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TEXTBOOK MULTIGRID EFFICIENCY FOR FLUID SIMULATIONS*

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■ **Abstract** Recent advances in achieving textbook multigrid efficiency for fluid simulations are presented. Textbook multigrid efficiency is defined as attaining the solution to the governing system of equations in a computational work that is a small multiple of the operation counts associated with discretizing the system. Strategies are reviewed to attain this efficiency by exploiting the factorizability properties inherent to a range of fluid simulations, including the compressible Navier-Stokes equations. Factorizability is used to separate the elliptic and hyperbolic factors contributing to the target system; each of the factors can then be treated individually and optimally. Boundary regions and discontinuities are addressed with separate (local) treatments. New formulations and recent calculations demonstrating the attainment of textbook efficiency for aerodynamic simulations are shown.

1. INTRODUCTION

This paper addresses the attainment of textbook multigrid efficiency (TME) in solving the fluid dynamics equations for simulation. A multigrid method is defined by Brandt (1984, 2000) as having TME if the solutions to the governing system of equations are attained in a computational work which is a small (less than 10) multiple of the operation count in the discretized system of equations (residual evaluations). The TME descriptor stems from the efficiency that has been demonstrated

for elliptic problems and is available in multigrid textbooks (Wesseling 1992, Trottenberg et al. 2000, Briggs et al. 2000). The compressible Navier-Stokes (NS) equations are a set of coupled nonlinear equations that are not fully elliptic, even for subsonic Mach numbers, but contain hyperbolic partitions. Across the Mach number range, the equations are of mixed type and the solutions exhibit discontinuities (shocks, slip lines, etc.) and flow- or grid-induced singularities. Thus, there are many difficulties in developing TME solvers for fluid simulations.

Existing computational fluid dynamics (CFD) solvers for the Euler and NS equations can accommodate a broad range of Mach numbers and are quite robust. Much of the robustness stems from the realization of nonlinear schemes that have their roots in the numerical solution of the time-dependent Riemann problem, extended to multiple dimensions on a dimension-by-dimension basis (Roe 1986). The flux-difference-splitting scheme is an example of this approach; central differencing schemes with scalar/matrix artificial viscosity are closely related to this approach. At low Mach numbers, preconditioning is typically required for accuracy and efficiency. The equations are solved using a time-dependent (multistage time-stepping methods) or a quasi-Newton framework (approximate implicit or residual minimization schemes), often embedded in a full approximation scheme (FAS) multigrid framework. The current state of the art is summarized by Mavriplis (1997) for aerodynamic applications using unstructured grids in Volume 29 of this series.

The efficiency of CFD solvers used in current practice is far from TME, especially for viscous simulations. Typically, more than 1000 residual evaluations are required to converge lift and drag values, even for relatively benign cruise configurations intended to minimize viscous and shock-wave losses. Even more computational work is required as the geometry and/or the flow simulation becomes more complex. Thus, there is a potential gain of more than two orders of magnitude in operation count reduction if TME could be attained for CFD simulations. For staggered-grid formulations of incompressible-flow equations, robust and relatively efficient multigrid solvers have already been developed (Oosterlee et al. 1988, 2000; Montero et al. 2000); their efficiency is still about an order of magnitude behind TME.

To be efficient, multigrid solvers for general systems of partial differential equations must adequately address three types of errors: (*a*) high-frequency error components, (*b*) uniformly smooth error components, and (*c*) characteristic error components. The latter are (usually smooth) error components that are much smoother in characteristic directions than in other directions. Standard multigrid methods that are efficient for elliptic problems recognize and separate the treatment of high-frequency and smooth error components. The former are efficiently reduced in relaxation; the latter are well approximated on coarse grids and, hence, eliminated through the coarse-grid correction. The efficiency of classical multigrid methods severely degrades for nonelliptic problems because characteristic components cannot be adequately approximated on coarse grids (Brandt 1981, Brandt & Yavneh 1992, Brandt & Diskin 2000).

If the target discretization is *h*-elliptic (or semi-*h*-elliptic), the high frequency error components can still be reduced by a local (or blockwise) relaxation procedure. By definition (Brandt 1981, Trottenberg et al. 2000), a discrete scalar (not necessarily elliptic) operator $L[u]$ possesses a good measure of *h*-ellipticity if the absolute value of its symbol $|L(\bar{\theta})| = |e^{-i(\bar{\theta}\cdot\mathbf{j})}L[e^{i(\bar{\theta}\cdot\mathbf{j})}]|$ is well separated from zero for all high-frequency Fourier modes. Here, $\mathbf{j} = (j_x, j_y, j_z)$ are the grid indexes and $\bar{\theta} = (\theta_x, \theta_y, \theta_z)$, $0 \le |\theta_x|, |\theta_y|, |\theta_z| \le \pi$ are normalized Fourier frequencies. High-frequency Fourier modes are the modes satisfying $\max(|\theta_x|, |\theta_y|, |\theta_z|) \ge \frac{\pi}{2}$. A system of equations possesses a good measure of *h*-ellipticity if the *h*-ellipticity measure of the determinant of its operator matrix is good.

Standard coarse-grid corrections are efficient for uniformly smooth error components, even for nonelliptic problems. An effective reduction of characteristic error components can be achieved either by designing a proper relaxation scheme reducing not only high-frequency but smooth error components as well (e.g., by downstream ordering of relaxation steps) or by adjusting coarse-grid operators for a better characteristic-component approximation. Multigrid methods efficiently reducing all the three aforementioned types of error have been developed for scalar nonelliptic operators (Brandt & Diskin 2000; Diskin 1998, 2001). Efficient methods for solving the different factors encountered in the determinants of the flow equations in different regimes, as well as available analytical tools, are reviewed by Brandt et al. (2001).

TME for systems of equations can be attained by exploiting the factorizability property of the governing equations. The factorizability of the NS equations is manifested by the fact that in smooth regions (i.e., neglecting shocks), the determinant of the matrix of differential operators consists of separable factors. Exploiting the factorizability property in discrete computations reduces the problem of relaxing a complicated system of discretized coupled differential equations to relaxation of simpler factors constituting the system determinant.

This approach is quite distinct from most approaches to accelerate convergence because, for steady-state flows, the factors are treated directly rather than through pseudo-time marching methods. Time-dependent flow solvers can be constructed within this approach as well and in principle are simpler to develop than steady-state solvers. A list of envisioned difficulties and possible solutions in attaining TME for CFD simulations is discussed elsewhere by Brandt (2000) and Brandt et al. (2001).

This paper is organized as follows: The foundations of the methodology for attaining TME are discussed in Section 2, including the concept of principal linearization and illustrations of the factorizability of various fluid dynamic equations. Two strategies for exploiting the factorizability are presented in the next two sections: Reformulation of the differential equations is discussed in Section 3. An alternative, more general, distributed relaxation approach is discussed in Section 4. Some recent advances in formulation and demonstration are shown in Section 5. Concluding remarks are given in Section 6.

2. FOUNDATIONS

The basic framework for nonlinear TME solvers is full multigrid (FMG) algorithms (Brandt 1977, Trottenberg et al. 2000, Briggs et al. 2000). In FMG algorithms, the solution process is started on a very coarse grid where the computational cost of solution is negligible. The coarse-grid solution is then interpolated to the next fine grid to form an initial approximation. A few multigrid FAS cycles (possibly just one) are performed to obtain an improved solution approximation. This process continues to finer grids until the solution on the target finest grid is achieved.

