_\mathfrak{h} = T_\mathfrak{h}^2 \in \mathcal{S}(\mathfrak{h}_0)$ and $\deg t_\mathfrak{h} = \dim \mathfrak{h}_{\overline{1}}$ one has $\deg_{1/2} t_{\mathfrak{h}} = \dim \mathfrak{h}_{\overline{1}}$. Therefore $\deg_{1/2} T_{\mathfrak{h}} = \frac{\dim \mathfrak{h}_{\overline{1}}}{2}$. Using (18) we obtain the following estimation

$$\deg u_z \leq \frac{\dim \mathfrak{g}_{\overline{1}}}{2} + \deg z - \deg_{1/2} T_{\mathfrak{h}} = \dim \mathfrak{n}^+_{\overline{1}} + \deg z. \tag{20}$$

We claim that u_z is divisible by $h_{\overline{\alpha}}$ for any root $\alpha \in \Delta^+_{\overline{0}}$. Since $t_{\mathfrak{h}}$ is not divisible by $h_{\overline{\alpha}}$, it suffices to show that $p := \mathrm{HC}(\vartheta(z))^2 = t_{\mathfrak{h}} u_z^2$ is divisible by $h_{\overline{\alpha}}$ for any root $\alpha \in \Delta^+_{\overline{0}}$. Since $\vartheta(z)^2 \in \mathcal{Z}(\mathfrak{g})$ Corollary 10.3.2 gives $p \in \mathcal{S}(\mathfrak{h}_0)^W$. As a consequence, it is enough to verify that p is divisible by $h_{\overline{\alpha}}$ for any $\alpha \in \pi$. Fix $\alpha \in \pi$. In the notation of 3.3.1 one has $h_{\overline{\alpha}} = h_i + h_{i+1}$ for some i. Take $\lambda \in \mathfrak{h}^*_{\overline{0}}$ such that $h_i(\lambda) = h_{i+1}(\lambda) = 0$. Observe that $t_{\mathfrak{h}}(\lambda) = 0$ and so $p(\lambda) = 0$. For any $c \in \mathbb{C}$ one has $h_{\overline{\alpha}}(\lambda - c\alpha) = 0$ and therefore, by Corollary 7.4 (ii), $N(\lambda - c\alpha)$ has a primitive vector of the weight $\lambda - (c+1)\alpha$. This gives $p(\lambda - c\alpha) = p(\lambda - (c+1)\alpha)$ because $\vartheta(z)^2$ is central. Now $p(\lambda) = 0$ forces $p(\lambda - c\alpha) = 0$ for all $c \in \mathbb{C}$. Any $\lambda' \in \mathfrak{h}^*_{\overline{0}}$ satisfying $h_{\overline{\alpha}}(\lambda') = 0$ takes form $\lambda' = \lambda'' - c\alpha$ where $c = -\lambda'(h_i)$ and λ'' is such that $h_i(\lambda) = h_{i+1}(\lambda) = 0$. Thus $p(\lambda') = 0$ for all λ' satisfying $h_{\overline{\alpha}}(\lambda') = 0$. This means that p is divisible by $h_{\overline{\alpha}}$ and the claim follows.

Hence $u_z = \prod_{\alpha \in \Delta^+_{\overline{0}}} h_{\overline{\alpha}} u'_z$. By (20), $\deg u'_z \leq \deg z$. In particular, for $z \in \mathbb{C}$ one has $\deg u'_z = 0$ that is $u'_z \in \mathbb{C}$ and thus $u_1 = \prod_{\alpha \in \Delta^+_{\overline{0}}} h_{\overline{\alpha}}$ up to a scalar. Since $T_{\mathfrak{g}} = \vartheta(1)$ up to an invertible scalar, this implies (i).

Now we have $\mathrm{HC}(\vartheta(z)) = \mathrm{HC}(T_{\mathfrak{g}}) u'_z$. One can easily deduce from Corollary 7.4 (i) that $u'_z \in \mathcal{S}(\mathfrak{h}_0)^W$. Define the map $\vartheta' : \mathcal{Z}(\mathfrak{g}_{\overline{0}}) \to \mathcal{S}(\mathfrak{h}_0)^W$ by $z \mapsto u'_z$. Obviously, ϑ' is a linear injective map. Since $\dim\{z \in \mathcal{Z}(\mathfrak{g}_{\overline{0}}) |\ \deg z = m\} = \dim\{s \in \mathcal{S}(\mathfrak{h}_0)^W |\ \deg s = m\}$ and $\deg u'_z \leq \deg z$, the map ψ is surjective. This proves (ii). $\square$

10.4.1. We obtain

$$t_{\mathfrak{g}} := \mathrm{HC}(T_{\mathfrak{g}}^2) = t_{\mathfrak{h}} \Big(\prod_{\alpha \in \Delta^+_{\overline{0}}} h_{\overline{\alpha}} \Big)^2. \tag{21}$$

where $t_{\mathfrak{h}} = T_{\mathfrak{h}}^2$ is given by the formula (19).

Since the product of two anticentral elements is central, Theorem 10.4 implies the following corollary.

10.4.2. **Corollary.** *The restriction of* HC *induces an algebra monomorphism* $\mathcal{Z}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h}_0)^W$ *whose image contains* $t_{\mathfrak{g}} \mathcal{S}(\mathfrak{h}_0)^W$.

11. Computation of Shapovalov determinants

In this section we calculate Shapovalov determinants, see Theorem 11.1. Then, in 11.4, we study Weyl modules $N(\lambda)$ for λ having the smallest possible degeneracy. Results of 11.4 are used for the calculation of $\mathcal{Z}(\mathfrak{g})$ in Sect. 13.

11.1. **Theorem.** *Up to a non-zero scalar,*

$$\det B_\nu = \big(\prod_{\alpha\in\Delta^+_{\overline{0}}} (h_{\overline{\alpha}})^{\tau_\alpha(\nu-\alpha)} \prod_{\alpha\in\Delta^+_{\overline{0}}}\prod_{r\geq 1}(h_\alpha - r)^{\tau(\nu-r\alpha)}\big)^{2^{\dim\mathfrak{h}_{\overline{1}}}}.$$

Corollary.

$$\operatorname{Norm} S_\nu = \prod_{\alpha\in\Delta^+_{\overline{0}}} (h_{\overline{\alpha}})^{\tau_\alpha(\nu-\alpha)} \prod_{\alpha\in\Delta^+_{\overline{0}}}\prod_{r\geq 1}(h_\alpha - r)^{\tau(\nu-r\alpha)}.$$

This theorem is proven in 11.2,11.3 below. First in 11.2 we deduce from Corollary 10.4.2 and Claim 8.1.2 that $\det B_\nu$ admits a linear factorization. Then using the sum formula (16) we compute the multiplicities of linear factors of $\det B_\nu$ in 11.3.

11.1.1. For $h\in\mathfrak{h}_{\overline{0}}, c\in\mathbb{C}$ set

$$\gamma_h := \{\zeta\in\mathfrak{h}^*_{\overline{0}}|\ h(\zeta)=0\},\quad \gamma_{h,c} := \{\zeta\in\mathfrak{h}^*_{\overline{0}}|\ h(\zeta)=c\}.$$

Set

$$\Gamma := \{\gamma_{\overline{\alpha}}, \gamma_{\alpha,r} : \alpha\in\Delta^+_{\overline{0}}, r=1,2,\ldots\}.$$

Say that $\lambda\in\mathfrak{h}^*_{\overline{0}}$ is *regular* if $\lambda\not\in\gamma$ for any $\gamma\in\Gamma$. Note that $N(\lambda)$ is simple iff λ is regular.

Say that $\lambda\in\mathfrak{h}^*_{\overline{0}}$ is *subregular* if it belongs to exactly one of the hyperplanes from Γ.

Theorem 11.1 implies that $N(\lambda)$ is simple iff λ is regular.

In 11.4 we will show that the Jantzen filtration of $M(\lambda)$ has length two if λ is subregular. This fact is used in the computation of the centre (see Sect. 13).

11.2. **Proof of Theorem 11.1: set of zeros.** In this subsection we show that $\det B_\nu$ admits a linear factorization of the form (22).

11.2.1. Fix $\nu\in Q^+$. Assume that $\lambda\in\mathfrak{h}^*_{\overline{0}}$ is such that $t_{\mathfrak{g}}(\lambda)\neq 0$ and $\det B_\nu(\lambda)=0$. By Corollary 4.4.3, $\det B_\nu(\lambda)=0$ iff $\overline{N(\lambda)}_{\lambda-\nu}\neq 0$. The last implies the existence of a primitive vector in $N(\lambda)_{\lambda-\mu}$ for some $0<\mu\leq\nu$. Then $\mathrm{HC}(z)(\lambda)=\mathrm{HC}(z)(\lambda-\mu)$ for any $z\in\mathcal{Z}(\mathfrak{g})$. By Corollary 10.4.2, $\mathrm{HC}(\mathcal{Z}(\mathfrak{g}))\supset t_{\mathfrak{g}}\mathcal{S}(\mathfrak{h}_0)^W$. Now $t_{\mathfrak{g}}(\lambda)\neq 0$ gives $\lambda-\mu=w\lambda$ for some $w\in W$. In particular, $(\lambda-\mu,\lambda-\mu)=(\lambda,\lambda)$ where $(-,-)$ stands for the standard W-invariant bilinear form on $\mathfrak{h}^*_{\overline{0}}$. We conclude that $(\lambda,\mu)-\frac{1}{2}(\mu,\mu)=0$ for some $0<\mu\leq\nu$.

Using the formula $t_{\mathfrak{g}}=t_{\mathfrak{h}}\prod_{\alpha\in\Delta^+} h^2_{\overline{\alpha}}$ we get

$$\det B_\nu(\lambda)=0 \implies t_{\mathfrak{h}}(\lambda)=0 \ \text{ or } \lambda\in\bigcup_{\gamma\in\tilde{\Gamma}}\gamma$$

where $\tilde{\Gamma}$ is the following set of hyperplanes

$$\tilde{\Gamma} := \bigcup_{\mu\in Q^+,\mu\neq 0} \{\zeta \in \mathfrak{h}_{\bar 0}^*|\ (\zeta,\mu) - \frac{1}{2}(\mu,\mu) = 0\} \cup \bigcup_{\alpha\in\Delta^+} \{\zeta \in \mathfrak{h}_{\bar 0}^*|\ h_{\overline{\alpha}}(\zeta) = 0\}.$$

As a consequence, the hypersurface $\det B_\nu = 0$ is the union of some hyperplanes from $\tilde{\Gamma}$ and some irreducible components of the hypersurface $t_{\mathfrak{h}} = 0$. Consider the case $\mathfrak{g} = \mathfrak{sq}(n), \mathfrak{psq}(n)$ where $n \geq 3$. Then $t_{\mathfrak{h}} = \pm \sum h_1 \dots \hat{h}_i \dots h_n$ and so the hypersurface $t_{\mathfrak{h}} = 0$ is irreducible. From the formula for leading term of $\det B_\nu$ given in 8.1.2 we see that the hypersurface $t_{\mathfrak{h}} = 0$ is not a component of the hypersurface $\det B_\nu = 0$. For $\mathfrak{g} = \mathfrak{sq}(2)$ the hypersurface $t_{\mathfrak{h}} = 0$ coincides with the hyperplane $h_{\overline{\alpha}} = 0$. Now consider the case $\mathfrak{g} = \mathfrak{q}(n), \mathfrak{pq}(n)$. Then $t_{\mathfrak{h}} = \pm h_1 \dots\dots h_n$ and from 8.1.2 we see that the hyperplanes $h_i = 0$ are not components of the hypersurface $\det B_\nu = 0$. Finally we get

$$\det B_\nu(\lambda) = 0 \implies \lambda \in \bigcup_{\gamma\in\tilde{\Gamma}} \gamma.$$

