A GENERALIZATION OF A COMBINATORIAL THEOREM OF SPARRE ANDERSEN ABOUT SUMS OF RANDOM VARIABLES

ACHI BRANDT

1. Introduction.

We consider a finite or infinite sequence \(X_1, X_2, \ldots\) of real valued random variables. The random variables \(X_1, \ldots, X_n\) are said to be symmetrically dependent if

\[
\operatorname{Pr}\left[\bigcap_{i=1}^n \{X_i \leq x_i\}\right] = \operatorname{Pr}[X_i \leq x_i, \ i = 1, 2, \ldots, n]
\]

is a symmetric function of \(x_1, \ldots, x_n\). If an event \(C\) is invariant under permutation of the variables \(x_1, \ldots, x_n\) we say that the event is symmetric with respect to \(X_1, \ldots, X_n\). For \(n = 1, 2, 3, \ldots\), we introduce the following random variables:

\[
S_n = X_1 + \ldots + X_n, \quad S_0 = 0.
\]

\[
R_n = \max\{S_0, S_1, \ldots, S_n\}, \quad R_0 = 0.
\]

\(N_n(\gamma)\) = the number of sums \(S_1, \ldots, S_n\) which are greater than \(\gamma\).

\(L_n(\gamma)\); for \(\gamma \geq 0\) we define \(L_n(\gamma)\) as the first index \(k\), \(0 \leq k \leq n\), for which \(S_k \geq R_n - \gamma\), and for \(\gamma < 0\), \(L_n(\gamma)\) is defined to be the last index \(k\), \(0 \leq k \leq n\), for which \(S_k > R_n + \gamma\).

The main result of this note is

**Theorem 1.1.** If \(X_1, \ldots, X_n\) are symmetrically dependent random variables, and \(C\) is an event which is symmetric with respect to \(X_1, \ldots, X_n\), then

\[
(1.1) \quad \operatorname{Pr}[N_n(\gamma) = k, C] = \operatorname{Pr}[L_n(\gamma) = k, C], \quad k = 0, 1, \ldots, n, \quad -\infty < \gamma < \infty.
\]

The case \(\gamma = 0\) is a known theorem of E. Sparre Andersen, who proved it by induction (see [4, Theorem 1]). Other proofs for \(\gamma = 0\) have been given by F. Spitzer [5] and W. Feller [2]. Our method is entirely combinatorial, and related to the methods in [5] and [2].

The combinatorial tool required to prove Theorem 1.1 is developed in Section 2. Since this tool (ordered mixture) may have some interest

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in itself, we give the theorems in Sec. 2 in more general form than necessa-
ry for our purposes. In Section 3 we prove Theorem 1.1. In Section 4
certain corollaries, dealing with the order statistics of the sums $S_1, S_2, \ldots,$
are obtained. These are related to the works of Wendel [6] and Pol-
laczec [3].

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2. Ordered Mixture.

**Definition.** Let $A$ be a finite set of finite sequences,

$$A = \{(x_{i,1}, x_{i,2}, \ldots, x_{i,n_i}); \ i = 1, 2, \ldots, m\}.$$ 

A sequence $y = (y_1, \ldots, y_n)$ is called an ordered mixture (o.m.) of $A$ if

there is a 1–1 correspondence

$$(i,j) \leftrightarrow v = v(i,j),$$

$j = 1, \ldots, n_i, i = 1, \ldots, m, v = 1, \ldots, n$ so that $x_{i,j} = y_{v(i,j)}$ and $v(i,j + 1) >

v(i,j)$ for $j = 1, 2, \ldots, n_i - 1$; that is, the members of $y$ are exactly all the

$x_{i,j}$, and the order of the members of each sequence of $A$ is preserved

in $y$.

For example, the sequence $(1, 4, 5, 2, 7, 6, 8, 3)$ is an o.m. of the set

$\{(1, 2, 3), (4, 5, 6), (7, 8)\}$.