In the solution of highly nonlinear problems, a good initial guess is important. A general way to obtain such an initial guess is by continuation, in which the solution to the target problem is approached through the solutions of a sequence of parameterized problems. Usually the problem starting the continuation process is easy to solve, and difficulty gradually increases as the control parameter approaches the target value; this continuation process can often be integrated into an FMG solver. For example, with viscosity as the control parameter, at the coarse grids more artificial viscosity can be used, then gradually taken out as the algorithm proceeds to finer levels. Such FMG continuation is often attained in practice, even without explicit design, because larger numerical viscosity is naturally introduced on coarse grids.

The objective of FMG algorithms (and TME methods in particular) is an accurate approximation to the solution of the differential equations, not necessarily fast asymptotic residual convergence. An approximation is considered accurate if its algebraic error is below the level of the discretization error. The algebraic error is defined as the difference between the exact and approximate solutions of the discrete problem. The discretization error is defined as the difference between the exact solutions of discrete and differential problems.

On any grid in an FMG algorithm, we require only that the algebraic error after a few multigrid cycles be less than the discretization error. The latter can be quite accurately estimated by comparing solutions at different levels of the FMG algorithm. On the other hand, a fast residual convergence is considered as an important monitoring tool. In many practical cases, it is possible to develop a multigrid solver exhibiting fast residual convergence rates without compromising TME.

The goal of this paper is to review solution strategies leading to TME solution of fluid mechanics equations. The most general system we consider here is the time-dependent compressible NS equations written as

$$\partial_t \mathbf{Q} + \mathbf{R}(\mathbf{Q}) = 0, \tag{1}$$

where the conserved variables are $\mathbf{Q} \equiv (\rho u, \rho v, \rho w, \rho, \rho E)^T$, representing the momentum vector, density, and total energy per unit volume, and

$$\mathbf{R}(\mathbf{Q}) \equiv \partial_x \mathbf{F}(\mathbf{Q}) + \partial_y \mathbf{G}(\mathbf{Q}) + \partial_z \mathbf{H}(\mathbf{Q}), \tag{2}$$

$$\mathbf{F}(\mathbf{Q}) = \begin{pmatrix} \rho u^2 + p - 2\mu\partial_x u - \lambda(\nabla \cdot \mathbf{u}) \\ \rho uv - \mu(\partial_x v + \partial_y u) \\ \rho uw - \mu(\partial_x w + \partial_z u) \\ \rho u \\ \rho uE + up - \lambda u(\nabla \cdot \mathbf{u}) - \mu\tau_1 - \kappa\partial_x \epsilon \end{pmatrix},$$

$$\mathbf{G}(\mathbf{Q}) = \begin{pmatrix} \rho uv - \mu(\partial_x v + \partial_y u) \\ \rho v^2 + p - 2\mu\partial_y v - \lambda(\nabla \cdot \mathbf{u}) \\ \rho vw - \mu(\partial_y w + \partial_z v) \\ \rho v \\ \rho vE + vp - \lambda v(\nabla \cdot \mathbf{u}) - \mu\tau_2 - \kappa\partial_y \epsilon \end{pmatrix},$$

$$\mathbf{H}(\mathbf{Q}) = \begin{pmatrix} \rho uw - \mu(\partial_x w + \partial_z u) \\ \rho vw - \mu(\partial_y w + \partial_z v) \\ \rho w^2 + p - 2\mu\partial_z w - \lambda(\nabla \cdot \mathbf{u}) \\ \rho w \\ \rho wE + wp - \lambda w(\nabla \cdot \mathbf{u}) - \mu\tau_3 - \kappa\partial_z \epsilon \end{pmatrix},$$

where

$$\tau_1 = 2u\partial_x u + v(\partial_x v + \partial_y u) + w(\partial_x w + \partial_z u),$$

$$\tau_2 = 2v\partial_y v + u(\partial_x v + \partial_y u) + w(\partial_y w + \partial_z v),$$

$$\tau_3 = 2w\partial_z w + u(\partial_x w + \partial_z u) + v(\partial_y w + \partial_z v),$$

μ and λ are viscosity coefficients, and κ is the coefficient of heat conductivity.

A basic step in developing an efficient multigrid algorithm is to design an efficient relaxation procedure. For nonlinear problems, the relaxation updates to a current solution approximation are usually computed through Newton iterations. The full Newton linearization of the NS equations (as expressed by Equation 1) is a very complicated operator, and its solution (inversion) is too costly for practical applications. To reduce the computational cost without compromising efficiency, one can choose to relax a principal linearization of the system of equations. The principal linearization of a scalar equation contains the linearization terms that make a major contribution to the residual per unit change in the unknown variable. The principal terms thus generally depend on the scale, or mesh size, of interest. For example, the discretized highest derivative terms are principal on grids with small enough mesh size. For a discretized system of differential equations, the principal terms are terms that contribute to the principal terms of the system determinant.

To illustrate the idea of principal linearization, consider a nonlinear discrete thin-layer approximation of the convection-diffusion operator, in which the flow is parallel to the boundary (x-direction) and only the viscous terms associated with

variations in the y-coordinate normal to the boundary are retained:

$$N(u) \equiv u\partial_x^h u - \nu\partial_{yy}^h u, \tag{3}$$

where ∂_x^h and ∂_{yy}^h are discrete approximations to the first x-directional derivative and to the second y-directional derivative, respectively. A full Newton linearization (assuming constant viscosity) for a correction δu has three terms

$$\frac{\partial N}{\partial u}\delta u \equiv u\partial_x^h \delta u + \left(\partial_x^h u\right)\delta u - \nu\partial_{yy}^h \delta u. \tag{4}$$

To evaluate principality of the terms, one can start from an exact discrete solution; for example, the function $\delta u \equiv 0$ is the exact solution of the homogeneous equation $\frac{\partial N}{\partial u}\delta u = 0$. A unit change in the unknown variable, δu, is defined as a perturbation of the solution value at one grid point. Introducing this perturbation, the residual function becomes nonzero in the vicinity of the perturbed point. One can directly check which of the terms of the full linearization operator make major residual contributions. Three situations are encountered:

- High cell Reynolds number ($\frac{uh_y^2}{\nu h_x} \gg 1$). If the velocity function u is smooth and nondegenerate, i.e., the magnitude of velocity deviations in neighboring grid points is less than the local velocity magnitude, then the major contribution to the residual function is $O(u/h_x)$ and comes from the term $u\partial_x^h \delta u$; the second term contribution is $O(\partial_x^h u)$; and the viscous term contributes $O(\nu/h_y^2)$ that is much less than $O(u/h_x)$. Thus, only the first term $u\partial_x^h \delta u$ is principal. However, if the velocity field is not smooth, either because of a coarse mesh or proximity to a discontinuity, or if the absolute velocity value is small (stagnation flows), then the second term becomes principal as well.
- Low cell Reynolds number ($\frac{uh_y^2}{\nu h_x} \ll 1$). The major residual contribution comes from the viscous term, which is the only principal term.
- Medium cell Reynolds number ($\frac{uh_y^2}{\nu h_x} = O(1)$). This situation corresponds to the usual boundary layer assumption when convection balances diffusion. For smooth nondegenerate flows, the only subprincipal term is the second; the first and the third terms are principal. For nonsmooth or degenerate flows, all three terms are principal.