11.2.2. Let us analyze the set of zeros of $\det B_\nu$ more carefully.

Let $\mu \in \mathfrak{h}_{\bar 0}^*$ be such that the hyperplane $\gamma := \{\zeta \in \mathfrak{h}_{\bar 0}^*|\ (\zeta,\mu) - \frac{1}{2}(\mu,\mu) = 0\}$ is a component of the hypersurface $\det B_\nu = 0$ and is not a component of the hypersurface $t_{\mathfrak{g}} = 0$. By 11.2.1 for any $\zeta \in \gamma$ satisfying $t_{\mathfrak{g}}(\zeta) \neq 0$ there exists $w \in W$ such that $\zeta - \mu = w\zeta$; in other words, $\gamma \subset \cup_{w\in W} X_w \cup \{\zeta|\ t_{\mathfrak{g}}(\zeta) = 0\}$ where $X_w := \{\eta \in \mathfrak{h}_{\bar 0}^*|\ \eta - \mu = w\eta\}$. Since each X_w is a proper linear subspace of $\mathfrak{h}_{\bar 0}^*$ we get $\gamma = X_w$ for some $w \in W$. Writing $\gamma = \mu/2 + \gamma'$ where $\gamma' := \{\zeta \in \mathfrak{h}_{\bar 0}^*|\ (\zeta,\mu) = 0\}$ we obtain $w(\mu/2 + \zeta) = -\mu/2 + \zeta$ for any $\zeta \in \gamma'$. In particular, $w\mu = -\mu$ and so $w\zeta = \zeta$ for all $\zeta \in \gamma'$. Hence w is the reflection with respect to μ. Since the only reflections in the symmetric group S_n are transpositions (ij) which correspond to the reflections with respect to the roots, we conclude that $\mu = r\alpha$ for some $r \in \mathbb{C}, \alpha \in \Delta^+$. Taking into account that $0 < \mu \leq \nu$, one obtains $r \in \mathbb{Z}_+$. Consequently,

$$\{\zeta \in \mathfrak{h}_{\bar 0}^*|\ (\zeta,\mu) - \frac{1}{2}(\mu,\mu) = 0\} = \{\zeta \in \mathfrak{h}_{\bar 0}^*|\ (\zeta,\alpha) - r = 0\}.$$

11.2.3. Summarizing 11.2.1 and 11.2.2 we conclude that up to a non-zero scalar

$$\det B_\nu = \prod_{\alpha\in\Delta_{\bar 0}^+} h_{\overline{\alpha}}^{d'_\alpha(\nu)} \prod_{\alpha\in\Delta_{\bar 0}^+, r\geq 1} (h_\alpha - r)^{d_{r\alpha}(\nu)}. \tag{22}$$

Comparing the above formula with the formula for leading term of $\det B_\nu$ (see 8.1.2) we obtain

$$d'_\alpha(\nu) = 2^{\dim \mathfrak{h}_{\bar 1}} \tau_\alpha(\nu - \alpha), \quad \sum_{r\geq 1} d_{r\alpha}(\nu) = 2^{\dim \mathfrak{h}_{\bar 1}} \sum_{m\geq 1} \tau(\nu - m\alpha).$$

for all $\alpha \in \Delta_{\bar 0}^+$.

11.3. **Proof of Theorem 11.1: computation of multiplicities.** In this subsection we compute the multiplicities $d_{r\alpha}(\nu)$. Fix $\alpha\in\Delta^+_{\bar 0}, r\in\mathbb{Z}_{>0}$ and set $\gamma:=\gamma_{\alpha,r}$. Let $\breve{\gamma}$ be the set of subregular points λ in γ satisfying $t_{\mathfrak{h}}(\lambda)\neq 0$. Observe that $\breve{\gamma}$ is dense in γ.

Let $s_\alpha\in W$ be the reflection with respect to α. For $\lambda\in\gamma$ one has $s_\alpha\lambda=\lambda-r\alpha$.

11.3.1. **Lemma.** *If $\lambda\in\breve{\gamma}$ then*

(i) $\mathrm{Hom}_{\mathfrak{g}}(N(\mu),N(\lambda))=0$ *for all* $\mu\neq\lambda, s_\alpha\lambda=\lambda-r\alpha$*;*
(ii) $N(s_\alpha\lambda)$ *is simple and* $t_{\mathfrak{h}}(s_\alpha\lambda)\neq 0$*.*

Proof. Suppose that $\mathrm{Hom}_{\mathfrak{g}}(N(\mu),N(\lambda))\neq 0$ and $\mu\neq\lambda$. Then $\lambda-\mu\in Q^+$ and moreover $\mu\in W\lambda$ by Corollary 10.4.2 since $t_{\mathfrak{g}}(\lambda)\neq 0$. Take $w\in W$ such that $\mu=w\lambda$. There exist linearly independent positive roots $\beta_1,\dots,\beta_k$ such that $w=s_{\beta_1}\dots s_{\beta_k}$ (see, for instance, [J], A.1.18). One has $\lambda-w\lambda=(\lambda,\beta_k)\beta_k+(\lambda,s_{\beta_k}\beta_{k-1})\beta_{k-1}+\ldots\in Q^+$. Recall that $(\lambda,\beta)\in\mathbb{Z}$ for $\beta\in\Delta^+$ forces $\beta=\alpha$. Therefore $k=1$ and $\beta_1=\alpha$. Thus $\mu=s_\alpha\lambda$ and (i) follows.

Since $t_{\mathfrak{g}}$ is W-invariant one has $t_{\mathfrak{g}}(s_\alpha\lambda)\neq 0$. Now to verify the simplicity of $N(s_\alpha\lambda)$, we need to check that for any $\beta\in\Delta^+$ the value $h_\beta(s_\alpha\lambda)=(s_\alpha\beta,\lambda)$ is not a positive integer. If $s_\alpha\beta\in\Delta^+$ then $(s_\alpha\beta,\lambda)\not\in\mathbb{Z}_{>0}$ since λ is subregular. One has $(s_\alpha\alpha,\lambda)=-r\not\in\mathbb{Z}_{>0}$. Finally, if $s_\alpha\beta\not\in\Delta^+$ and $\alpha\neq\beta$ then $s_\alpha\beta=\beta-m\alpha$ for some $m>0$ and so $(s_\alpha\beta,\lambda)=(\beta,\lambda)-mr\not\in\mathbb{Z}_{>0}$ because $(\beta,\lambda)\not\in\mathbb{Z}_{>0}$. $\square$

11.3.2. Take $\lambda\in\breve{\gamma}$ and consider the Jantzen filtration on $M(\lambda)$. Since $\gamma=\gamma_{\alpha,r}$ is the only element of Γ which contains λ, the sum formula (16) gives

$$d_{r\alpha}(\nu)=\sum_{j\geq 1}\dim F^j(M(\lambda))_{\lambda-\nu}\tag{23}$$

for any $\nu\in Q^+$. Lemma 11.3.1 implies that any proper submodule N of $N(\lambda)$ satisfies $\mathrm{ch}\, N=i\,\mathrm{ch}\, M(s_\alpha\lambda)$ for some $i\geq 0$. Therefore for each $j\geq 1$ we have

$$\mathrm{ch}\, F^j(M(\lambda))=k_j\,\mathrm{ch}\, N(\lambda-r\alpha)$$

for some $k_j\geq 0$. Putting $m_r:=\sum_{j\geq 1}k_j$ one obtains

$$d_{r\alpha}(\nu)=m_r\tau(\nu-r\alpha)$$

for all $\nu\in Q^+$.

11.3.3. By 11.2.3 we have $\sum_{r\geq 1}d_{r\alpha}(\nu)=2^{\dim\mathfrak{h}_{\bar 1}}\sum_{j\geq 1}\tau(\nu-j\alpha)$ that is

$$\sum_{r\geq 1}m_r\tau(\nu-r\alpha)=2^{\dim\mathfrak{h}_{\bar 1}}\sum_{j\geq 1}\tau(\nu-j\alpha).$$

Let us show that $m_r = 2^{\dim \mathfrak{h}_{\overline{1}}}$ by induction on r. Indeed, substituting $\nu := \alpha$ we get $m_1 = 2^{\dim \mathfrak{h}_{\overline{1}}}$. Now assuming that $m_1 = \ldots = m_{i-1} = 2^{\dim \mathfrak{h}_{\overline{1}}}$ and putting $\nu = i\alpha$ one concludes $m_i = 2^{\dim \mathfrak{h}_{\overline{1}}}$ as required. Hence

$$d_{r\alpha}(\nu) = 2^{\dim \mathfrak{h}_{\overline{1}}}\tau(\nu - r\alpha)$$

for all $\alpha \in \Delta^+_{\overline{0}}, r \in \mathbb{Z}_{>0}$, and $\nu \in Q^+$.

This completes the proof of Theorem 11.1.

11.4. **The Jantzen filtration in subregular points.** In this subsection we show that the Jantzen filtration in the points having the smallest possible degeneracy has length two.

Retain notation of 11.1.1. Set

$$V^{\oplus}(\mu) := M(\mu)/\overline{M(\mu)}.$$

Recall that, by A.3, Verma modules $M(\lambda)$ are the direct sum of Weyl modules if λ are such that $t_{\mathfrak{h}}(\lambda) \neq 0$.

11.4.1. For $\gamma = \gamma_{\alpha,r}$, let $\breve{\gamma}$ be the set of subregular points of γ satisfying $t_{\mathfrak{h}}(\lambda) \neq 0$ (as in 11.3). Clearly, $\breve{\gamma}$ is dense in γ.

Recall that $\mathfrak{g} \neq \mathfrak{psq}(2)$ throughout the paper. For $\gamma = \gamma_{\overline{\alpha}}$ and $\mathfrak{g} \neq \mathfrak{sq}(2)$, let $\breve{\gamma}$ be the maximal set of subregular points of γ which does not meet the hypersurface $t_{\mathfrak{h}} = 0$ and is invariant under the shift by α: $\lambda \in \breve{\gamma} \implies \lambda - \alpha \in \breve{\gamma}$. The condition $\mathfrak{g} \neq \mathfrak{sq}(2), \mathfrak{psq}(2)$ ensures that the hypersurface $t_{\mathfrak{h}} = 0$ does not contain hyperplanes parallel to γ. As a consequence, $\breve{\gamma}_{\overline{\alpha}}$ is dense in $\gamma_{\overline{\alpha}}$.

For $\mathfrak{g} = \mathfrak{sq}(2)$ let $\breve{\gamma}_{\overline{\alpha}}$ be the set of subregular points of $\gamma_{\overline{\alpha}}$.

11.4.2. **Proposition.** *For any $\lambda \in \breve{\gamma}$ one has*

(i) *A Jordan-Hölder filtration of module $N(\lambda)$ has length two.*
(ii) *The Jantzen filtration of $M(\lambda)$ has length two.*
(iii) $\overline{M}(\lambda) = V^{\oplus}(\tilde{\lambda})$ *where* $\tilde{\lambda} = \lambda - r\alpha$ *if* $\gamma = \gamma_{\alpha,r}$ *and* $\tilde{\lambda} = \lambda - \alpha$ *if* $\gamma = \gamma_{\overline{\alpha}}$.

Proof. The case $\mathfrak{g} = \mathfrak{sq}(2)$ was treated in 9.3. Suppose $\mathfrak{g} \neq \mathfrak{sq}(2)$.

From the formula for Shapovalov determinants we know that $N(\lambda)$ is not simple if $\lambda \in \gamma$. Since λ is subregular, $\dim \overline{N(\lambda)}_{\tilde{\lambda}} \neq 0$. Let $E(\mu)$ be a simple $\mathcal{C}\ell(\lambda)$-module ($E(\mu) = N(\mu)_{\mu}$). One has $\dim \overline{N(\lambda)}_{\tilde{\lambda}} = ke$ where $e := \dim E(\tilde{\lambda})$ and k is a positive integer. By the construction of $\breve{\gamma}$ one has $t_{\mathfrak{h}}(\lambda), t_{\mathfrak{h}}(\tilde{\lambda}) \neq 0$. Therefore $\mathcal{C}\ell(\lambda), \mathcal{C}\ell(\tilde{\lambda})$ are non-degenerate. Then $M(\lambda)$ is the direct sum of copies of $N(\lambda)$ and of $\Pi(N(\lambda))$ and $M(\tilde{\lambda})$ has the similar form with the same number of summands. Then $\dim \overline{M(\lambda)}_{\tilde{\lambda}} = 2^{\dim \mathfrak{h}_{\overline{1}}}k$. However $\det B_{r\alpha}$ at λ has zero of order $2^{\dim \mathfrak{h}_{\overline{1}}}$ so $\dim \overline{M(\lambda)}_{\lambda - r\alpha} \leq 2^{\dim \mathfrak{h}_{\overline{1}}}$. Therefore $k = 1$

so $\overline{N(\lambda)}_{\tilde{\lambda}}$ is a simple $\mathfrak{h}$-module and $\overline{M(\lambda)}_{\tilde{\lambda}} \cong \mathcal{C}\ell(\tilde{\lambda})$ as $\mathfrak{h}$-module. In particular, $\overline{M(\lambda)}$ has a subquotient isomorphic to $V^{\oplus}(\tilde{\lambda})$. Observe that $\overline{M(\lambda)} \cong V^{\oplus}(\lambda-\alpha)$ iff $\overline{N(\lambda)}$ is simple. Hence (i) and (iii) become equivalent.