**Theorem 2.1.** Let $(E_1, \ldots, E_m)$ be a disjoint sequence of sets of real

numbers, and suppose $A$ is any sequence of real sequences

$$A = (\xi_1, \xi_2, \ldots, \xi_m), \quad \xi_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n_i}), \quad 1 \leq i \leq m.$$ 

Then there is at most one o.m. of $A$, say $y = (y_1, \ldots, y_n)$, for which

$$\sum_{m=1}^{v(i,j)} y_m \in E_i, \quad v(i,j) = 1, 2, \ldots, n.$$ 

**Proof.** The proof is easily accomplished by induction with respect to

$\Sigma_i^m n_i$ ($m$ being fixed throughout the induction; $n_i$ may be zero). The

case $\Sigma n_i = 1$ is trivial. We assume now that the theorem holds for

$\Sigma n_i = n - 1$ and shall prove the case $\Sigma n_i = n$. Put

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} x_{i,j} = S.$$ 

If an o.m. $(y_1, \ldots, y_n)$ with the above property does exist, then $\Sigma_{\mu=1}^{m} y_\mu = S$,

which implies $S \in E_k$ and $y_n \in \xi_k$ for some $k$ with $1 \leq k \leq m$. Thus there

are two possibilities: Either $S \notin \bigcup_{i=1}^{m} E_i$, in which case no o.m. exists at

all; or $S \in E_k$ in which case we must have $y_n = x_{k,n_k}$, and then by the
induction assumption there is at most one admissible arrangement of
$y_1, \ldots, y_{n-1}$. This completes the proof.

A similar theorem with a similar proof is clearly valid for any group
instead of the real numbers.

**Definition.** The ordered mixture $(y_1, \ldots, y_n)$ in Theorem 2.1 is cal-
led the ordered mixture of $(\xi_1, \ldots, \xi_m)$ with respect to $(E_1, \ldots, E_m)$. Or,
when abbreviated: $(y_1, \ldots, y_n)$ is the o.m. $(E_1, \ldots, E_m)$ of $(\xi_1, \ldots, \xi_m)$.

**Theorem 2.2.** Let $(E_1, \ldots, E_m)$ be a disjoint sequence of sets of real
numbers, and let $X_1, \ldots, X_n$ be symmetrically dependent random variables.
Furthermore, let $n_1, \ldots, n_m$ be non-negative integers, with $n_1 + \ldots + n_k = N_k$,
$N_m = n$, $N_0 = 0$. Then, for the partial sums $S_1, \ldots, S_n$ and the sequences

$$
\xi_k = (X_{N_{k-1}+1}, X_{N_{k-1}+2}, \ldots, X_{N_k}), \quad k = 1, 2, \ldots, m,
$$

we have

$$
\Pr \left[ \bigcap_{i=1}^m \{ \text{exactly } n_i \text{ partial sums are in } E_i \} \right] = \Pr[(\xi_1, \ldots, \xi_m) \text{ has an o.m. } (E_1, \ldots, E_m)] .
$$

**Proof.** Let $\Sigma^*$ denote summation extending over all the $n$-tuples
$\{j_{i,k} \mid 1 \leq k \leq n_i, 1 \leq i \leq m \}$ for which $1 \leq j_{i,1} < \ldots < j_{i,n_i} \leq n$, $i = 1, 2, \ldots, m$, and whose members $j_{i,k}$ are all distinct. Then

$$
\Pr \left[ \bigcap_{i=1}^m \{ \text{exactly } n_i \text{ partial sums are in } E_i \} \right] = \Sigma^* \Pr \left[ \bigcap_{i=1}^m \bigcap_{k=1}^{n_i} \{ S_{j_{i,k}} \in E_i \} \right] = \Sigma^* \Pr[X_1, \ldots, X_n \text{ is the o.m. } (E_1, \ldots, E_m) \text{ of }
(X_{j_{1,1}, X_{j_{1,2}}, \ldots, X_{j_{1,n_1}}}, (X_{j_{2,1}, X_{j_{2,2}}, \ldots, X_{j_{2,n_2}}}, \ldots,
\ldots, (X_{j_{m,1}, X_{j_{m,2}}, \ldots, X_{j_{m,n_m}}})]\n = \Sigma^* \Pr[(\xi_1, \ldots, \xi_m) \text{ has an o.m. } (E_1, \ldots, E_m) \text{ in }
\text{which } x_{N_{k-1}+1} \text{ appears in the } j_{k,1}\text{-th place}])
= \Pr[(\xi_1, \ldots, \xi_m) \text{ has an o.m. } (E_1, \ldots, E_m)] .
$$

In the third equality we made use of the symmetrical dependence of
the random variables $X_1, \ldots, X_n$.