As an example of a system of nonlinear flow equations, we consider a discrete operator corresponding to a one-dimensional steady-state compressible inviscid flow:

$$N(\mathbf{q}) \equiv \begin{pmatrix} u\partial_x^h u + \dfrac{(\gamma-1)\epsilon}{p}\partial_x^h p \\ \gamma p\partial_x^h u + u\partial_x^h p \\ (\gamma-1)\epsilon\partial_x^h u + u\partial_x^h \epsilon \end{pmatrix}, \tag{5}$$

where $\mathbf{q} = (u, p, \epsilon)^T$ represents velocity, pressure, and internal energy and γ is

the ratio of specific heats. The full Newton linearization of this operator is given by

$$\frac{\partial N}{\partial \mathbf{q}}\delta\mathbf{q} \equiv$$

$$\begin{pmatrix} \left(\partial_x^h u\right) + u\partial_x^h & (\gamma-1)\epsilon\left(\frac{1}{p}\partial_x^h - \frac{\left(\partial_x^h p\right)}{p^2}\right) & (\gamma-1)\frac{\left(\partial_x^h p\right)}{p} \\ \gamma p\partial_x^h + \left(\partial_x^h p\right) & \gamma\left(\partial_x^h u\right) + u\partial_x^h & 0 \\ (\gamma-1)\epsilon\partial_x^h + \left(\partial_x^h \epsilon\right) & 0 & (\gamma-1)\left(\partial_x^h u\right) + u\partial_x^h \end{pmatrix}$$

$$\times \begin{pmatrix} \delta u \\ \delta p \\ \delta\epsilon \end{pmatrix}, \tag{6}$$

Assuming a smooth nondegenerate solution $\mathbf{q}$, the first simplification step is to eliminate lower derivative terms from each entry of the matrix in Equation 6. This simplification leads to an approximate linearization as

$$\begin{pmatrix} u\partial_x^h & (\gamma-1)\frac{\epsilon}{p}\partial_x^h & (\gamma-1)\frac{\left(\partial_x^h p\right)}{p} \\ \gamma p\partial_x^h & u\partial_x^h & 0 \\ (\gamma-1)\epsilon\partial_x^h & 0 & u\partial_x^h \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta\epsilon \end{pmatrix}. \tag{7}$$

The determinant of the matrix in Equation 7 is

$$u\partial_x^h\left(u^2\partial_x^h\partial_x^h\partial_x^h - c^2\partial_x^h\partial_x^h\partial_x^h - (\gamma-1)^2\frac{\epsilon}{p}\left(\partial_x^h p\right)\partial_x^h\right),$$

where the sound speed c relates to the internal energy ϵ as $c^2 = \gamma(\gamma-1)\epsilon$. Because p is nondegenerate, the last term in the parentheses is subprincipal in comparison to the other two terms. The third element in the first row of the matrix in Equation 7 does not contribute to the principal part of the determinant operator; therefore the principal linearization is defined as

$$\begin{pmatrix} u\partial_x^h & (\gamma-1)\frac{\epsilon}{p}\partial_x^h & 0 \\ \gamma p\partial_x^h & u\partial_x^h & 0 \\ (\gamma-1)\epsilon\partial_x^h & 0 & u\partial_x^h \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \\ \delta\epsilon \end{pmatrix}. \tag{8}$$

The notion of principal linearization is essentially based on the discrete formulation. The principal part of a differential operator may be defined as the limit of the principal part of the corresponding discrete operator as the mesh size h tends

to zero. So for smooth nondegenerate flows, the principal terms of the differential equations are the highest derivatives.

TME for solution of the NS system of differential equations can be achieved by exploiting the system factorizability. To illustrate the factorizability property, examples are given below for various fluid mechanics regimes. In all cases, we assume a smooth nondegenerate solution as defined previously.

2.1. Incompressible Navier-Stokes Equations

The steady-state incompressible NS equations can be written as

$$\begin{aligned} &Q_\nu \mathbf{u} + \nabla p = 0, \\ &\nabla \cdot \mathbf{u} = 0, \end{aligned} \tag{9}$$

where $\mathbf{u} = (u, v, w)^T$ is the velocity vector and $Q_\nu = \mathbf{u} \cdot \nabla - \nu\Delta$ is a convection-diffusion operator. The principal linearization operator is given by

$$\mathbf{L} \begin{pmatrix} \delta u \\ \delta v \\ \delta w \\ \delta p \end{pmatrix} = \begin{bmatrix} Q_\nu & 0 & 0 & \partial_x \\ 0 & Q_\nu & 0 & \partial_y \\ 0 & 0 & Q_\nu & \partial_z \\ \partial_x & \partial_y & \partial_z & 0 \end{bmatrix} \begin{pmatrix} \delta u \\ \delta v \\ \delta w \\ \delta p \end{pmatrix}, \tag{10}$$

where velocity $\mathbf{u}$ is fixed in the linearized convection-diffusion operator Q_ν. The determinant of the matrix operator $\mathbf{L}$ is

$$\det \mathbf{L} = -Q_\nu^2 \, \Delta. \tag{11}$$

2.2. Compressible Euler Equations

A nonconservative form of the Euler equations is given by

$$\begin{aligned} Q\mathbf{u} + \frac{1}{\rho} \nabla p = 0, \\ \rho c^2 \nabla \cdot \mathbf{u} + Qp = 0, \\ \frac{c^2}{\gamma} \nabla \cdot \mathbf{u} + Q\epsilon = 0, \end{aligned}$$

where $Q \equiv Q_0$ denotes the particular case of Q_ν with zero ($\nu = 0$) physical diffusion, and density, ρ, pressure, p, sound speed, c, and internal energy, ϵ, are related as

$$p = (\gamma - 1)\rho\epsilon, \tag{12}$$

$$c^2 = \gamma p/\rho. \tag{13}$$

The principal linearization is given by

$$\mathbf{L}\begin{pmatrix}\delta u\\ \delta v\\ \delta w\\ \delta p\\ \delta\epsilon\end{pmatrix}=\begin{bmatrix}Q & 0 & 0 & \frac{1}{\rho}\partial_x & 0\\ 0 & Q & 0 & \frac{1}{\rho}\partial_y & 0\\ 0 & 0 & Q & \frac{1}{\rho}\partial_z & 0\\ \rho c^2\partial_x & \rho c^2\partial_y & \rho c^2\partial_z & Q & 0\\ \frac{c^2}{\gamma}\partial_x & \frac{c^2}{\gamma}\partial_y & \frac{c^2}{\gamma}\partial_z & 0 & Q\end{bmatrix}\begin{pmatrix}\delta u\\ \delta v\\ \delta w\\ \delta p\\ \delta\epsilon\end{pmatrix}. \tag{14}$$

The determinant of the matrix operator $\mathbf{L}$ is

$$\det\mathbf{L}=Q^3[Q^2-c^2\Delta], \tag{15}$$

where Δ is the Laplace operator, and $Q^2-c^2\Delta$ represents the full-potential operator.