Consider the case $\gamma = \gamma_{\alpha,r}$. Combining Lemma 11.3.1 with the above conclusions we obtain (i), (iii) and moreover that $\overline{M(\lambda)} \cong M(\lambda - r\alpha)$. The sum formula gives

$$\sum_{j=1}^{\infty} \operatorname{ch} F^j(M(\lambda)) = \sum_{\mu} e^{\lambda - r\alpha - \mu}\tau(\mu) = \operatorname{ch} M(\lambda - r\alpha) = \operatorname{ch} F^1(M(\lambda)).$$

The equality $F^1(M(\lambda)) \cong M(\lambda - r\alpha)$ forces $F^j(M(\lambda)) = 0$ for $j > 1$. This completes the proof for $\gamma = \gamma_{\alpha,r}$.

Fix $\gamma := \gamma_{\overline{\alpha}}$ and take $\lambda \in \breve{\gamma}$. Let us prove (iii). Observe that for a subregular point $\lambda' \in \gamma$ the sum formula gives

$$\sum_{j=1}^{\infty} \operatorname{ch} F^j(M(\lambda')) = \sum_{\mu} e^{\lambda' - \alpha - \mu}\tau_\alpha(\mu) \tag{24}$$

and therefore

$$\begin{aligned}&\operatorname{ch} \overline{M(\lambda)} \leq \textstyle\sum_\mu e^{\lambda-\alpha-\mu}\tau_\alpha(\mu),\\ &\operatorname{ch} \overline{M(\lambda-\alpha)} \leq \textstyle\sum_\mu e^{\lambda-2\alpha-\mu}\tau_\alpha(\mu) = \sum_\mu e^{\lambda-\alpha-\mu}\tau_\alpha(\mu-\alpha).\end{aligned}$$

where $\sum a_\mu e^\mu \geq \sum b_\mu e^\mu$ means that $a_\mu \geq b_\mu$ for all μ. Using the formula $\tau(\nu) = \tau_\alpha(\nu) + \tau_\alpha(\nu - \alpha)$ we get

$$\operatorname{ch} \overline{M(\lambda-\alpha)} + \operatorname{ch} \overline{M(\lambda)} \leq \operatorname{ch} M(\lambda - \alpha).$$

On the other hand, since $V^{\oplus}(\lambda-\alpha)$ is a subquotient of $\overline{M(\lambda)}$, we obtain

$$\operatorname{ch} \overline{M(\lambda)} \geq \operatorname{ch} V^{\oplus}(\lambda-\alpha) = \operatorname{ch} M(\lambda-\alpha) - \operatorname{ch} \overline{M(\lambda-\alpha)}.$$

Comparing the above inequalities, we conclude

$$\operatorname{ch} \overline{M(\lambda)} = \operatorname{ch} V^{\oplus}(\lambda-\alpha) = \sum_\mu e^{\lambda-\alpha-\mu}\tau_\alpha(\mu) \tag{25}$$

so $\overline{M(\lambda)} \cong V^{\oplus}(\lambda-\alpha)$ as required. This proves (iii) and (i). Finally, combining (25) with (24) we obtain $F^j(M(\lambda)) = 0$ for $j > 1$. □

11.4.3. **Corollary.** *Assume that $\lambda \in \mathfrak{h}^*_{\overline{0}}$ is subregular. Then the Jantzen filtration of $M(\lambda)$ has length two: $F^2(M(\lambda)) = 0$.*

Proof. Recall that the Jantzen filtration of $M(\lambda)$ has length two iff, for any ν the order of zero of $\det B_\nu$ at point λ is equal to the corank of B_ν at point λ. Fix a hyperplane $\gamma \in \Gamma$ and take a subregular point $\lambda \in \gamma$. Observe that for all subregular points of $\gamma \in \Gamma$ the order of zero of $\det B_\nu$ is the same number, say, $r(\nu)$. From Proposition 11.4.2 (ii), the corank of B_ν at point λ is at least $r(\nu)$. Since the corank does not exceed the order of zero, it is equal to $r(\nu)$. The assertion follows. □

11.4.4. Take $\gamma \in \Gamma$ and set $\nu := r\alpha$ if $\gamma = \gamma_{\alpha,r}$ and $\nu := \alpha$ if $\gamma = \gamma_{\overline{\alpha}}$. By Theorem 11.4.3, $M(\lambda)$ has $2^{\dim \mathfrak{h}_{\overline{1}}}$ linearly independent primitive vectors of the weight $\lambda - \nu$ if $\lambda \in \gamma$ is a subregular point. It is easy to see that $M(\lambda)$ has k linearly independent primitive vectors of a given weight iff a certain matrix with entries in $\mathcal{S}(\mathfrak{h}_0)$ at the point λ has corank equal to k. Since the set of subregular points is dense in γ, $M(\lambda')$ has at least $2^{\dim \mathfrak{h}_{\overline{1}}}$ linearly independent primitive vectors of the weight $\lambda' - \nu$ for any $\lambda' \in \gamma$.

11.4.5. **Corollary.** *The module $N(\lambda)$ is simple iff $h_{\overline{\alpha}}(\lambda) \neq 0$ and $h_\alpha(\lambda) \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Delta^+_{\overline{0}}$.*

If $h_{\overline{\alpha}}(\lambda) = 0$ then $\overline{N(\lambda)}$ has a subquotient isomorphic to $V(\lambda - \alpha)$ or to $\Pi(V(\lambda - \alpha))$.

If $r := h_\alpha(\lambda) \in \mathbb{Z}_{>0}$ then $\overline{N(\lambda)}$ has a subquotient isomorphic to $V(s_\alpha\lambda)$ or to $\Pi(V(s_\alpha\lambda))$.

12. On the centre of completion $\hat{U}$ of $\mathcal{U}(\mathfrak{g})$

A classical theorem by Chevalley states that for a semisimple Lie algebra the restriction of a non-zero $\mathfrak{g}$-invariant regular function on $\mathfrak{g}$ to $\mathfrak{h}$ is non-zero, that is for $f \in \mathcal{S}(\mathfrak{g}^*)^{\mathfrak{g}}$, $f \neq 0$ forces $f_{\mathfrak{h}} \neq 0$. This theorem was generalized by A. Sergeev (see [S3]) to all finite-dimensional Lie superalgebras. If $\mathfrak{g}$ admits an even non-degenerate invariant bilinear form, then $\mathfrak{g} \cong \mathfrak{g}^*$ as $\mathfrak{g}$-modules. Since $\operatorname{gr} z \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ for $z \in \mathcal{Z}(\mathfrak{g})$ the Chevalley theorem can be reformulated as follows: $\deg \mathrm{HC}(z) = \deg(z)$ for any $z \in \mathcal{Z}(\mathfrak{g})$. In this section we will prove a similar statement for a certain completion $\hat{U}$ of $\mathcal{U}(\mathfrak{g})$ where $\mathfrak{g}$ is a Kac-Moody superalgebra with a symmetrizable Cartan matrix or Q-type Lie superalgebra. For a finite-dimensional case this implies $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$.

The centre of $\hat{U}$ for $\mathfrak{g}$ being a Kac-Moody superalgebra with a symmetrizable Cartan matrix was described by Kac in [K3]. We elucidate his approach to Q-type algebras in Section 13; this will give us a description of $\mathcal{Z}(\mathfrak{g})$.

We denote by $\mathfrak{g}$ a Q-type Lie superalgebra. However, all results of this section are valid for a Kac-Moody superalgebra with a symmetrizable Cartan matrix (see Remark 12.6 for details).

12.1. **A construction $\hat{U}$.** Define on $\mathcal{U}(\mathfrak{g})$ a topology where a basis of neighborhoods of zero are left ideals $J(\nu) := \mathcal{U}(\mathfrak{g})\mathcal{U}(\mathfrak{n}^+)_{\geq \nu}$ for $\nu \in Q(\pi)$. Clearly, $J(\nu) = \mathcal{U}(\mathfrak{g})$ for $\nu \leq 0$ and $J(\nu) \subset J(\nu')$ if $\nu > \nu'$. Let $\hat{U}$ be the completion of $\mathcal{U}(\mathfrak{g})$ with respect to this topology. It is easy to see that the structure of associative algebra and the adjoint action of $\mathfrak{g}$ can be uniquely extended to $\hat{U}$. Clearly, $\mathcal{Z}(\hat{U}) = \hat{U}^{\mathfrak{g}}$.

Let N be a $\mathfrak{g}$-module with locally nilpotent action of $\mathfrak{n}^+$ [1]. Then the action of $\mathfrak{g}$ can be canonically extended to the action of $\hat{U}$. In particular, the $\mathfrak{g}$–$\mathfrak{h}$ bimodule $\mathcal{M}$ introduced in 4.1.2 can be viewed as a $\hat{U}$–$\mathfrak{h}$ bimodule.

12.1.1. Set

$$J'(\nu) := \mathcal{U}(\mathfrak{b}^-)\mathcal{U}(\mathfrak{n}^+)_\nu.$$

Then, by PBW theorem, $J(\nu) = \sum_{\mu\geq\nu} J'(\mu)$ and

$$\mathcal{U}(\mathfrak{g}) = \oplus_{\nu\in Q^+} J'(\nu), \quad \hat{U} = \prod_{\nu\in Q^+} J'(\nu). \tag{26}$$

We write the elements of $\hat{U}$ in the form $u = \sum_{\nu\in Q^+} u_\nu$ where $u_\nu \in J'(\nu)$.

12.1.2. Put $\mathfrak{b}^- := \mathfrak{n}^- + \mathfrak{h}$. Observe that $\mathcal{U}(\mathfrak{h})$ and $\mathcal{U}(\mathfrak{b}^-)$ are closed with respect to our topology. One has $\hat{U} = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-) + \hat{U}\mathfrak{n}^+)$. Extend the Harish-Chandra projection to $\hat{U}$ along the above decomposition.

Take $u = \sum_{\nu\in Q^+} u_\nu$ where $u_\nu \in J'(\nu)$. Then $\mathrm{HC}(u) = \mathrm{HC}(u_0)$ and $\mathrm{HC}(u) = u_0$ if u has weight zero, because $J'(0) = \mathcal{U}(\mathfrak{b}^-)$.

12.1.3. The canonical filtration $\mathcal{F}$ on $\mathcal{U}(\mathfrak{g})$ can be naturally extended to $\hat{U}$, however, the resulted filtration is not exhausting: $\cup_{r=0}^{\infty}\mathcal{F}^r(\hat{U}) \neq \hat{U}$. Let $\deg u$ be the degree of $u \in \hat{U}$ with respect to this filtration ($\deg u = \infty$ if $u \notin \cup_{r=0}^{\infty}\mathcal{F}^r(\hat{U})$).

12.2. **Results.** The following theorem was suggested to the author by J. Bernstein.

12.2.1. **Theorem.** *For any $z \in \mathcal{Z}(\hat{U})$ one has*

$$\deg z = \deg \mathrm{HC}(z).$$

In other words, $z \in \mathcal{Z}(\hat{U})$ takes form $z = \sum_{\nu\in Q^+} z(\nu)$ where $z(\nu) \in J'(\nu)$ is such that $\deg z(\nu) \leq \deg z(0)$.