**Remark.** All the above equalities remain valid if every event in
brackets $[\ldots]$ is intersected with an event $C$ which is symmetric with
respect to $X_1, \ldots, X_n$. 

In the following theorem \((\gamma, \infty)\) and \((-\infty, \gamma]\) denote the real intervals \(\gamma < x < \infty\) and \(-\infty < x \leq \gamma\), respectively.

**Theorem 2.3.** Let \(x = (x_1, \ldots, x_L)\) and \(y = (y_1, \ldots, y_M)\) be two sequences of real numbers. Then for every real number \(\gamma\), \((x, y)\) has an o.m. \(((\gamma, \infty), (-\infty, \gamma])\) which begins with \(x_1\), if and only if

\[Q + P \leq \gamma \quad \text{and} \quad P > \gamma,
\]

where

\[Q = \max_{1 \leq m \leq M} \sum_{1}^{m} y_r, \quad P = \min_{1 \leq l \leq L} \sum_{1}^{l} x_r.
\]

**Proof.** Suppose \((x, y)\) has an o.m. \(((\gamma, \infty), (-\infty, \gamma])\) of the form

\[x_{i_1}, x_{i_2}, \ldots, x_{i_k}; \quad y_{j_1}, y_{j_2}, \ldots, y_{j_1}; \quad x_{i_3}, x_{i_4}, \ldots, x_{i_3};
\]

\[y_{j_1+1}, y_{j_1+2}, \ldots, y_{j_2}; \quad \ldots
\]

where \(0 < i_1 < i_2 < \ldots, 0 = j_0 < j_1 < j_2 < \ldots\). By the definition of o.m. we have

\[\sum_{1}^{i_3} x_r + \sum_{1}^{j_3} y_r \leq \gamma
\]

and, for \(i_3 < k \leq j_{x+1},
\]

\[\sum_{1}^{k} x_r + \sum_{1}^{j_3} y_r > \gamma,
\]

which together imply, for \(\alpha \geq 1,
\]

\[\sum_{i_3+1}^{k} x_r > 0.
\]

Thus, if \(\sum_{1}^{j} x_r = P\), then necessarily \(\lambda \leq i_1,\) and therefore \(\sum_{1}^{j} x_r\) is a partial sum of the o.m., which implies \(\sum_{1}^{j} x_r = P > \gamma\).

Let \(Q = \sum_{1}^{j} y_r\). For some \(\alpha, j_{x+1} < \mu \leq j_{x}\). Then, by the requirements of o.m.,

\[P + Q \leq \sum_{1}^{i_3} x_r + \sum_{1}^{\mu} y_r \leq \gamma.
\]

To prove the "if" assertion of the theorem, we construct the o.m. \((z_1, z_2, \ldots, z_{M+L})\) starting at the right end, by the following procedure: We put

\[z_{M+L} = x_L \quad \text{or} \quad z_{M+L} = y_M
\]

according as

\[\sum_{1}^{L} x_r + \sum_{1}^{M} y_r \quad \text{falls in} \ (\gamma, \infty) \text{or in} \ (-\infty, \gamma].
\]

Assume we have already defined \(z_{m+1}, z_{m+1+1}, \ldots, z_{M+L}\) using \(x_{i+1}, x_{i+2}, \ldots, x_L\) and \(y_{m+1}, y_{m+2}, \ldots, y_{M}\). We then choose \(z_{m+1} = x_{i}\) if \(\sum_{1}^{i} x_r + \sum_{1}^{m} y_r > \gamma\) and \(z_{m+1} = y_{m}\) otherwise. The sequence \(z_1, \ldots, z_{M+L}\) so
built is clearly an o.m. of \((x,y)\), but we have to prove that we can indeed proceed with the above construction until we get \(z_1 = x_1\).

There are no difficulties to define \(z_{m+1}\) as long as both \(m > 0\) and \(l > 0\). If \(m = 0\), the choice \(z_1 = x_1\) agrees with the requirements, since \(\Sigma_1^l x_r \geq P > \gamma\). Thus, to complete the proof, we only have to show that it is impossible to reach a point where \(l = 0, m > 0\), i.e. the case \(z_{k+1} = x_1, k \geq 1\). Such a situation indeed leads to a contradiction: Let \(P = \Sigma_1^i x_r\). For some \(r \geq k \geq 1\) we have \(x_i = z_{r+1}\), which would imply \(\Sigma_1^i y_r + \Sigma_1^i x_r > \gamma\), in contradiction to the assumption \(Q + P \leq \gamma\). This completes the proof of Theorem 2.3.