2.3. Compressible Navier-Stokes Equations

The nonconservative formulation corresponding to the steady-state version of Equation 1 is given by

$$\left((\mathbf{u}\cdot\nabla)-\frac{\mu}{\rho}\Delta-\frac{\hat{\lambda}}{\rho}\partial_{xx}\right)u-\frac{\hat{\lambda}}{\rho}(\partial_{xy}v+\partial_{xz}w)+\frac{1}{\rho}\partial_x p=0,$$

$$\left((\mathbf{u}\cdot\nabla)-\frac{\mu}{\rho}\Delta-\frac{\hat{\lambda}}{\rho}\partial_{yy}\right)v-\frac{\hat{\lambda}}{\rho}(\partial_{xy}u+\partial_{yz}w)+\frac{1}{\rho}\partial_y p=0,$$

$$\left((\mathbf{u}\cdot\nabla)-\frac{\mu}{\rho}\Delta-\frac{\hat{\lambda}}{\rho}\partial_{zz}\right)w-\frac{\hat{\lambda}}{\rho}(\partial_{xz}u+\partial_{yz}v)+\frac{1}{\rho}\partial_z p=0,$$

$$\rho c^2(\nabla\cdot\mathbf{u})+(\mathbf{u}\cdot\nabla)p+(\gamma-1)(-\kappa\Delta\epsilon+\Phi)=0,$$

$$\frac{c^2}{\gamma}(\nabla\cdot\mathbf{u})+(\mathbf{u}\cdot\nabla)p-\frac{\kappa}{\rho}\Delta\epsilon+\rho\Phi=0,$$

where

$$\Phi\equiv\mu(2(\partial_x u)^2+2(\partial_y v)^2+2(\partial_z w)^2+(\partial_x v+\partial_y u)^2+(\partial_x w+\partial_z u)^2 +(\partial_y w+\partial_z v)^2)+\lambda(\partial_x u+\partial_y v+\partial_z w)^2.$$

Assuming constant viscosity and heat conduction coefficients, the principal linearization operator $\mathbf{L}$, keeping the terms principal on both the viscous and

inviscid scales, is given by

$$\mathbf{L} = \begin{bmatrix} Q_{\frac{\mu}{\rho}} - \frac{\hat{\lambda}}{\rho}\partial_{xx} & -\frac{\hat{\lambda}}{\rho}\partial_{xy} & -\frac{\hat{\lambda}}{\rho}\partial_{xz} & \frac{1}{\rho}\partial_x & 0 \\ -\frac{\hat{\lambda}}{\rho}\partial_{xy} & Q_{\frac{\mu}{\rho}} - \frac{\hat{\lambda}}{\rho}\partial_{yy} & -\frac{\hat{\lambda}}{\rho}\partial_{yz} & \frac{1}{\rho}\partial_y & 0 \\ -\frac{\hat{\lambda}}{\rho}\partial_{xz} & -\frac{\hat{\lambda}}{\rho}\partial_{yz} & Q_{\frac{\mu}{\rho}} - \frac{\hat{\lambda}}{\rho}\partial_{zz} & \frac{1}{\rho}\partial_z & 0 \\ \rho c^2\partial_x & \rho c^2\partial_y & \rho c^2\partial_z & Q & (1-\gamma)\kappa\Delta \\ \frac{c^2}{\gamma}\partial_x & \frac{c^2}{\gamma}\partial_y & \frac{c^2}{\gamma}\partial_z & 0 & Q_{\frac{\kappa}{\rho}} \end{bmatrix}, \quad (16)$$

$$\det \mathbf{L} = Q_{\frac{\mu}{\rho}}^2 \left[\frac{\kappa c^2}{\gamma\rho}\Delta^2 + Q\left(-c^2\Delta + \frac{\kappa(\hat{\lambda}+\mu)}{\rho^2}\Delta^2 \right) - Q^2 \frac{\kappa+\hat{\lambda}+\mu}{\rho}\Delta + Q^3 \right], \quad (17)$$

where, nondimensionalizing by density and sound speed and applying Stokes hypothesis for the bulk viscosity term, the coefficients become $\mu/\rho = M_\infty/(\rho\ \mathrm{Re})$, $\kappa = M_\infty\gamma/(\mathrm{Re\ Pr})$, and $\hat{\lambda} = \lambda + \mu = \mu/3$; M_∞ is the freestream Mach number, and Re and Pr are Reynolds and Prandtl numbers respectively.

The approaches exploiting the factorizability property for efficient solution of the NS equations may be divided into two categories: (*a*) reformulating the target differential equations so that the principal linearization of the new formulation becomes uncoupled (usually triangular with the factors of the determinant on the main diagonal) and (*b*) modifying the equations for computing relaxation updates while keeping the original formulation for computing residuals.

For the subsonic compressible Euler equations, the first TME solvers exploiting factorizability of the system have been developed by Ta'asan (1993, 1994, 1995). These solvers represent examples of the reformulation approach. New canonical variables have been introduced, and in these variables, the Euler system of equations becomes block upper triangular with the main diagonal blocks consisting of the basic components of the system. Another reformulation approach toward achieving TME for solution of the Euler and incompressible NS equations is based on the pressure-equation formulation which effectively separates elliptic and hyperbolic factors of the system (Sidilkover & Ascher 1995, Sidilkover 1999b).

The approaches from the second category are more general, allowing considerable freedom in relaxation-scheme design because different schemes may be applied to different flow regions. However, the design itself is relatively simple only if the target discretization scheme is also factorizable, i.e., the determinant of the discrete principal linearization can be represented as a product of discrete factors, each of them approximating a corresponding factor of the determinant of the differential equations. A stumbling block that has prevented fast progress in developing TME

solvers was the lack of factorizable discretizations for many important application areas in fluid mechanics. Among the widely known discretization schemes, only staggered-grid formulations for incompressible and subsonic compressible-flow regimes are conveniently factorizable. Some centrally differenced collocated-grid formulations are factorizable as well, but the factors obtained in the corresponding discrete determinant are not often easily treated. The search for new factorizable discretization schemes is chiefly motivated by the need to derive discrete schemes with the resulting discretizations of scalar factors satisfying some desired properties (e.g., stability, correct alignment with the physical anisotropies, compactness, availability of an efficient relaxation scheme, etc.) Development of suitable factorizable discrete schemes for the NS equations is a challenging task. Much of the recent progress in achieving TME for CFD simulations is because new families of general factorizable collocated-grid discretization schemes are emerging (Sidilkover 1999a, Sidilkover 2001, Roberts et al. 2000, Roberts 2000, Diskin & Thomas 2002). The next two sections present some details of methods from the two categories.

3. EQUATION REFORMULATION STRATEGIES

As mentioned previously, the first TME solvers exploiting factorizability of the system have been developed by Ta'asan (1993, 1994, 1995). The original equations were reformulated in terms of canonical forms, in which the subsystems governed by hyperbolic operators are distinguished and treated separately, both in discretization and relaxation, from those governed by elliptic operators. The canonical variables for two dimensions are velocity (u, v), entropy s, and total enthalpy H. The elliptic operators are discretized with $h-$elliptic centered differences and solved with point relaxation and coarse grid corrections; the hyperbolic operators are discretized with upwind schemes and solved by marching techniques. Ta'asan (1994) was able to demonstrate solutions for the subsonic compressible Euler equations that converged with the same rates as the solution of the scalar full-potential equation. An additional advantage shown for this formulation was that the total artificial viscosity error was smaller than with other schemes because the upwinding was only used for the hyperbolic subsystems. The main disadvantage of this formulation is that it is not easily generalized to viscous and unsteady problems, especially in three dimensions.