In particular, HC induces an algebra embedding $\mathcal{Z}(\hat{U}) \to \mathcal{U}(\mathfrak{h})$. The image of the embedding lie in the centre of $\mathcal{U}(\mathfrak{h})$ which is $\mathcal{U}(\mathfrak{h}_{\overline{0}})$. We describe this image in Theorem 13.1 below.

12.2.2. **Corollary.** *One has $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$.*

Proof. From Theorem 12.2.1, $z(\nu) \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g}))$ where $r := \deg z(0)$. Then $z(\nu) = 0$ if ν is "sufficiently large" that is $\nu \notin \Delta^{(r)}$ where $\Delta^{(r)} := \{\sum_{i=1}^{s} \alpha_i |\ \alpha_i \in \Delta^+, s \leq r\}$. Since Δ is finite, $\Delta^{(r)}$ is also finite and thus z is a finite sum of elements of $\mathcal{U}(\mathfrak{g})$. Hence $z \in \mathcal{U}(\mathfrak{g})$. □

[1] this means that $\dim \mathcal{U}(\mathfrak{n}^+)v < \infty$ for any $v \in N$.

12.2.3. View $\mathcal{M}$ as a $\hat{U}$-$\mathfrak{h}$ bimodule.

Proposition.

(i) *$\mathcal{M}$ is a faithful $\hat{U}$-module.*
(ii) *For $z \in \hat{U}$ one has*

$$z \in \mathcal{Z}(\hat{U}) \iff zv = v\,\mathrm{HC}(z) \quad \textit{for any } v \in \mathcal{M}\,.$$

In other words, we have an embedding of $\mathfrak{g}$-modules $\hat{U} \to \mathrm{End}(\mathcal{M})$ and the image of a central element is an endomorphism induced by the right action.

12.3. **Proof of Theorem 12.2.1.** Take a non-zero $z \in \hat{U}^{\mathfrak{g}}$ and write $z = \sum_{\nu\in Q^+} z_\nu$ where $z_\nu \in J'(\nu)$. Set $r := \deg z_0$. Let us prove the inequality

$$\deg z_\nu \leq r \tag{27}$$

by induction on $\mathrm{ht}(\nu)$ where the function height on Q^+ is given by $\mathrm{ht}(\sum_{\alpha_i\in\pi} k_i\alpha_i) = \sum k_i$. The inequality trivially holds for $\mathrm{ht}(\nu) = 0$.

12.3.1. Suppose that the inequality (27) holds for all ν such that $\mathrm{ht}(\nu) \leq k$ and let us show that (27) holds for all weights of height $k+1$. Take μ of height $k+1$ and write $\mu = \nu + \alpha_0$ where α_0 is a simple root and $\nu \in Q^+$ is such that $\mathrm{ht}(\nu) = k$.

Denote by p_η the projection $\hat{U} \to J'(\eta)$ with respect to the decomposition (26). Since $z \in \hat{U}^{\mathfrak{g}}$ one has $[e_\beta, z] = 0$ for any simple root β and any $e_\beta \in \mathfrak{g}_\beta$. Therefore

$$0 = p_\mu\big([e_\beta, z]\big) = p_\mu\big([e_\beta, z_{\mu-\beta}]\big) + p_\mu\big([e_\beta, z_\mu]\big).$$

The canonical filtration is $\mathrm{ad}\,\mathfrak{g}$-stable and p_η-stable; thus, by the induction hypothesis, the first summand has degree at most r. Hence

$$p_\mu([e_\beta, z_\mu]) \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g})). \tag{28}$$

for all simple roots β. We need to deduce that $z_\mu \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g}))$.

12.3.2. Fix a basis $\{\mathbf{e}^{\mathbf{k}}\}$ in $\mathcal{U}(\mathfrak{n}^+)_\mu$, $\{\mathbf{h}^{\mathbf{s}}\}$ in $\mathcal{U}(\mathfrak{h})$ and write $z_\mu = \sum_{\mathbf{k},\mathbf{s}} a_{\mathbf{k},\mathbf{s}}\mathbf{h}^{\mathbf{s}}\mathbf{e}^{\mathbf{k}}$ where $a_{\mathbf{k},\mathbf{s}}$ are elements of $\mathcal{U}(\mathfrak{n}^-)_{-\mu}$. One has

$$p_\mu([e_\beta, z_\mu]) = \sum_{\mathbf{k},\mathbf{s}} [e_\beta, a_{\mathbf{k},\mathbf{s}}]\mathbf{h}^{\mathbf{s}}\mathbf{e}^{\mathbf{k}} \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g})). \tag{29}$$

by (28). Recall that $\mathcal{S}(\mathfrak{g}) = \mathcal{S}(\mathfrak{b}^-)\otimes\mathcal{S}(\mathfrak{n}^+)$. As a consequence, for any linearly independent elements $\{s_i^+\} \subset \mathcal{S}(\mathfrak{n}^+)$ and any elements $\{s_i^-\} \subset \mathcal{S}(\mathfrak{b}^-)$ the degree of $\sum_i s_i^- s_i^+$ is the maximum of $\deg s_i^- + \deg s_i^+$. Denoting the degree of $\mathbf{e}^{\mathbf{k}}$ by $|\mathbf{k}|$ we obtain from (29)

$$\deg \sum_{\mathbf{s}} [e_\beta, a_{\mathbf{k},\mathbf{s}}]\mathbf{h}^{\mathbf{s}} \leq r - |\mathbf{k}| \tag{30}$$

for any $\mathbf{k}$. Identify $\mathfrak{b}^-$ with $\mathfrak{g}/\mathfrak{n}^+$. Then $\sum_{\mathbf{s}}[e_\beta, a_{\mathbf{k},\mathbf{s}}]\mathbf{h}^{\mathbf{s}}$ identifies with $(\mathrm{ad}\, e_\beta)\sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}}\mathbf{h}^{\mathbf{s}}$ where the adjoint action $\mathfrak{n}^+$ on $\mathfrak{g}/\mathfrak{n}^+$ is induced by the usual adjoint action.

Suppose that $\deg z_\mu > r$. Then

$$\deg \sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}} \mathbf{h}^{\mathbf{s}} > r - |\mathbf{k}| \tag{31}$$

for some $\mathbf{k}$. In the light of Lemma 12.4, the image of $\sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}}\mathbf{h}^{\mathbf{s}}$ in $\mathcal{S}(\mathfrak{g}/\mathfrak{n}^+)$ is not $\mathfrak{n}^+$-invariant and thus there exists a simple root β such that $\deg \sum_{\mathbf{s}} [e_\beta, a_{\mathbf{k},\mathbf{s}}]\mathbf{h}^{\mathbf{s}} = \deg \sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}}\mathbf{h}^{\mathbf{s}}$. Comparing (30) with (31) we get a contradiction. The statement follows. □

12.4. Put $\mathfrak{n} := \mathfrak{n}^+, \mathfrak{b} := \mathfrak{b}^+$.

Lemma. *View $\mathfrak{g}/\mathfrak{n}$ as $\mathfrak{n}$-module via the adjoint action. Then $\mathcal{S}(\mathfrak{g}/\mathfrak{n})^{\mathfrak{n}} = \mathcal{S}(\mathfrak{h})$ where the embedding $\mathfrak{h} \to \mathfrak{g}/\mathfrak{n}$ is induces by the isomorphism $\mathfrak{b}/\mathfrak{n} \xrightarrow{\sim} \mathfrak{h}$.*

12.4.1. *Remark.* Assume that $\mathfrak{g}$ is finite-dimensional Lie algebra. Then $\mathfrak{g}/\mathfrak{n}$ identifies with $\mathfrak{b}^*$ via the invariant bilinear form and $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$ identifies with the set of regular functions on $\mathfrak{b}$, and $\mathcal{S}(\mathfrak{g}/\mathfrak{n})^{\mathfrak{n}}$ identifies with the invariant functions. The formula $\mathcal{S}(\mathfrak{g}/\mathfrak{n})^{\mathfrak{n}} = \mathcal{S}(\mathfrak{b}/\mathfrak{n})$ means that for any invariant function ϕ one has $\phi(h+n) = \phi(h)$ for any $h \in \mathfrak{h}, n \in \mathfrak{n}$. This is a standard fact. Indeed, let ϕ be an invariant function and N be the Lie group corresponding to $\mathfrak{n}$. If h is a generic element of $\mathfrak{h}$ then the orbit $N.h$ is dense in $h + \mathfrak{n}$ and thus $\phi(h+n) = \phi(h)$ for generic h. Since the set of generic elements is dense in $\mathfrak{h}$, $\phi(h+n) = \phi(h)$ for any $h \in \mathfrak{h}, n \in \mathfrak{n}$.

Proof of Lemma 12.4. Identify the image of $\mathfrak{g}/\mathfrak{n}$ in $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$ with $\mathfrak{b}^-$ and $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$ with $\mathcal{S}(\mathfrak{b}^-)$. For an algebra S and its subspaces X, Y denote by XY the span of $xy, x \in X, y \in Y$.

12.4.2. First, let us check that for any $\alpha \in \Delta^+$ one has $\mathcal{S}(\mathfrak{g}_{-\alpha} + \mathfrak{h})^{\mathfrak{n}} = \mathcal{S}(\mathfrak{h})$. Let $\mathfrak{g}$ be a Q-type. Taking a natural basis (see Sect. 7) in $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sq}(2)$ we obtain $(\operatorname{ad} e)(u) = h\frac{\partial u}{\partial f} + H\frac{\partial u}{\partial F}$ and $(\operatorname{ad} E)(u) = H\frac{\partial u}{\partial f} + h'\frac{\partial u}{\partial F}$ where $H^2 = h'$. Thus $(\operatorname{ad} e)(u) = (\operatorname{ad} E)(u) = 0$ forces $\frac{\partial u}{\partial f} = \frac{\partial u}{\partial F} = 0$ that is $u \in \mathcal{S}(\mathfrak{h})$ as required.

12.4.3. Now let us verify the statement of lemma. Extend the partial order on the root lattice $Q(\pi)$ to a total order compatible with the addition that is

$$\beta > \beta' \implies \beta + \alpha > \beta' + \alpha$$

(for instance, take an embedding $Q(\pi)$ into $\mathbb{R}$). Let Δ^+ be the set (not the multiset) of positive roots and let $\mathcal{P} := \sum_{\alpha \in \Delta^+} \mathbb{Z}_{\geq 0}\alpha$ be the positive lattice generated by Δ^+. For $\mathbf{k} \in \mathcal{P}$ set $X_{\mathbf{k}} := \prod_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}^{k_\alpha} \mathcal{S}(\mathfrak{h}) \subset \mathcal{S}(\mathfrak{b}^-)$. Then $\mathcal{S}(\mathfrak{b}^-) = \sum_{\mathbf{k} \in \mathcal{P}} X_{\mathbf{k}}$; let $p_{\mathbf{k}}$ be the projection with respect to this decomposition. For $u \in \mathcal{S}(\mathfrak{b}^-)$ set $\operatorname{supp} u := \{\mathbf{k} |\ p_{\mathbf{k}}(u) \neq 0\}$.

Define a lexicographic order on $\mathcal{P}$ by putting $\mathbf{k} > \mathbf{m}$ if for some $\beta \in \Delta^+$ one has $k_\beta > m_\beta$ and $k_\alpha = m_\alpha$ for all $\alpha > \beta$. Suppose that $u \in \mathcal{S}(\mathfrak{b}^-)^{\mathfrak{n}}$ is such that $u \notin \mathcal{S}(\mathfrak{h})$.