The proof of the following theorem is essentially the same as that of the last one.

**Theorem 2.4.** In the above notation, \((x,y)\) has an o.m. \(((\gamma, \infty), (-\infty, \gamma)]\) that begins with \(y_1\), if and only if

\[
Q + P > \gamma, \quad Q \leq \gamma.
\]

**3. Proof of Theorem 1.1.**

The events \([N_n(\gamma) = 0]\) and \([L_n(\gamma) = 0]\) are identical, while \([N_n(\gamma) = n]\) is the image of the event \([L_n(\gamma) = n]\) under the permutation \(X_i \rightarrow X_{n-i+1}\), \(i = 1, \ldots, n\). Thus we may restrict ourselves to the case \(0 < k < n\).

**Lemma 3.1.** If \(X_1, \ldots, X_n\) are symmetrically dependent random variables with respect to which \(C\) is a symmetric event, then, for \(0 < k < n\),

\[
\Pr[N_n(\gamma) = k, C] \quad \Pr \left[ \max_{k+1 \leq m \leq n} \sum_{k+1}^m X_r + \min_{1 \leq l \leq k} \sum_{1}^l X_r \leq \gamma, \min_{1 \leq l \leq k} \sum_{1}^l X_r > \gamma, C \right] + \\
+ \Pr \left[ \max_{k+1 \leq m \leq n} \sum_{k+1}^m X_r + \min_{1 \leq l \leq k} \sum_{1}^l X_r > \gamma, \max_{k+1 \leq m \leq n} \sum_{k+1}^m X_r \leq \gamma, C \right].
\]

This lemma is a direct corollary of Theorems 2.2 (with the subsequent remark), 2.3 and 2.4.

To prove Theorem 1.1, let us first consider the case \(\gamma \geq 0\). By the permutation \(X_i \rightarrow X_{k-i+1}, i = 1, 2, \ldots, k\), we derive from Lemma 3.1 that

\[
\Pr[N_n(\gamma) = k, C] = \Pr[\{Q + P \leq \gamma, P > \gamma, C\} \cup \{Q + P > \gamma, Q \leq \gamma, C\}],
\]

where

\[
Q = Q_k = \max_{k+1 \leq m \leq n} \sum_{k+1}^m X_r, \\
P = P_k = \min_{1 \leq l \leq k} \sum_{1}^l X_r = S_k - R_{k-1}.
\]
Therefore, and from the fact that $Q + P \leq \gamma$ and $P > \gamma$ implies $Q < 0$:

\[
\Pr[N_n(\gamma) = k, C] = \\
= \Pr[(\{Q + P \leq \gamma, P > \gamma\} \cup \{Q + P > \gamma, Q \leq \gamma, Q < 0\} \cup \{Q + P > \gamma, Q \leq \gamma, Q \leq 0\}) C] \\
= \Pr[(\{Q + P \leq \gamma, P > \gamma, Q < 0\} \cup \{Q + P > \gamma, Q < 0, P > \gamma\} \cup \{Q + P > \gamma, 0 \leq Q \leq \gamma\}) C] \\
= \Pr[(\{P > \gamma, Q < 0\} \cup \{P + Q > \gamma, 0 \leq Q \leq \gamma\}) C].
\]

Consequently, when $\gamma \geq 0$, the theorem results from the following two trivial lemmas:

**Lemma 3.2.** Let $\gamma \geq 0, 0 < k < n$. Then $Q_k < 0$ and $P_k > \gamma$ if and only if $L_n(\gamma) = k$ and $S_k$ is the last maximum of the sequence $S_1, S_2, \ldots, S_n$.

**Lemma 3.3.** Let $\gamma \geq 0, 0 < k < n$. Then $Q_k + P_k > \gamma$ and $0 \leq Q_k \leq \gamma$ if and only if $L_n(\gamma) = k$ and $S_k$ is not the last maximum of $S_1, S_2, \ldots, S_n$.