An alternative pressure-equation formulation for the incompressible NS Equations 9 effectively separates the elliptic and hyperbolic factors of the system. The continuity equation is replaced with an equation for the pressure, as

$$\nabla (Q_v \mathbf{u} + \nabla p) - Q_v(\nabla \cdot \mathbf{u}) = 0. \tag{18}$$

Assuming a smooth nondegenerate flow, the principal linearization taken in the limit as mesh size h tends to zero is an upper triangular matrix with the main diagonal composed of the linearized convection-diffusion and Laplace operators,

$$\mathbf{L} \equiv \begin{bmatrix} Q_v & 0 & 0 & \partial_x \\ 0 & Q_v & 0 & \partial_y \\ 0 & 0 & Q_v & \partial_z \\ 0 & 0 & 0 & \Delta \end{bmatrix}. \tag{19}$$

The relaxation scheme is defined as

$$\mathbf{L}\delta\mathbf{q} = -R(\mathbf{q}), \tag{20}$$

where $R(\mathbf{q})$ is the new nonlinear formulation of the incompressible NS equations.

The determinant of the reformulated system is $Q_v^3\Delta$ as compared to $Q_v^2\Delta$ of the original system. Thus, additional boundary conditions that enforce zero-divergence need to be applied. The equations are uncoupled everywhere except at the boundaries, and some local relaxation is needed to relax the equations in this region. Some two-dimensional results are shown subsequently, demonstrating the efficiency of this approach. Although unexploited as yet, the approach applies equally well to time-dependent flows. This approach has met difficulties in generalizing to viscous compressible flows.

4. DISTRIBUTED RELAXATION STRATEGIES

The most general procedure exploiting factorizability of the target NS equations is distributed relaxation used earlier for special cases (Brandt & Dinar 1979, Brandt & Yavneh 1992). The general framework, first introduced by Thomas et al. (2002), can be outlined as follows:

In general, the simplest form of the differential NS equations corresponds to nonconservative equations expressed in primitive variables, e.g., taken as the set composed of velocity, pressure, and internal energy, $\mathbf{q} = (u, v, w, p, \epsilon)^T$. For a perfect gas, the primitive and conservative variables are connected through Equations 12, 13, and

$$\epsilon = E - \frac{1}{2}(u^2 + v^2 + w^2). \tag{21}$$

The time-dependent nonconservative equations are found readily by transforming the time-dependent conservative equations:

$$\frac{\partial \mathbf{q}}{\partial \mathbf{Q}}\left[\partial_t \mathbf{Q} + \mathbf{R}\right] = 0,$$

$$\partial_t \mathbf{q} + \frac{\partial \mathbf{q}}{\partial \mathbf{Q}}\mathbf{R} = 0,$$

where $\frac{\partial \mathbf{q}}{\partial \mathbf{Q}}$ is the Jacobian matrix of the transformation. For steady-state equations, the time derivative is dropped. In an iterative procedure, the correction $\delta\mathbf{q} \equiv \mathbf{q}^{n+1} - \mathbf{q}^n$, where n is an iteration counter, can be computed from the equation

$$\mathbf{L}\delta\mathbf{q} = -\frac{\partial \mathbf{q}}{\partial \mathbf{Q}}\mathbf{R}, \tag{22}$$

where the right side is a linear combination of the conservative residuals, and $\mathbf{L}$ is the principal linearization of the nonconservative operator at the scale h.

Although significantly simplified by retaining only principal terms, the system (Equation 22) is still a set of coupled equations containing elliptic and hyperbolic components. Therefore, collective Gauss-Seidel relaxation of $\mathbf{L}$ is not often effective. The distributed-relaxation method replaces $\delta\mathbf{q}$ in Equation 22 by $\mathbf{M}\delta\mathbf{w}$.

$$\mathbf{LM}\delta\mathbf{w} = -\frac{\partial \mathbf{q}}{\partial \mathbf{Q}}\mathbf{R}. \tag{23}$$

The resulting matrix $\mathbf{LM}$ becomes lower triangular. The diagonal elements of $\mathbf{LM}$ are composed ideally of the separable factors of the matrix $\mathbf{L}$ determinant. These factors are scalar differential operators of first or second order, so their efficient relaxation is a much simpler task than relaxing the entire system associated with $\mathbf{L}$. In relaxing scalar factors, any change introduced in the variables $\delta\mathbf{w}$ during relaxation is distributed, with the pattern of distribution matrix $\mathbf{M}$, to the primitive variables, $\delta\mathbf{q} = \mathbf{M}\delta\mathbf{w}$. The variables $\delta\mathbf{w}$ need not be explicitly used in relaxation; therefore, they are sometimes referred to as "ghost" variables. To obtain the optimal (textbook) efficiency, relaxation of each factor should incorporate the essential part of an efficient multigrid solver for its corresponding operator: Sometimes this essential part is just the relaxation part of that solver, sometimes this may even be an entire separate multigrid cycle of that solver applied over subdomains.

4.1. Distribution Matrices

For incompressible NS equations, an appropriate distribution matrix corresponding to the operator $\mathbf{L}$ of Equation 19 is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -\partial_x \\ 0 & 1 & 0 & -\partial_y \\ 0 & 0 & 1 & -\partial_z \\ 0 & 0 & 0 & Q_\nu \end{bmatrix}, \tag{24}$$

yielding the lower triangular operator

$$\mathbf{LM} = \begin{bmatrix} Q_\nu & 0 & 0 & 0 \\ 0 & Q_\nu & 0 & 0 \\ 0 & 0 & Q_\nu & 0 \\ \partial_x & \partial_y & \partial_z & -\Delta \end{bmatrix}. \tag{25}$$

A possible distribution matrix for the compressible Euler equations with the principal linearization operator $\mathbf{L}$ of Equation 14 is given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{\rho}\partial_x & 0 \\ 0 & 1 & 0 & -\frac{1}{\rho}\partial_y & 0 \\ 0 & 0 & 1 & -\frac{1}{\rho}\partial_z & 0 \\ 0 & 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

with

$$\mathbf{LM} = \begin{bmatrix} Q & 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 \\ \rho c^2 \partial_x & \rho c^2 \partial_y & \rho c^2 \partial_z & Q^2 - c^2\Delta & 0 \\ \frac{c^2}{\gamma}\partial_x & \frac{c^2}{\gamma}\partial_y & \frac{c^2}{\gamma}\partial_z & -\frac{c^2}{\rho\gamma}\Delta & Q \end{bmatrix}. \tag{27}$$

For compressible NS equations, one of the factors of the principal-linearization determinant in Equation 17 is very complicated. Instead of devising a suitable relaxation method for this scalar factor, one can employ a distributed relaxation partially decoupling the linear system associated with operator **L** as given by Equation 16. In particular, the distribution matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{\rho}\partial_x & 0 \\ 0 & 1 & 0 & -\frac{1}{\rho}\partial_y & 0 \\ 0 & 0 & 1 & -\frac{1}{\rho}\partial_z & 0 \\ \hat{\lambda}\partial_x & \hat{\lambda}\partial_y & \hat{\lambda}\partial_z & Q_{\frac{\lambda+\mu}{\rho}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

results in

$$\mathbf{LM} = \begin{bmatrix} Q_{\frac{\mu}{\rho}} & 0 & 0 & 0 & 0 \\ 0 & Q_{\frac{\mu}{\rho}} & 0 & 0 & 0 \\ 0 & 0 & Q_{\frac{\mu}{\rho}} & 0 & 0 \\ \mathcal{P}\partial_x & \mathcal{P}\partial_y & \mathcal{P}\partial_z & Q Q_{\frac{\lambda+\mu}{\rho}} - c^2\Delta & (1-\gamma)\kappa\Delta \\ \frac{c^2}{\gamma}\partial_x & \frac{c^2}{\gamma}\partial_y & \frac{c^2}{\gamma}\partial_z & -\frac{c^2}{\gamma\rho}\Delta & Q_{\frac{\kappa}{\rho}} \end{bmatrix}, \tag{29}$$

where $\mathcal{P} \equiv \rho c^2 + \hat{\lambda} Q$. The last two equations remain coupled, requiring a block 2×2 matrix solution in relaxation. This distributed relaxation scheme is still much less expensive than direct relaxation of matrix $\mathbf{L}$ requiring solution for a block 5×5 matrix.