Let $\mathbf{k}$ be the maximal element in $\operatorname{supp} u$ and let $\alpha \in \Delta^+$ be the minimal root satisfying $k_\alpha \neq 0$. Put $\mathbf{k}' := \mathbf{k} - \alpha$ and let us compute $p_{\mathbf{k}'}(\operatorname{ad} e)(u)$ for $e \in \mathfrak{g}_\alpha$. One easily sees that

$$p_{\mathbf{k}'}(\operatorname{ad} e)(u) = p_{\mathbf{k}'}(\operatorname{ad} e)\big(p_{\mathbf{k}}(u)\big).$$

Fix a basis $\{f_i\}$ in $\mathfrak{g}_{-\alpha}^{k_\alpha}$ and write

$$p_{\mathbf{k}}(u) = \sum f_i a_i,\ \ a_i \in X_{\mathbf{k} - k_\alpha \alpha}.$$

Then for any $e \in \mathfrak{g}_\alpha$

$$0 = p_{\mathbf{k}'}(\operatorname{ad} e)\big(p_{\mathbf{k}}(u)\big) = \sum (\operatorname{ad} e)(f_i) a_i.$$

One has $\mathcal{S}(\mathfrak{b}^-) = \mathcal{S}(\mathfrak{g}_{-\alpha}) \otimes \mathcal{S}(\mathfrak{h}) \otimes \mathcal{S}'$ where $\mathcal{S}' := \otimes_{\beta \neq \alpha} \mathcal{S}(\mathfrak{g}_{-\beta})$. Notice that $(\operatorname{ad} e)(f_i) \in \mathcal{S}(\mathfrak{g}_{-\alpha})\mathcal{S}(\mathfrak{h})$ and $a_i \in \mathcal{S}(\mathfrak{h})\mathcal{S}'$. Thus $\sum (\operatorname{ad} e)(f_i) a_i = 0$ forces $\sum (\operatorname{ad} e)(f_i) b_i = 0$ for some non-zero $b_i \in \mathcal{S}(\mathfrak{h})$ which is impossible by 12.4.2. $\square$

12.5. **Proof of Proposition 12.2.3.** For (i) take a non-zero $u \in \mathcal{U}(\mathfrak{g})_\lambda$ and write $u = \sum u_\nu$ where $u_\nu \in J'(\nu)$. Let $\mu \in Q^+$ be a minimal element satisfying $u_\mu \neq 0$; note that $\mu \geq \lambda$. Retain notation of 8.1 and fix a PBW basis $\mathbf{f}^{\mathbf{k}}, \mathbf{k} \in \mathcal{P}(\mu - \lambda)$ in $\mathcal{U}(\mathfrak{n}^-)_{\lambda - \mu}$. Write $u_\mu = \sum_{\mathbf{k} \in \mathcal{P}(\mu - \lambda)} \mathbf{f}^{\mathbf{k}} a_{\mathbf{k}}$ where $a_{\mathbf{k}}$ are elements of $\mathcal{U}(\mathfrak{b}^+)$. Identify $\mathcal{U}(\mathfrak{b}^-)$ and $\mathcal{M}$ as $\mathfrak{b}^-$-modules. Thanks to the minimality of μ, for $v \in \mathcal{M}_{-\mu}$ one has $uv = u_\mu v$. Then

$$u_\mu v = \sum_{\mathbf{k} \in \mathcal{P}(\mu - \lambda)} \mathbf{f}^{\mathbf{k}} a_{\mathbf{k}} v = \sum_{\mathbf{k}} \mathbf{f}^{\mathbf{k}} \operatorname{HC}(a_{\mathbf{k}} v).$$

Take $\mathbf{k}$ such that $a_{\mathbf{k}} \neq 0$. Since the restriction of Shapovalov map to $\mathcal{M}_\mu$ is non-degenerate, there exists $v \in \mathcal{M}_{-\mu}$ satisfying $\operatorname{HC}(a_{\mathbf{k}} v) \neq 0$. Hence $uv = u_\mu v \neq 0$ and this gives (i).

For (ii) recall that $\mathcal{M}$ is generated by the image of $1 \in \mathcal{U}(\mathfrak{h})$ which is annihilated by $\mathfrak{n}^+$. This gives the implication

$$z \in \mathcal{Z}(\hat{U}) \implies zv = v \operatorname{HC}(z) \ \text{ for any } v \in \mathcal{M}\,.$$

The inverse implication follows from (i) and the fact that $\mathcal{M}$ is a $\hat{U}$-$\mathfrak{h}$ bimodule and so the map $v \mapsto v \operatorname{HC}(z)$ lies in $\operatorname{End}_{\hat{U}}(\mathcal{M})$. $\square$

12.6. *Remark.* All constructions and results of this section are valid for a Kac-Moody superalgebra with a symmetrizable Cartan matrix. In particular, Theorem 12.2.1 gives $\deg z = \deg \operatorname{HC}(z)$ for any $z \in \mathcal{Z}(\hat{U})$ and thus HC induces an algebra embedding $\mathcal{Z}(\hat{U}) \to \mathcal{U}(\mathfrak{h})$. The image $\operatorname{HC}\big(\mathcal{Z}(\hat{U})\big)$ was described by V. Kac in [K3], Remark 3 and Section 8. Corollary 12.2.2 gives

$$\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$$

if $\mathfrak{g}$ is a finite-dimensional Kac-Moody (contragredient) superalgebra.

All proofs except for the proof of Lemma 12.4 work for the Kac-Moody case. The only difference occurs in the proof of the formula $\mathcal{S}(\mathfrak{g}_{-\alpha} + \mathfrak{h})^{\mathfrak{n}} = \mathcal{S}(\mathfrak{h})$, see 12.4.2. If $\mathfrak{g}$ is a Kac-Moody superalgebra with a symmetrizable Cartan matrix, this step can be done as follows. The algebra $\mathfrak{g}$ admits a non-degenerate invariant bilinear form $(-,-)$ and there exists $h \in \mathfrak{h}$ such that $[e,f] = (f,e)h$ for any $f \in \mathfrak{g}_{-\alpha}, e \in \mathfrak{g}_\alpha$. Let $\{e_i\} \subset \mathfrak{g}_\alpha$ and $\{f_i\} \subset \mathfrak{g}_{-\alpha}$ form dual bases with respect to $(-,-)$. Viewing $u \in \mathcal{S}(\mathfrak{g}_{-\alpha} + \mathfrak{h})$ as a polynomial in $\{f_i\}$ we obtain $(\mathrm{ad}\, e_i)(u) = h\frac{\partial u}{\partial f_i}$. Thus $(\mathrm{ad}\, e_i)(u) = 0$ for all i gives $u \in \mathcal{S}(\mathfrak{h})$ as required.

13. The centre of a Q-type Lie superalgebra

In this section we describe the centre of a Q-type Lie superalgebra (see Theorem 13.1). The central elements correspond to the polynomials in $\mathcal{S}(\mathfrak{h}_0)$ which have the same values at λ and λ' provided that λ is subregular and λ' is the highest weight of $\overline{M(\lambda)}$.

We also show that $\mathcal{Z}(\mathfrak{q}(n)) = \mathcal{Z}(\mathfrak{sq}(n))$ and $\mathcal{Z}(\mathfrak{pq}(n)) = \mathcal{Z}(\mathfrak{psq}(n))$ (see Corollary 13.3). Notice that $\mathcal{Z}(\mathfrak{q}(n))$ was described in [S2].

Throughout the section $\mathfrak{g}$ is a Q-type Lie superalgebra and $\mathfrak{g} \neq \mathfrak{pq}(2), \mathfrak{psq}(2)$.

13.1. **Theorem.** *Let $\mathfrak{g}$ be a Q-type Lie superalgebra, $\mathfrak{g} \neq \mathfrak{pq}(2), \mathfrak{psq}(2)$. The restriction of* HC *to $\mathcal{Z}(\mathfrak{g})$ is an algebra isomorphism $\mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} Z$ where Z is the set of W-invariant polynomial functions on $\mathfrak{h}_{\overline{0}}^*$ which are constant along each straight line parallel to a root α and lying in the hyperplane $h_{\overline{\alpha}}(\lambda) = 0$. In other words,*

$$Z := \mathcal{S}(\mathfrak{h}_0)^W \cap \bigcap_{\alpha\in\Delta} Z_\alpha,$$

where

$$Z_\alpha := \{f \in \mathcal{S}(\mathfrak{h}_0) |\ h_{\overline{\alpha}}(\lambda) = 0 \implies f(\lambda) = f(\lambda - c\alpha)\ \forall c \in \mathbb{C}\}.$$

We prove this theorem using a modification of Kac method presented in [K3]. Actually we prove that $\mathrm{HC}(\mathcal{Z}(\hat{U})) = Z$ (see Section 12 for the definition of $\hat{U}$) and then use the equality $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$ obtained in Corollary 12.2.2.

13.2. **Proof of Theorem 13.1.** By Theorem 12.2.1 the restriction of HC to $\mathcal{Z}(\hat{U})$ is injective. For $z \in \mathcal{Z}(\hat{U})$ the image $\mathrm{HC}(z)$ is central in $\mathcal{U}(\mathfrak{h})$ so $\mathrm{HC}(z) \in \mathcal{S}(\mathfrak{h}_0)$. By Proposition 12.2.3, z acts on $M(\lambda)$ by $\mathrm{HC}(z)(\lambda)$. Using Corollary 11.4.5 we conclude that $\mathrm{HC}(\mathcal{Z}(\hat{U})) \subseteq Z$. To prove the opposite inclusion $\mathrm{HC}(\mathcal{Z}(\hat{U})) \supseteq Z$ we assign to each $\phi \in Z$ an element $z \in \hat{U}$ with the property: $\mathrm{HC}(z) = \phi$ and $zv = v\phi$ for all $v \in \mathcal{M}$. By Proposition 12.2.3, z is central.

We construct the element $z = \sum z_\nu$ by a recursive procedure; the summands $z_\nu \in J'(\nu)$ (see 12.1.1 for the notation) are chosen to fulfill the condition $\sum_{\mu\leq\nu} z_\mu v = v\phi$ for all $v \in \mathcal{M}_{-\nu}$. For any $\nu \in Q^+$ and any $v \in \mathcal{M}_{-\nu}$ one has $zv = \sum_{\mu\leq\nu} z_\mu v$. Hence $zv = v\phi$ for all $v \in \mathcal{M}$.

Putting $z_{<\nu} := \sum_{\mu<\nu} z_\mu$ we can rewrite the above condition as

$$z_\nu v = v\phi - z_{<\nu} v,\ \forall v \in \mathcal{M}_{-\nu}\,. \tag{32}$$

In the rest of the proof we show the existence of z_ν satisfying (32).

13.2.1. The term z_ν lies in $\mathcal{U}(\mathfrak{b}^-)_{-\nu} \otimes_{\mathfrak{h}} \mathcal{U}(\mathfrak{b}^+)_\nu$. Identify $\mathcal{U}(\mathfrak{b}^-)$ with $\operatorname{Ind}(R) = \mathcal{M}$ and $\mathcal{U}(\mathfrak{b}^+)$ with $\operatorname{Ind}_+(R)$. Under these identifications, z_ν lies in $\mathcal{M}_{-\nu} \otimes_R \operatorname{Ind}_+(R)_\nu$. The action of $\mathcal{M}_{-\nu} \otimes_R \operatorname{Ind}_+(R)_\nu$ on $\mathcal{M}_{-\nu}$ takes form

$$(a \otimes b)v = a\,\mathrm{HC}(\sigma(b)v) = aS^*(b)(v) \tag{33}$$

where $S^* : \operatorname{Ind}_+(R) \to \operatorname{Hom}_{R_r}(\mathcal{M}, R^\sigma)$ is induced by the Shapovalov map $S : \mathcal{M} \to \operatorname{Hom}_{R_r}(\operatorname{Ind}_+(R), R^\sigma)$. Consider the chain of homomorphisms

$$\mathcal{M} \otimes_R \operatorname{Ind}_+(R) \xrightarrow{\operatorname{id} \otimes S^*} \mathcal{M} \otimes_{R_r} \operatorname{Hom}_{R_r}(\mathcal{M}, R^\sigma) \xrightarrow{\iota} \operatorname{End}_{R_r}(\mathcal{M})$$

where ι is the natural map ($\iota(v \otimes f)(v') := vf(v')$). Let ψ be the composed map $\psi := \iota \circ (\operatorname{id} \otimes S^*)$ and $\psi_\nu : \mathcal{M}_{-\nu} \otimes_R \operatorname{Ind}_+(R)_\nu \to \operatorname{End}_{R_r}(\mathcal{M}_{-\nu})$ be the restriction of ψ. In the light of (33), an element $x \in \mathcal{M}_{-\nu} \otimes_{\mathfrak{h}} \operatorname{Ind}_+(R)_\nu$ acts on $v \in \mathcal{M}_{-\nu}$ by the formula

$$xv = \psi_\nu(x)(v).$$

Thus the existence of z_ν satisfying (32) is equivalent to the inclusion $C \in \operatorname{Im}\psi_\nu$ where $C \in \operatorname{End}(\mathcal{M}_{-\nu})$ is given by $Cv = v\phi - z_{<\nu}v$; notice that $C \in \operatorname{End}_{R_r}(\mathcal{M}_{-\nu})$ because $\phi \in \mathcal{S}(\mathfrak{h}_0)$ belongs to the centre of $R = \mathcal{U}(\mathfrak{h})$. The condition $C \in \operatorname{Im}\psi_\nu$ can be rewritten as $\operatorname{Im} C^* \subset \operatorname{Im} S^*_\nu$ where $C^* \in \operatorname{End}_{\mathfrak{h}}(\operatorname{Hom}_{R_r}(\mathcal{M}_{-\nu}, R^\sigma))$ is transpose to C. Thus it remains to verify the inclusion

$$\operatorname{Im} C^* \subset \operatorname{Im} S^*_\nu, \quad \text{where } Cv = v\phi - z_{<\nu}v. \tag{34}$$

13.2.2. Since S^*_ν and C^* are linear maps, the inclusion (34) is equivalent to the linear equation $C^* = S^*_\nu X$ over the polynomial algebra $\mathcal{S}(\mathfrak{h}_0)$. Rewrite $C^* = S^*_\nu X$ as $YS_\nu = C$ for $Y := X^*$.