Note that $Q_k + P_k = \max_{k+1 \leq i \leq n} S_i - \max_{0 \leq i \leq k-1} S_i$.

Similarly, for $\gamma < 0$ we have

\[
\Pr[N_n(\gamma) = k, C] = \\
= \Pr[(\{Q + P \leq \gamma, P > \gamma, C\} \cup \{Q + P > \gamma, Q \leq \gamma, C\})] \\
= \Pr[(\{Q + P \leq \gamma, Q < 0\} \cup \{Q + P \leq \gamma, P > \gamma, P \geq 0\} \cup \{Q + P > \gamma, Q \leq \gamma\}) C] \\
= \Pr[(\{Q + P \leq \gamma, Q < 0\} \cup \{Q + P \leq \gamma, P \geq 0, Q \leq \gamma\} \cup \{Q + P > \gamma, Q \leq \gamma, P \geq 0\}) C] \\
= \Pr[(\{Q + P \leq \gamma, Q < 0\} \cup \{P \geq 0, Q \leq \gamma\}) C],
\]

and the corresponding lemmas:

**Lemma 3.4.** Let $\gamma < 0, 0 < k < n$. Then $P_k \geq 0$ and $Q_k \leq \gamma$ if and only if $L_n(\gamma) = k$ and $S_k = R_n$.

**Lemma 3.5.** Let $\gamma < 0, 0 < k < n$. Then $P_k + Q_k \leq \gamma$ and $\gamma < P_k < 0$ if and only if $L_n(\gamma) = k$ and $S_k < R_n$.

This completes the proof of Theorem 1.1 for all $\gamma$. Obviously, a similar theorem holds for $N_n^*(\gamma)$ the number of sums $S_1, \ldots, S_n$ which are greater than or equal to $\gamma$.


We introduce the notation: $S_k^+ = \max[S_k, 0]$. The order statistics of $S_1^+, \ldots, S_n^+$ are designated by $R_{n,1} \geq R_{n,2} \geq R_{n,3} \geq \ldots \geq R_{n,n}$.

**Theorem 4.1.** If $X_1, \ldots, X_n$ are symmetrically dependent random variables with respect to which $C$ is a symmetric event, then

\[
\Pr[R_{n,m+1} \leq \gamma, C] = \Pr[R_n - R_m \leq \gamma, C], \quad 0 \leq m \leq n - 1, \quad -\infty < \gamma < \infty.
\]
Proof. For $\gamma < 0$ the two sides of the equality vanish. For $\gamma \geq 0$ we just have to sum over $k = 0, 1, 2, \ldots, m$ in (1.1).

The following theorem, concerning the characteristic functions of $(R_{n,k}, S_n)$ was first proved in [3] by function theoretic methods. In [6] an algebraic proof appeared which is closely connected with a recent proof in [1].

**Theorem 4.2.** (Pollaczec and Wendel.) Let $X_1, X_2, \ldots$ be an infinite sequence of independent and identically distributed random variables. Let $\Phi(\sigma) = E(\exp i\sigma X_k),$

$$\psi_n = \psi_n(\varphi, \sigma) = E(\exp i[\varphi S_n + \sigma S_n]),$$

$$\zeta_{n,k} = \zeta_{n,k}(\varphi, \sigma) = E(\exp i[\varphi R_{n,k} + \sigma S_n]).$$

Then, the following identity holds, provided $|\omega|, |z| < 1:$$$

$$\sum_{n=1}^{\infty} \omega^n \sum_{k=1}^{n} z^{k-1} \zeta_{n,k} = \frac{1}{(1-z)(1-\omega \Phi(\sigma))} \left[ \exp \left\{ \sum_{n=1}^{\infty} \frac{\omega^n}{n} (1-z^n) \psi_n \right\} - 1 \right].$$

It is not too difficult to prove this theorem from Theorem 1.1 (for $C$ we take the event $S_n \leq t$) and a theorem of Spitzer. Spitzer's theorem is simply the case $z = 0$ in Theorem 4.2, but he has proved it independently (in [5]), using pure combinatorial considerations. Thus we obtain a purely combinatorial proof of Theorem 4.2. Conversely, combinatorial results may be deduced from Theorem 4.2 (see [5] and [7]) as well as from Spitzer's theorem (see [5]).

**References**


University of Jerusalem, Israel