4.2. Boundaries and Discontinuities

Boundaries and discontinuities introduce some additional complexity in distributed relaxation. The determinant of $\mathbf{LM}$ is usually of higher order than the determinant of $\mathbf{L}$. Thus, as a set of new variables, $\delta\mathbf{w}$ would generally need additional boundary conditions. In relaxation, it is usually possible to determine suitable boundary conditions for $\delta\mathbf{w}$ that satisfy the original boundary conditions for the primitive variables.

Distributed relaxation is applied throughout the entire computational domain, having the full effect away from boundaries in the regular (smoothly varying) flow field. The discrete equations near the boundaries are usually different from the interior equations; the relaxation equations are coupled near the boundaries, not decoupled as they are in the interior of the domain, even if expressed in terms of $\delta\mathbf{w}$. Thus, some local procedures should supplement the distributed-relaxation pass. The coupled near-boundary equations can be separated from other equations and solved (relaxed) with an appropriate method, such as direct solution or block-Newton-Kacmarcz relaxation. The smoothing by such general robust relaxation may be much slower per pass than the smoothing of the interior relaxation, so more passes will be needed near boundaries. However, the additional work will not seriously affect the overall complexity because the number of boundary (discontinuity) points is usually very small in comparison with the number of interior points.

Solution (or extensive relaxation) of the coupled near-boundary equations serves two purposes. The first is to provide convenient and reliable boundary conditions for the distributed-relaxation equations in the interior. The second purpose arises because in the outer multigrid cycle, efficient fine-to-coarse restriction of residuals near the boundaries is difficult to design; it depends on many factors, such as the shape of the boundary, the type of the boundary conditions, etc., and differs from the residual restriction in the interior. A general way to avoid efficiency degradation is to reduce residuals near the boundaries before restriction to a level that is significantly below the residual level characterizing the interior field. Having small residuals near the boundaries makes the precise form of the restriction operator less important.

In regions near discontinuities, more relaxations than in other places are required because of the large differences between the conservative and nonconservative operators. Additional local general relaxation sweeps should be applied in these regions, relaxing directly the conservative equations. One does not have to detect discontinuities explicitly other than through monitoring the residuals. A general

rule is to apply relaxation wherever the residuals are large in comparison with the rest of the field, which is a general consideration for attaining TME (Brandt 1977).

5. RECENT ADVANCES

5.1. Pressure-Equation Discretization

The original pressure-equation formulation of Sidilkover & Asher (1995) has been extended to general coordinates and implemented for lifting airfoils in inviscid flow by Roberts et al. (1997, 1999, 2002) and viscous flow by Swanson (2001). The results for viscous flow over a lifting airfoil at low Reynolds number are shown in Figure 1. An alternating line-implicit Gauss-Seidel relaxation is used to treat the mesh anisotropy that generally occurs in resolving viscous boundary layers on stretched grids. The computed pressure distributions are nearly indistinguishable from each other on the finer grids. The convergence rate actually increases as grids are refined and more levels in the FAS cycle are used; the 16×8 grid is always the coarsest grid in the multigrid computations. The convergence rates are comparable to the rates obtained for fully elliptic problems.

5.2. Staggered-Grid Factorizable Discretization

The first TME solver applying the distributed relaxation approach for solution to an entering flow problem for the incompressible NS equations was developed using a staggered-grid formulation (Brandt & Yavneh 1992). This formulation was extended to the compressible NS equations, and fast convergence rates were demonstrated (Thomas et al. 1999) for several viscous model problems. This latter work was the first experience with distributed relaxation in computation of compressible viscous flows in which a 2×2 block was relaxed simultaneously in line-implicit Gauss-Seidel relaxation. The coupling of boundary and interior relaxation was not completely treated at the time. A more complete study on TME for the incompressible equations at high-Reynolds-number conditions was performed (Thomas et al. 2001). In all these calculations, a staggered arrangement of variables on Cartesian grids has been used. With distributed relaxation, the system of equations has been decomposed (i.e., factored) everywhere, except near boundaries where the equations remained coupled. The results of the calculation are shown in Figure 2 for the viscous flow over a finite flat plate. The convergence of residuals and the algebraic-to-discretization error ratios in drag are shown versus multigrid cycles. The residual convergence rate is about the same as for the underlying Laplacian factor. The FMG solver, with just one FAS multigrid cycle per grid level and a total computational work equivalent to about 10 target-grid residual evaluations, converged the drag to the discretization accuracy.

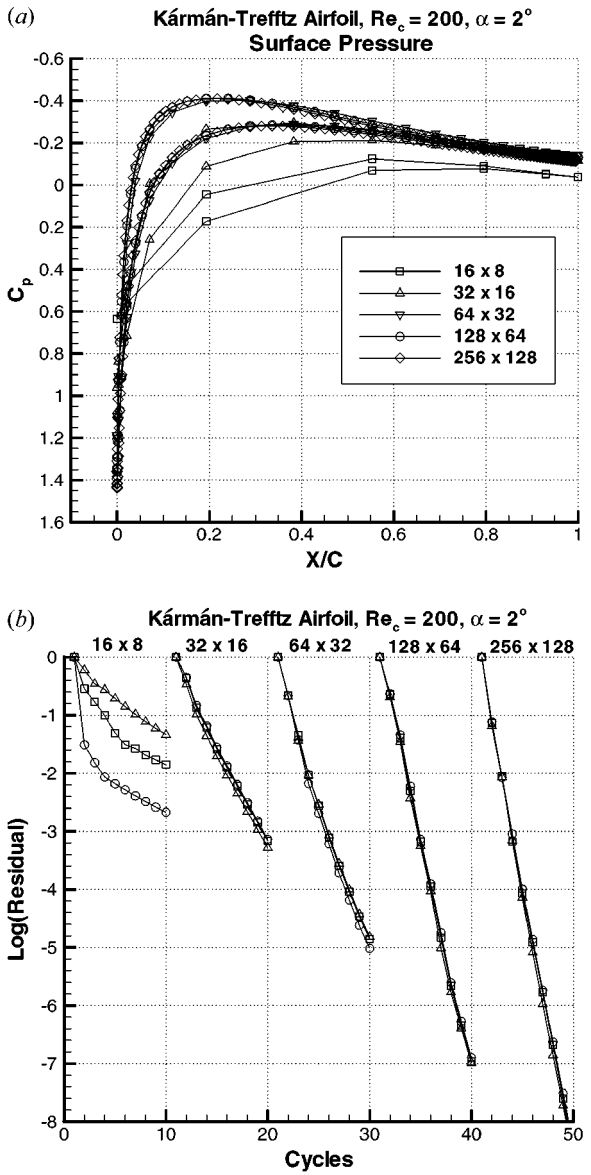

Figure 1 Computational results for the incompressible viscous flow over a lifting Kármán-Trefftz airfoil at $Re = 200$ and $\alpha = 2^\circ$. (*a*) Pressure distribution on the upper and lower surfaces for a sequence of grids. (*b*) Residual convergence with an FMG method using 10 FAS cycles on each grid from the coarsest 16×8 to the finest 256×128 (squares: *x*–momentum; triangles: *y*–momentum; circles: pressure equation).