The equation $YS_\nu = C$ has a solution over the field of fractions of $\mathcal{S}(\mathfrak{h}_0)$, because S_ν is a monomorphism between free $\mathcal{S}(\mathfrak{h}_0)$-modules of the same finite rank and thus it is invertible over $F := \operatorname{Fract}\mathcal{S}(\mathfrak{h}_0)$. View $\mathfrak{h}_{\bar{0}}$ as an affine space $\mathbb{C}^n$. Let us check that $Y := CS_\nu^{-1}$ is regular.

Retain terminology of 11.1.1. If $\lambda \in \mathfrak{h}^*_{\bar{0}}$ is a regular point, $S(\lambda)$ is bijective and so Y is regular at λ. Since the union of regular and subregular points in $\mathfrak{h}^*_{\bar{0}}$ is a set of codimension two, it is enough to verify that Y is regular in a neighbourhood of any subregular point. Here we need the following lemma.

13.2.3. **Lemma.** *Let $D=(d_{ij})_{i=1,N}^{j=1,N}$ and $E=(d_{ij})_{i=1,N}^{j=1,M}$ be two matrices, where d_{ij},e_{ij} are functions in $z_1,\dots,z_m$ which are regular on some neighbourhood U of the origin. Put $V:=U\cap\{z_1=0\}$. Suppose that D is invertible on $U\setminus V$ and that for any $\lambda\in V$ one has*

(a) The order of zero of $\det D$ *at the point* λ *is equal to* $\dim\operatorname{Ker} D(\lambda)$,

(b) $\operatorname{Ker} D(\lambda)\subset\operatorname{Ker} E(\lambda)$.

Then ED^{-1} is regular on U.

The first assumption means that all poles of D^{-1} at λ have order one. The proof is completely similar to one given in [K3].

13.2.4. Let $\lambda\in\mathfrak{h}_{\bar{0}}^*$ be a subregular point. In the light of 9.1.3, the first assumption of the lemma for the matrix B_ν follows from Corollary 11.4.3. Recall that $B=\int\circ S$ where $\int$ is an invertible map (see Lemma A.4.7 (ii)). Hence the first assumption holds for S_ν.

For the second assumption, recall that $\operatorname{Ker} S_\nu(\lambda)=\overline{M(\lambda)}_{\lambda-\nu}$. By Proposition 11.4.2, $\overline{M(\lambda)}\cong V^{\oplus}(\tilde\lambda)$ for some $\tilde\lambda<\lambda$. The condition $\phi\in Z$ ensures that $\phi(\lambda)=\phi(\tilde\lambda)$. Take $v\in\overline{M(\lambda)}_{\lambda-\nu}$. One has $v\in V^{\oplus}(\tilde\lambda)_{\tilde\lambda-\nu'}$ where $\lambda-\nu=\tilde\lambda-\nu'$. Therefore $z_{<\nu}v=z_{\leq\nu'}v$. Since $\nu'<\nu$ the induction hypothesis gives $z_{\leq\nu'}v=v\phi=\phi(\tilde\lambda)v$. Thus $z_{<\nu}v=\phi(\lambda)v$ and so $v\in\operatorname{Ker} C(\lambda)$. This implies the second assumption of Lemma 13.2.3.

Finally, Y is regular and this completes the proof of Theorem 13.1. $\square$

13.3. **Corollary.**

(i) $\mathcal{Z}(\mathfrak{q}(n))=\mathcal{Z}(\mathfrak{sq}(n))$.
(ii) $\mathcal{Z}(\mathfrak{pq}(n))=\mathcal{Z}(\mathfrak{psq}(n))$.

Proof. Observe that $\mathfrak{q}(n)=\mathfrak{sq}(n)\oplus\mathbb{C}H$ where H is odd and $[H,\mathfrak{sq}(n)]\subset\mathfrak{sq}(n)$ (in the notation of 3.3.1 $H=H_1+\dots+H_n$).

Take $z\in\mathcal{Z}(\mathfrak{q}(n))$ and write $z=a+bH$ where $a,b\in\mathcal{U}(\mathfrak{sq}(n))$. Recall that $\mathcal{Z}(\mathfrak{q}(n))$ is pure even and so a is even and b is odd. For any $x\in\mathfrak{sq}(n)$ one has

$$0=[x,a+bH]=[x,a]\pm b[x,H]+[x,b]H.$$

Since $[x,a]\pm b[x,H]\in\mathcal{U}(\mathfrak{sq}(n))$ and $[x,b]H\in\mathcal{U}(\mathfrak{sq}(n))H$, we conclude

$$[x,b]=0.$$

On the other hand,

$$0=[H,a+bH]=[H,a]\pm 2bH^2+[H,b]H$$

which gives

$$[H,b]=0.$$

Therefore $b \in \mathcal{Z}(\mathfrak{q}(n))$ and so $b = 0$ because b is odd. Hence $z = a \in \mathcal{U}(\mathfrak{sq}(n))$. This shows that $\mathcal{Z}(\mathfrak{q}(n)) \subset \mathcal{U}(\mathfrak{sq}(n))$ and so $\mathcal{Z}(\mathfrak{q}(n)) \subset \mathcal{Z}(\mathfrak{sq}(n))$. The even parts of Cartan subalgebras of $\mathfrak{q}(n)$ and $\mathfrak{sq}(n)$ coincide. By Theorem 13.1 one has $\mathrm{HC}\big(\mathcal{Z}(\mathfrak{q}(n))\big) = \mathrm{HC}\big(\mathcal{Z}(\mathfrak{sq}(n))\big)$. This proves (i). The proof of (ii) is completely similar. $\square$

Appendix A. The algebra $\mathcal{U}(\mathfrak{h})$

In this section we study the algebra $\mathcal{U}(\mathfrak{h})$ which is a Clifford algebra over $\mathcal{S}(\mathfrak{h}_0)$. In A.2 we describe the centre and the anticentre of $\mathcal{U}(\mathfrak{h})$. In A.3 we recall some basic facts on complex Clifford algebras. In A.4 we introduce a map $\int : \mathcal{U}(\mathfrak{h}) \to \mathcal{S}(\mathfrak{h}_0)$. In A.5 we compare various constructions of dual R-modules. In A.6 we introduce a reduced norm of an endomorphisms of R-module. This is an analogue of the reduced norm for endomorphisms of modules over an Azumaya algebra (see [Kn]).

In A.2— A.6 we put $A = \mathcal{S}(\mathfrak{h}_0)$ and $R = \mathcal{U}(\mathfrak{h})$. Set

$$n := \dim \mathfrak{h}_{\bar{1}}.$$

A.1. **Notation.** Let A be a commutative ring, M be a projective finitely generated A-module endowed with a quadratic form $q : M \to A$. *The Clifford algebra* $\mathcal{C}\ell(M, q)$ defined by these data is the A-algebra generated by M with the relations: $x^2 = q(x),\ x \in M$. The Clifford algebra corresponding to a non-degenerate form is called non-degenerate. We view $\mathcal{C}\ell(M, q)$ as a superalgebra by letting the elements of M to be odd.

A.1.1. One has $\mathcal{C}\ell(M \oplus M', q + q') = \mathcal{C}\ell(M, q) \otimes_A \mathcal{C}\ell(M', q')$ where the right-hand side is the tensor product of superalgebras.

In what follows M will be a free A-module of finite rank. In this case $\mathcal{C}\ell(M, q)$ is free as A-module and admits a PBW basis. We say that a Clifford algebra $\mathcal{C}\ell(M, q)$ has rank r if M has rank r over A.

All $\mathcal{C}\ell(M, q)$-modules we consider below are assumed to be $\mathbb{Z}_2$-graded and free over A.

A.1.2. Set $A := \mathcal{U}(\mathfrak{h}_{\bar{0}})$, $R := \mathcal{U}(\mathfrak{h})$. Denote by σ the antiautomorphism defined by $x \mapsto -x$ for $x \in \mathfrak{h}$. It provides an equivalence between the categories of left and right R-modules.

We can view R as a Clifford algebra over A: M is a free A-module spanned by $\mathfrak{h}_{\bar{1}}$ and $q : M \to A$ is given by $q(H) = \frac{1}{2}[H, H]$ for $H \in \mathfrak{h}_{\bar{1}}$ (the corresponding bilinear form is given on $\mathfrak{h}_{\bar{1}}$ by the formula $B(H, H') = [H, H']$). The antiautomorphism σ defined above coincides with the restriction of $\sigma : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ defined in 3.2.1.

A.2. **The anticentre of $\mathcal{U}(\mathfrak{h})$.** Retain notation and definitions of Sect. 10. It is easy to see that $\mathcal{Z}(\mathfrak{h}) = \mathcal{S}(\mathfrak{h}_0)$. We describe the anticentre $\mathcal{A}(\mathfrak{h})$ below.

A.2.1. In the notation of 10.2, $T_{\mathfrak{h}}=(\mathrm{ad}'\,u_0)(1)$ and $\mathcal{A}(\mathfrak{h})=(\mathrm{ad}'\,u_0)(\mathcal{S}(\mathfrak{h}_0))$ for some $u_0\in\mathcal{U}(\mathfrak{h})$. Any $z\in\mathcal{S}(\mathfrak{h}_0)$ is central in $\mathcal{U}(\mathfrak{h})$ and so $(\mathrm{ad}'\,u_0)(z)=z(\mathrm{ad}'\,u_0)(1)$. Therefore

$$\mathcal{A}(\mathfrak{h})=\mathcal{S}(\mathfrak{h}_0)T_{\mathfrak{h}}.$$

A.2.2. Retain notation of 3.3.1. The elements $H_1,\dots,H_n$ form a basis of $\mathfrak{h}_{\bar{1}}$ for $\mathfrak{g}=\mathfrak{q}(n),\mathfrak{pq}(n)$ and $H_1-H_2,H_2-H_3,\dots,H_{n-1}-H_n$ form a basis of $\mathfrak{h}_{\bar{1}}$ for $\mathfrak{g}=\mathfrak{q}(n),\mathfrak{psq}(n)$. We can put

$$T_{\mathfrak{h}}:=\left\{\begin{array}{ll} H_1\dots H_n & \text{for } \mathfrak{g}=\mathfrak{q}(n),\mathfrak{pq}(n)\\ \sum_{i=1}^n(-1)^iH_1\dots\hat{H}_i\dots H_n & \text{for } \mathfrak{g}=\mathfrak{sq}(n),\mathfrak{psq}(n)\end{array}\right. \tag{35}$$

since the right-hand side has the degree $\dim\mathfrak{h}_{\bar{1}}$ and is invariant with respect to the twisted adjoint action of the basis elements; the invariance easily follows from the formulas

$$(\mathrm{ad}'\,H_i)(H_{i_1}\dots H_{i_r})=\left\{\begin{array}{ll} 0 & \text{if } i\in\{i_1,\dots,i_r\},\\ 2H_iH_{i_1}\dots H_{i_r} & \text{if } i\not\in\{i_1,\dots,i_r\}.\end{array}\right.$$

One has

$$t_{\mathfrak{h}}:=T_{\mathfrak{h}}^2=\left\{\begin{array}{ll} \pm h_1\dots h_n & \text{for } \mathfrak{g}=\mathfrak{q}(n),\mathfrak{pq}(n)\\ \pm\sum h_1\dots\hat{h}_i\dots h_n & \text{for } \mathfrak{g}=\mathfrak{sq}(n),\mathfrak{psq}(n).\end{array}\right.$$

Observe that $t_{\mathfrak{h}}$ is equal to the determinant of the bilinear form B introduced in A.1.2.

