Information Theory: Exercise V

1) Let $X \in \{1, \ldots, m\}$ be a random variable with distribution $q = (q_1, \ldots, q_m)$. Let E be an event that depends only on X, such that, $\mathrm{Prob}[E] = \alpha$. Let $p = (p_1, \ldots, p_m)$ be the distribution of X conditioned on the event E. What can you say about $D(p\|q)$?

2) Let $X \in \{1, \ldots, m\}$ be a random variable with distribution $q = (q_1, \ldots, q_m)$. Let E be any event, such that, $\mathrm{Prob}[E] = \alpha$. Let $p = (p_1, \ldots, p_m)$ be the distribution of X conditioned on the event E. What can you say about $D(p\|q)$?

3) Let $X \in \{1, \ldots, m\}$ be a random variable with distribution $q = (q_1, \ldots, q_m)$ and let $Y \in \{0, 1\}$ be a random variable. Let $p_0 = (p_{0,1}, \ldots, p_{0,m})$ be the distribution of X conditioned on $Y = 0$ and let $p_1 = (p_{1,1}, \ldots, p_{1,m})$ be the distribution of X conditioned on $Y = 1$. Show that
$\mathrm{Prob}[Y = 0] \cdot D(p_0\|q) + \mathrm{Prob}[Y = 1] \cdot D(p_1\|q) = I(Y; X)$.

4) Let $X_1, \ldots, X_n \in \{0, 1\}$ be independent random variables with distribution $(p, 1 - p)$. Show that with high probability, the Kolmogorov complexity of the sequence of bits $X_1, \ldots, X_n$ is $n \cdot H(p, 1 - p) \pm o(n)$.
(That is, show that for every $\epsilon > 0$ there exists n_0 such that if $n > n_0$ then with high probability the Kolmogorov complexity of $X_1, \ldots, X_n$ is between $n \cdot [H(p, 1 - p) - \epsilon]$ and $n \cdot [H(p, 1 - p) + \epsilon]$).