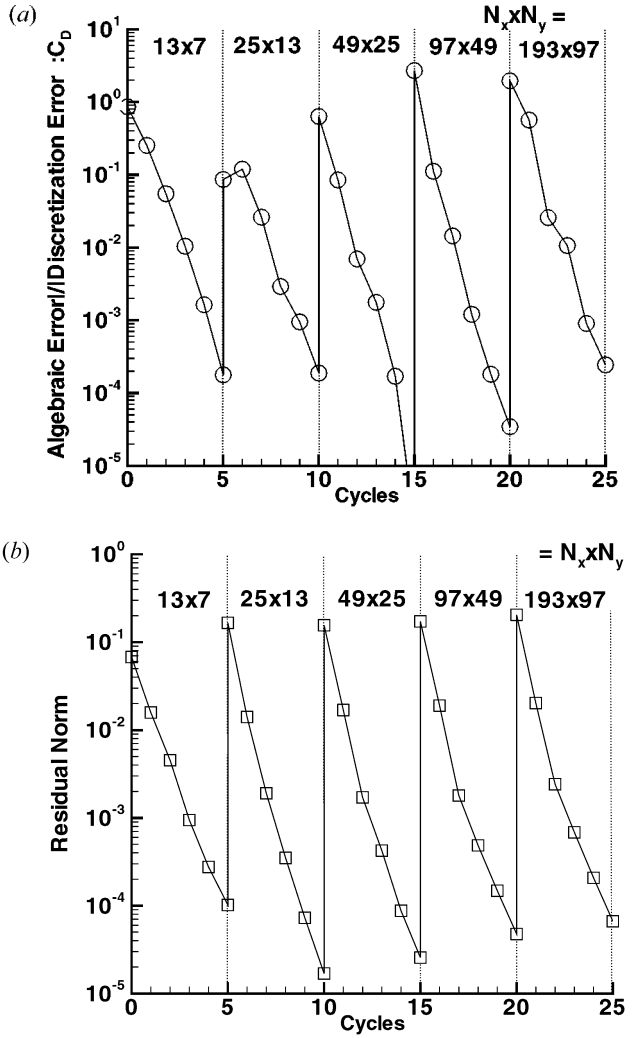

Figure 2 Convergence of errors in an FMG cycle using five FAS cycles on each grid from the coarsest 13×7 to the finest 193×97 for the incompressible viscous flow over a flat plate at $Re = 10{,}000$. (Reproduced from Thomas et al. 2001 with permission.) (*a*) Algebraic-to-discretization errors in drag, C_D. (*b*) L_2-norm of the residual.

5.3. New Factorizable Collocated-Grid Discretizations

Recently, a new multidimensional factorizable scheme for the Euler equations has been developed (Sidilkover 1999a) for Cartesian coordinates and extended through generalized coordinates to external lifting flows around airfoils with both subcritical and supercritical freestream Mach numbers (Roberts et al. 2000, Roberts 2001). The starting point for the scheme is the first-order discretization of the

flux-difference-splitting scheme (Roe 1986). Correction terms are added in the form of mixed derivatives to make the scheme both second-order accurate and discretely factorizable. The resulting scheme is second-order accurate and compact in comparison with other schemes; it is the first flux-difference-splitting scheme that is discretely factorizable in multiple dimensions. Discrete factorizability is achieved by using some nonstandard wide approximations for spatial derivatives to ensure that the identities

$$\begin{aligned} \partial_{xx}\partial_{yy} &= \partial_{xy}\partial_{xy}, \\ \partial_{xx}\partial_{y} &= \partial_{xy}\partial_{x}, \\ \partial_{yy}\partial_{x} &= \partial_{xy}\partial_{y} \end{aligned}$$

are satisfied on the discrete level. The determinant of the resulting scheme is composed of an upwind differenced convection factor and an *h*-elliptic approximation for the full-potential factor. The distributed relaxation is possible by using a left and right distribution matrix, although this has not been applied as yet.

In numerical tests performed for this scheme, the multigrid solver employed symmetric point collective Gauss-Seidel relaxation. Computations for subsonic and transonic channel flows with essentially grid-independent convergence rates have been presented (Roberts et al. 2000). Grid-independent convergence rates have also been attained for a flow with stagnation points (Roberts 2001). For lifting airfoil problems, the subsonic-flow convergence rates observed in multigrid V-cycles were quite fast (about 0.3 per cycle) and only slightly grid dependent. The rates somewhat deteriorate in transonic/supersonic computations, emphasizing the need for distributed relaxation. The scheme applies at low Mach numbers although it has yet to be extended to viscous flows. Multigrid results for the transonic flow over a lifting Kármán-Trefftz airfoil with a shock are shown in Figure 3. The pressure distribution shows the weak shock that is captured by the scheme. The residual convergence indicates some deterioration of the rate on the finer grids but the lift and drag coefficients are converged to below discretization error levels in only a few cycles.

Another example of a factorizable collocated-grid scheme is a scheme for the Euler equations with second-order central differencing for the off-diagonal first derivatives in Equation 14:

$$\mathbf{L^h} = \begin{bmatrix} Q^h & 0 & 0 & \frac{1}{\rho}\partial_x^{2h} & 0 \\ 0 & Q^h & 0 & \frac{1}{\rho}\partial_y^{2h} & 0 \\ 0 & 0 & Q^h & \frac{1}{\rho}\partial_z^{2h} & 0 \\ \rho c^2 \partial_x^{2h} & \rho c^2 \partial_y^{2h} & \rho c^2 \partial_z^{2h} & \bar{Q}^h & 0 \\ \frac{c^2}{\gamma}\partial_x^{2h} & \frac{c^2}{\gamma}\partial_y^{2h} & \frac{c^2}{\gamma}\partial_z^{2h} & 0 & Q^h \end{bmatrix}, \tag{30}$$