A.2.3. *The centre of* $\mathcal{U}(\mathfrak{h})$. View $\mathcal{U}(\mathfrak{h})$ as a non-graded algebra and denote its centre by Z. The definition of the anticentre immediately gives $Z=\mathcal{Z}(\mathfrak{h})_{\bar{0}}\oplus\mathcal{A}(\mathfrak{h})_{\bar{1}}$. Thus $Z=\mathcal{U}(\mathfrak{h}_{\bar{0}})$ if n is even and $Z=\mathcal{U}(\mathfrak{h}_{\bar{0}})\oplus\mathcal{U}(\mathfrak{h}_{\bar{0}})T_{\mathfrak{h}}$ if n is odd.

A.3. **Clifford algebras over** $\mathbb{C}$**.** Set $n:=\dim\mathfrak{h}_{\bar{1}}, A:=\mathcal{S}(\mathfrak{h}_0), R:=\mathcal{U}(\mathfrak{h})$. For $\lambda\in\mathfrak{h}_{\bar{0}}^*$ denote by $I(\lambda)$ the maximal ideal of A corresponding to λ. Set

$$\mathcal{C}\ell(\lambda):=R/(RI(\lambda)).$$

Denote by $\mathrm{sMat}_{r,s}(\mathbb{C})$ the superalgebra of the endomorphisms of a superspace of dimension $r+s\epsilon$. The elements of $\mathfrak{q}(n)$ forms a subalgebra in the superalgebra $\mathrm{sMat}_{n,n}(\mathbb{C})$; in this section we denote this (associative) algebra by $Q(n)$.

A.3.1. Let $\mathcal{C}\ell(m)$ be the standard Clifford algebra: it is generated by $\xi_1,\dots,\xi_m$ subject to the relations $\xi_i^2=1,\xi_i\xi_j+\xi_j\xi_i=0$ for $i\neq j$.

If the symmetric form $B(\lambda):(H,H')\mapsto\lambda([H,H'])$ is non-degenerate then $\mathcal{C}\ell(\lambda)$ is isomorphic to the standard complex Clifford algebra generated by the image of $\mathfrak{h}_{\bar{1}}$. The observation in A.2.2 gives

$$\mathcal{C}\ell(\lambda)\cong\mathcal{C}\ell(\dim\mathfrak{h}_{\bar{1}})\ \text{ if } t_{\mathfrak{h}}(\lambda)\neq 0.$$

A.3.2. It is well known that $\mathcal{C}\ell(m)$ is a simple superalgebra: it has at most two simple modules which differ by grading (E and $\Pi(E)$) and all graded $\mathcal{C}\ell(m)$-modules are completely reducible.

If m is even then E is simple as a non-graded module, $\dim E = 2^{\frac{m}{2}}, \dim E_{\overline{0}} = \dim E_{\overline{1}}$. One has $E \not\cong \Pi(E)$ and $\mathcal{C}\ell(m) = \mathrm{End}(E) = \mathrm{End}(\Pi(E))$. Thus, $\mathcal{C}\ell(m)$ is isomorphic to the superalgebra $\mathrm{sMat}_{r,r}(\mathbb{C})$ where $r := 2^{\frac{m}{2}-1}$.

If m is odd then E is not simple as a non-graded module, $\dim E = 2^{\frac{m+1}{2}}$ and $E \cong \Pi(E)$. Put

$$r := 2^{\frac{m-1}{2}}.$$

We claim that image of $\mathcal{C}\ell(m)$ in $\mathrm{End}(E) \cong \mathrm{sMat}_{r,r}(\mathbb{C})$ coincides with $Q(r)$ in a suitable basis. Indeed, let $\dim \mathfrak{h}_{\overline{1}} = m$. Take λ such that $t_{\mathfrak{h}}(\lambda) \neq 0$. Then $\mathcal{C}\ell(\lambda) \cong \mathcal{C}\ell(m)$ and the image of $T_{\mathfrak{h}}$ in $\mathcal{C}\ell(\lambda)$ is a central odd element whose square is the non-zero scalar $t_{\mathfrak{h}}(\lambda)$. Therefore the centre of $\mathcal{C}\ell(m)$ contains an odd element z such that $z^2 = 1$. Clearly, we can choose a basis in E in such a way that the matrix corresponding to z is $X_{0,\mathrm{id}}$ in the notation of 3.2. It is easy to verify that

$$\{Y \in \mathrm{sMat}_{r,r}(\mathbb{C}) |\ YX_{0,\mathrm{id}} = X_{0,\mathrm{id}}Y\} = Q(r).$$

Since $\dim \mathcal{C}\ell(m) = 2^m = \dim Q(r)$ we conclude

$$\mathcal{C}\ell(m) = Q(r).$$

A.3.3. Take an arbitrary $\lambda \in \mathfrak{h}_{\overline{0}}^*$. One easily sees that $\mathcal{C}\ell(\lambda) \cong \mathcal{C}\ell(m) \otimes \bigwedge(\mathrm{Ker}\, B(\lambda))$ where $m := \dim \mathfrak{h}_{\overline{1}} - \dim \mathrm{Ker}\, B(\lambda)$.

As a result, $\mathcal{C}\ell(\lambda)$ has at most two simple modules (E and $\Pi(E)$) which are simple as $\mathcal{C}\ell(m)$-modules (where $\mathcal{C}\ell(m) = \mathcal{C}\ell(m) \otimes 1 \subset \mathcal{C}\ell(m) \otimes \bigwedge(\mathrm{Ker}\, B(\lambda)) \cong \mathcal{C}\ell(\lambda)$).

View $\mathcal{C}\ell(\lambda)$ as a right module over itself via the multiplication. One has $\Pi(\mathcal{C}\ell(\lambda)) \cong \mathcal{C}\ell(\lambda)$ iff $\lambda \neq 0$. Indeed, if $\lambda \neq 0$ there exists $H \in \mathfrak{h}_{\overline{1}}$ satisfying $H^2 = 1$; the map $x \mapsto Hx$ provides an isomorphism $\mathcal{C}\ell(\lambda) \xrightarrow{\sim} \Pi(\mathcal{C}\ell(\lambda))$. On the other hand, $\mathcal{C}\ell(0) = \bigwedge(\mathfrak{h}_{\overline{1}})$ has a one-dimensional socle $\bigwedge^{\mathrm{top}} \mathfrak{h}_{\overline{1}}$ which forces $\Pi(\mathcal{C}\ell(0)) \not\cong \mathcal{C}\ell(0)$.

A.3.4. Any simple R-module is annihilated by $I(\lambda)$ for some $\lambda \in \mathfrak{h}_{\overline{0}}^*$. We see that for each $\lambda \in \mathfrak{h}_{\overline{0}}^*$ there are at most two simple R-modules annihilated by $I(\lambda)$; we may (and will) denote them by $E(\lambda), \Pi(E(\lambda))$. The total dimension $\dim E(\lambda)$ depends on the rank m of the evaluated bilinear form $B(\lambda)$. For the case $\mathfrak{g} = \mathfrak{q}(n)$ one has $m = n - |\{i : \lambda(h_i) = 0\}|$.

A.4. **The map $\int$.** In this subsection we construct a linear map $\int : R \to A$ which satisfies the properties (i)-(v) of A.4.7. These properties ensure that $\mathcal{B}(x, y) := \int xy$ is a non-degenerate invariant bilinear form $\mathcal{B} : R \otimes_A R \to A$ (the invariance means that $\mathcal{B}([x, y], z) = \mathcal{B}(x, [y, z])$). This form is even (resp., odd) if n is even (resp., odd). Moreover supersymmetric that is $\mathcal{B}(x, y) = (-1)^{p(x)p(y)}\mathcal{B}(y, x)$ (if $\mathcal{B}$ is odd it is equivalent to the symmetricity). For each $\lambda \in \mathfrak{h}_{\bar{0}}^*$ the evaluated form $\mathcal{B}_\lambda : \mathcal{C}\ell(\lambda) \otimes_{\mathbb{C}} \mathcal{C}\ell(\lambda) \to \mathbb{C}$ is a non-degenerate invariant bilinear form.

A.4.1. In order to express the form $\mathcal{B}$ more explicitly, we calculate its evaluation in the case when $\mathcal{C}\ell(\lambda)$ is non-degenerate.

Recall that if $\mathcal{C}\ell(\lambda)$ is non-degenerate then $\mathcal{C}\ell(\lambda) \cong \mathrm{sMat}_{r,r}(\mathbb{C})$ for even n and $\mathcal{C}\ell(\lambda) \cong Q(n)$ for odd n. The superalgebra $\mathrm{sMat}_{r,r}(\mathbb{C})$ has an even non-degenerate invariant bilinear form $\mathcal{B}'(X, Y) := \mathrm{str}\, XY$. The superalgebra $Q(r)$ has an odd non-degenerate invariant bilinear form $\mathcal{B}''(X, Y) := \mathrm{tr}'\, XY$ (see 3.2 for notation). We show that for $\mathcal{C}\ell(\lambda)$ being non-degenerate, $\mathcal{B}' = c'(\lambda)\mathcal{B}_\lambda$ where $c'(\lambda)^2 = 4t_{\mathfrak{h}}(\lambda)$ if n is even, and $\mathcal{B}'' = c''(\lambda)\mathcal{B}_\lambda$ where $c''(\lambda)^2 = t_{\mathfrak{h}}(\lambda)$ if n is odd. Recall that $\mathcal{C}\ell(\lambda)$ is non-degenerate iff $t_{\mathfrak{h}}(\lambda) \neq 0$.

A.4.2. The Clifford algebra $R = \mathcal{U}(\mathfrak{h})$ over $A = \mathcal{S}(\mathfrak{h}_0)$ has the canonical filtration:

$$\mathcal{F}^0(R) := A,\ \mathcal{F}^1(R) := A\mathfrak{h}_{\bar{1}},\ \mathcal{F}^i(R) := (\mathcal{F}^1(R))^i.$$

The associated graded algebra $\mathrm{gr}\, R$ is a commutative superalgebra $A \otimes \bigwedge \mathfrak{h}_{\bar{1}}$ with the grading: $(\mathrm{gr}\, R)_i = A \bigwedge^i \mathfrak{h}_{\bar{1}}$. Take $T_{\mathfrak{h}}$ as in A.2.2 and observe that $T_{\mathfrak{h}} \not\in \mathcal{F}^{n-1}(R)$. There exists a unique A-homomorphism $\int : R \to A$ such that

$$\int (T_{\mathfrak{h}}) = 1, \quad \mathrm{Ker} \int = \mathcal{F}^{n-1}(R).$$

Since $T_{\mathfrak{h}}$ has the same parity as n, the map $\int$ is even (resp., odd) if n is even (resp., odd).

We will write $\int u$ instead of $\int(u)$.

Notice that

$$\int u = f(\mathrm{gr}\, u)$$

where $f : A \otimes \bigwedge \mathfrak{h}_{\bar{1}} \to A$ is the A-homomorphism given by

$$f(\mathrm{gr}\, T_{\mathfrak{h}}) = 1, \quad \mathrm{Ker}\, f = \sum_{i=0}^{n-1} (\mathrm{gr}\, R)_i = A \otimes \sum_{i=0}^{n-1} \bigwedge \mathfrak{h}_{\bar{1}}.$$

A.4.3. For each $\lambda \in \mathfrak{h}_{\bar{0}}^*$ the evaluated map $u \mapsto (\int u)(\lambda)$ induces a linear map $\mathcal{C}\ell(\lambda) \to \mathbb{C}$ which we also denote by $\int$. Thus

$$\int u(\lambda) := (\int u)(\lambda)$$

where $u(\lambda)$ stands for the image of $u \in R$ in $\mathcal{C}\ell(\lambda)$.

A.4.4. Define the bilinear form $\mathcal{B}: R\otimes_A R\to A$ by setting

$$\mathcal{B}(u,u'):=\int uu'.$$

Clearly, $\mathcal{B}$ is even (resp., odd) if n is even (resp., odd). Lemma A.4.7 below shows that $\mathcal{B}$ is a non-degenerate invariant form which is supersymmetric. It also shows that the evaluated form $\mathcal{B}_\lambda$ is non-degenerate *for all* λ.

A.4.5. Clearly, $\int(H'_1\ldots H'_n)\in\mathbb{C}^*$ if $H'_1,\ldots,H'_n$ is any basis of $\mathfrak{h}_{\bar 1}$. Choose a basis $\{H'_i\}_{i\in I}$ in such a way that $\int(H'_1\ldots H'_n)=1$. Set $I:=\{1,\ldots,n\}$. For $J\subset I$ set $H_J:=\prod_{j\in J}H'_j$ ($H_\emptyset=1$) where the factors are arranged by increasing of indices. The elements H_J form a system of free generators of R over A.

A.4.6. We claim that

$$\int\sigma(H_J)H_{J'}=\begin{cases}\pm1, & \text{if } J'=I\setminus J\\ 0 & \text{otherwise.}\end{cases}\tag{36}$$

Indeed, $\int\sigma(H_J)H_{J'}=f(\operatorname{gr}\sigma(H_J)H_{J'})=f(\operatorname{gr}\sigma(H_J)\operatorname{gr}H_{J'})$. Evidently, $\operatorname{gr}\sigma(H_J)\operatorname{gr}H_{I\setminus J}=\pm\operatorname{gr}H_I$ and $\operatorname{gr}\sigma(H_J)\operatorname{gr}H_{J'}\in\operatorname{Ker}f$ for $J'\neq I\setminus J$.

A.4.7. **Lemma.** *For all $a,b\in R$ one has*

(i) $\int[a,b]=0$.
(ii) *If $u\in R$ is such that $\int ua=0$ for all $a\in R$ then $u=0$.*
(iii) *If $u\in\mathcal{C}\ell(\lambda)$ is such that $(\int ua)(\lambda)=0$ for all $a\in\mathcal{C}\ell(\lambda)$ then $u=0$.*
(iv) $\int[a,b]c=\int a[b,c]$.
(v) $\int\sigma(a)=(-1)^n\int a$.

Proof. The assertion (i) follows from the formula $\int a=f(\operatorname{gr}a)$ and (iv) is a reformulation of (i). The formula (36) imply both (ii) and (iii). The last assertion is an immediate consequence of the formula $\sigma(T)=(-1)^nT$ which can be easily verified. $\square$

A.4.8. Retain notation of A.4.1. Take λ such that $t_{\mathfrak{h}}(\lambda)\neq0$ and thus $\mathcal{C}\ell(\lambda)$ is non-degenerate. Let $T(\lambda)\in\mathcal{C}\ell(\lambda)$ be the image of $T_{\mathfrak{h}}$. Recall that $T(\lambda)$ commutes with the even elements of $\mathcal{C}\ell(\lambda)$ and that $T(\lambda)^2=t_{\mathfrak{h}}(\lambda)\in\mathbb{C}$.

Let n be even. One can easily sees from Lemma A.4.7 (i) that the evaluation of $\int$ on $\mathcal{C}\ell(\lambda)\cong\operatorname{sMat}_{r,r}(\mathbb{C})$ is proportional to the supertrace. The element $T(\lambda)$ belongs to the centre of the algebra $\operatorname{sMat}_{r,r}(\mathbb{C})_{\bar 0}=\operatorname{Mat}_r(\mathbb{C})\times\operatorname{Mat}_r(\mathbb{C})$. Since $T(\lambda)\not\in\mathbb{C}$ and $T(\lambda)^2=t_{\mathfrak{h}}(\lambda)\in\mathbb{C}$ one has $T(\lambda)=c\operatorname{id}\times(-c\operatorname{id})$ where $c^2=t_{\mathfrak{h}}(\lambda)$ and id is the identity matrix in $\operatorname{Mat}_r(\mathbb{C})$. In particular, $\operatorname{str}T(\lambda)=2c$. Since $\int T(\lambda)=1$, we conclude $2c\int=\operatorname{str}$ and so $\mathcal{B}'=2c\mathcal{B}$.

Let n be odd. It is easy to deduce from Lemma A.4.7 (i) that the evaluation of $\int$ on $\mathcal{C}\ell(\lambda)\cong Q(r)$ is proportional to the map tr'. The element $T(\lambda)\in Q(r)_{\bar 1}$ belongs to the centralizer of $Q(r)_{\bar{0}}$. Consequently, $T(\lambda) = c\,\mathrm{id}'$ where $c^2 = t_{\mathfrak{h}}(\lambda)$ and $\mathrm{id}' \in Q(r)_{\bar{1}}$ corresponds to the identity matrix in $Q(r)_{\bar{1}} \cong \mathrm{Mat}_r(\mathbb{C})$. Since $\mathrm{tr}'\,T(\lambda) = c$ we obtain $c\int = \mathrm{tr}'$ and thus $\mathcal{B}'' = c\mathcal{B}$.

A.5. **Realization of N^*.** Define on R a new bimodule structure R^σ via

$$v.r := (-1)^{p(r)p(v)}\sigma(r)v, \quad r.v := (-1)^{p(r)p(v)}v\sigma(r)$$

where the dot stands for the new action, σ is the antiautomorphism introduced in 3.2.1, r is an element of the algebra R and $v \in R^\sigma$.

Let U be a superalgebra containing R such that the antiautomorphism σ can be extended to U. Let N be a bimodule over U-R. Let $N' := \mathrm{Hom}_{R_r}(N, R^\sigma)$ be the set of homomorphisms of *right* R-modules. Notice that N' has the natural structure of R-U-bimodule which we convert to U-R-bimodule structure via the antiautomorphism σ.

A.5.1. **Lemma.**

(i) *The map $R^\sigma \to \mathrm{Hom}_{R_r}(R, R)$ given by $t \mapsto l_t$ where $l_t(r) := tr$ is an even R-bimodule isomorphism.*
(ii) *The map $R \to \mathrm{Hom}_{R_r}(R, R^\sigma)$ given by $t \mapsto l'_t$ where $l'_t(r) := (-1)^{p(r)p(t)}\sigma(r)t$ is an even R-bimodule isomorphism.*

The proof is straightforward.

A.5.2. View $N^* := \mathrm{Hom}_{A_r}(N, A)$ as a $U - R$ bimodule via σ where by Hom_{A_r} we mean the set of homomorphisms of *right* A-modules. Convert $U - R$ bimodules to right $(U \otimes R)$-modules using σ. Define similarly R-bimodule structure on $R^* := \mathrm{Hom}_A(R, A)$.

A.5.3. **Lemma.**

(i) *The map $t \mapsto \int l_t$ where $\int l_t(r) := \int(tr)$ is an isomorphism of R-bimodules $R^\sigma \xrightarrow{\sim} R^*$ if $\int$ is even and $R^\sigma \xrightarrow{\sim} \Pi(R^*)$ if $\int$ is odd.*
(ii) *Let N be a U-R bimodule. The map $\psi \mapsto \int\psi$ where $(\int\psi)(x) := \int(\psi(x))$ is an isomorphism of $U - R$-bimodules $\mathrm{Hom}_{R_r}(N, R^\sigma) \to N^*$ if $\int$ is even and $\mathrm{Hom}_{R_r}(N, R^\sigma) \to \Pi(N^*)$ if $\int$ is odd.*

Proof. (i) is straightforward. Now (ii) follows from the Frobenius reciprocity. □

A.6. **Reduced norm.** Let A be a polynomial algebra and $R := \mathcal{C}\ell(M, q)$ be a Clifford algebra viewed as a non-graded algebra. Suppose that M is a free A-module of an even rank $2n$ and that the kernel of bilinear form corresponding to q is zero. Let N be an R-module which is free of finite rank over A.

We will construct a *reduced norm* i.e., a map $\mathrm{Norm} : \mathrm{End}_R(N) \to A$ satisfying the properties

$$\mathrm{Norm}(\mathrm{id}) = 1, \quad \mathrm{Norm}(\psi\psi') = \mathrm{Norm}(\psi)\,\mathrm{Norm}(\psi'), \quad \mathrm{Norm}(\psi)^{2^n} = \det\psi \tag{37}$$

where in the last formula ψ is viewed as an element of $\mathrm{End}_A(N)$ and $\det : \mathrm{End}_A(N) \to A$ is the determinant map. This map is an analogue of the reduced norm for endomorphisms of module over an Azumaya algebra (see [Kn]).

A.6.1. Set $F' := \mathrm{Fract}\, A$. The algebra $R_{F'} := R \otimes_A F'$ is an Azumaya algebra over F' (see [Kn]). Therefore, for a suitable Galois extension F/F', the algebra $R_F := R \otimes_A F$ is isomorphic to the matrix algebra $\mathrm{Mat}_r(F)$ where $r := 2^n$. The module $N_F := R_F \otimes_A N$ is an R_F-module and so there is an isomorphism

$$N_F \xrightarrow{\sim} E \otimes_F V$$

where E is the simple module over the matrix algebra $\mathrm{Mat}_r(F)$ and V is a finite dimensional vector space over F. Since E is simple over R_F any R_F-endomorphism of $N_F = E \otimes_F V$ takes form $\mathrm{id}_E \otimes \phi'$ where $\phi \in \mathrm{End}(V)$. In this way, we obtain the algebra isomorphism $\gamma : \mathrm{End}_{R_F}(N_F) \xrightarrow{\sim} \mathrm{End}(V)$. Viewing $\mathrm{End}_R(N)$ as an A-subalgebra of $\mathrm{End}_{R_F}(N_F)$ and set

$$\mathrm{Norm}\,\phi := \det\gamma(\phi).$$

Clearly, Norm satisfies the first two properties of (37). The last property follows from the formula $\dim E = r$. It remains to verify the

A.6.2. **Lemma.** $\mathrm{Norm}\,\phi \in A$.

Proof. The Galois group G acts on F, R_F, N_F leaving elements of $F', R_{F'}$ and $N_{F'}$ stable. This induces the action of G on $\mathrm{End}_{R_F}(N_F)$. On the other hand, G acts naturally on $\mathrm{End}(V)$. The map γ does not commute with these G-action; however, for each $g \in G$ the commutator $\gamma g^{-1}\gamma^{-1} g$ is an automorphism of the matrix algebra $\mathrm{End}(V)$ which leaves elements of F stable. Therefore $\gamma g^{-1}\gamma^{-1} g$ is an inner automorphism of $\mathrm{End}(V)$ and thus for any $\psi \in \mathrm{End}(V)$ one has $\det(\gamma g^{-1}\gamma^{-1} g(\psi)) = \det\psi$. In other words, for any $\phi \in \mathrm{End}_{R_F}(N_F)$ one has

$$\det\gamma(g\phi) = \det g\gamma(\phi) = g(\det\gamma(\phi)).$$

Take $\phi \in \mathrm{End}_R(N)$. One has $g(\phi) = \phi$ for all $g \in G$. This means that $\mathrm{Norm}\,\phi = \det\gamma(\phi)$ is G-stable that is $\mathrm{Norm}\,\phi \in F'$. Observe that $\det\phi \in A$ and so, by (37) $(\mathrm{Norm}\,\phi)^r \in A$. Since A is integrally closed in F', $\mathrm{Norm}\,\phi \in A$ as required. □

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