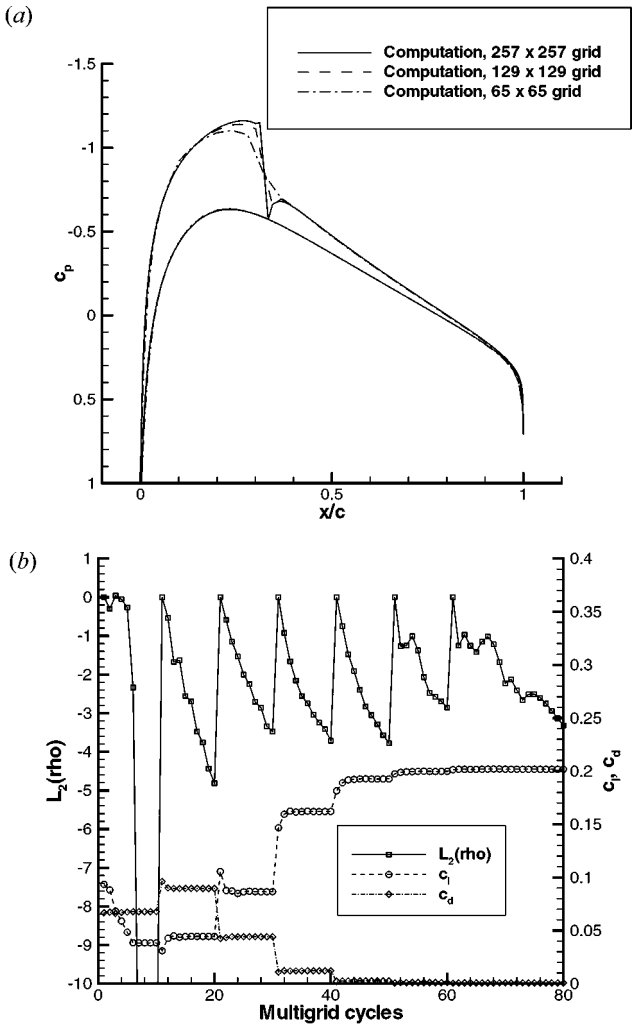

Figure 3 Computational results for compressible Euler flow over a lifting Kármán-Trefftz airfoil; $M_\infty = 0.70; \alpha = 1^\circ$. (Reproduced from Roberts 2001 with permission.) (*a*) Surface pressure (C_p) on a sequence of three grids. (*b*) Convergence of continuity equation residual (L_2(rho)), lift (c_l), and drag (c_d) in an FMG cycle with 20 FAS cycles on the finest 257×257 mesh and 10 FAS cycles on each coarser mesh.

where the discrete derivatives, ∂_x^{2h}, ∂_y^{2h}, ∂_z^{2h}, in all off-diagonal terms are the wide (with mesh spacing $2h$), second-order-accurate central-differencing approximations. All the diagonal terms, Q^h, except $\bar{Q}^h$ in the fourth equation, are discretized with the same second-order-accurate upwind (or upwind-biased) discretization scheme. In the subsonic regime ($|\mathbf{u}|^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2 < c^2$), the $\bar{Q}^h$-term is

discretized with a second-order-accurate downwind (or downwind-biased) discretization.

A typical difficulty associated with this type of scheme is a poor measure of *h*-ellipticity in the discrete approximation for the full-potential factor of the system determinant. The determinant of the matrix operator $\mathbf{L^h}$ is given by

$$(Q^h)^3[Q^h\bar{Q}^h - c^2\triangle^{2h}], \tag{31}$$

where $\triangle^{2h}$ is a wide discretization of the Laplace operator. The full-potential-operator approximation appearing in the brackets is not *h*-elliptic for subsonic Mach numbers.

Several approaches to cure the lack of *h*-ellipticity (mainly in applications to incompressible-flow equations) have been proposed in the literature (e.g., Armfield 1994, Brandt & Ta'asan 1985, Lang 2002). Some of the approaches are associated with introduction of additional terms increasing the measure of *h*-ellipticity in the system of equations, and others propose averaging (filtering) spurious oscillations.

An approach advocated by the authors is based on a mechanism that allows one to improve the *h*-ellipticity measure by obtaining any desired discretizations for the full-potential factor of the system determinant without compromising the discrete factorizability. The starting point is the discretization as given by Equation 30. The way proposed to improve the discrete full-potential operator is to change the discretization of $\bar{Q}^h$ to $\bar{Q}^h + \mathcal{A}^h$. Then the discrete full-potential operator is changed to

$$Q^h\mathcal{A}^h + Q^h\bar{Q}^h - c^2\triangle^{2h},$$

where $\mathcal{A}^h = (Q^h)^{-1}\mathcal{D}^h$, $\mathcal{D}^h = \mathcal{F}^h - (Q^h\bar{Q}^h - c^2\triangle^{2h})$, and $\mathcal{F}^h$ is a desired approximation for the full-potential factor. In smooth regions, $\mathcal{A}^h$ is second-order small (proportional to h^2); hence the overall second-order discretization accuracy is not compromised. The operator $(Q^h)^{-1}$ is a nonlocal operator, and its introduction can be effected through a new auxiliary variable ψ^h and a new discrete equation $Q^h\psi^h = \mathcal{D}^h p^h$.

Thus, the corrected discrete approximation to Equation 14 is defined as

$$\mathbf{L^h} = \begin{bmatrix} Q^h & 0 & 0 & 0 & \frac{1}{\rho}\partial_x^h & 0 \\ 0 & Q^h & 0 & 0 & \frac{1}{\rho}\partial_y^h & 0 \\ 0 & 0 & Q^h & 0 & \frac{1}{\rho}\partial_z^h & 0 \\ 0 & 0 & 0 & Q^h & -\mathcal{D}^h & 0 \\ \rho c^2\partial_x^h & \rho c^2\partial_y^h & \rho c^2\partial_z^h & 1 & \bar{Q}^h & 0 \\ \frac{c^2}{\gamma}\partial_x^h & \frac{c^2}{\gamma}\partial_y^h & \frac{c^2}{\gamma}\partial_z^h & 0 & 0 & Q^h \end{bmatrix} \tag{32}$$

The corresponding distribution matrix, $\mathbf{M^h}$, for distributed relaxation is defined as

$$\mathbf{M^h} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{\rho}\partial_x^h & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\rho}\partial_y^h & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{\rho}\partial_z^h & 0 \\ 0 & 0 & 0 & 1 & \mathcal{D}^h & 0 \\ 0 & 0 & 0 & 0 & Q^h & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{33}$$

so that the resulting matrix $\mathbf{L^h M^h}$ becomes lower triangular as

$$\mathbf{L^h M^h} = \begin{bmatrix} Q^h & 0 & 0 & 0 & 0 & 0 \\ 0 & Q^h & 0 & 0 & 0 & 0 \\ 0 & 0 & Q^h & 0 & 0 & 0 \\ 0 & 0 & 0 & Q^h & 0 & 0 \\ \rho c^2 \partial_x^h & \rho c^2 \partial_y^h & \rho c^2 \partial_z^h & 1 & \mathcal{F}^h & 0 \\ \frac{c^2}{\gamma}\partial_x^h & \frac{c^2}{\gamma}\partial_y^h & \frac{c^2}{\gamma}\partial_z^h & 0 & -\frac{c^2}{\gamma\rho}\Delta^{2h} & Q^h \end{bmatrix}. \tag{34}$$

The scheme as defined above is valid for nonconservative flows. A version to be used for distributed relaxation of conservative equations has also been designed (Diskin & Thomas 2001).

Numerical tests have been performed as yet only for a quasi-one-dimensional subsonic flow in a convergent/divergent channel. The accuracy was comparable to other schemes. With proper treatment of the distributed-relaxation equations in the regions adjacent to the boundaries, the convergence of the multigrid solver with a V-cycle and two relaxation sweeps per level is identical with the convergence of a similar multigrid solver for the discrete full-potential operator.

6. CONCLUDING REMARKS

Fundamentals and recent advances toward the development of TME solvers for fluid simulations have been presented. Accurate discrete approximations to the solution of the differential equations are obtained with FMG methods through fast reduction of algebraic errors below the discretization error level on each mesh. Strategies to attain TME for general fluid systems by exploiting factorizability of the governing differential equations are reviewed. These strategies include a reformulation of the target differential equations and a distributed-relaxation approach applied to the original equations. New discretizations and computations

demonstrating this methodology for inviscid and viscous flow simulations are presented.

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