Multigrid Analysis of Scattering by Large Quasi-Planar Structures

Oren E. Livne [†§] Achi Brandt [†¶] Amir Boag [‡]

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Abstract

We present a novel multigrid algorithm for the fast evaluation of the electrical field, required for the iterative analysis of three-dimensional scattering by a large but finite quasi-planar structure, whose height is small compared with its length and width. The algorithm is an extension of the two-dimensional version presented in [6], maintaining linear scaling with the number of spatial gridpoints.

Key terms: computational electromagnetics, fast multilevel summation, oscillatory kernels.

[†]Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel. Fax: +972-8-9344122

[‡]Department of Physical Electronics, Tel-Aviv University, Tel-Aviv, Israel. Email address: boag@eng.tau.ac.il

[§]Email address: livneo@wisdom.weizmann.ac.il

[¶]Email address: achi@wisdom.weizmann.ac.il

1 Introduction (from 1D Paper)

Wave scattering by periodic structures has been extensively treated by many researchers since the pioneering works of Lord Rayleigh. However, scattering by finite periodic and quasi-periodic geometries has received little attention, especially, in terms of numerically rigorous analysis. Examples of such structures are Fresnel lenses and planar reflector antennas as well as realistic finite Frequency Selective Surfaces (FSS) and patch antenna arrays.

Scattering by planar structures can be formulated in the integral equation form, which is conventionally discretized using the Method of Moments (MoM) [8]. The computational cost of solving matrix equations poses the main limitation on the electrical size of scattering problems that can be analyzed using MoM. The direct solution of the MoM matrix equations for electrically large geometries is impractical due to $O(N^3)$ complexity of direct solvers, N being the number of unknowns. The solution of very large linear systems is usually facilitated via iterative solvers, whose cost depends on the cost of matrix vector multiplication representing discretized evaluation of a field produced by a given current distribution. Recently, several fast direct and iterative algorithms for the solution of the planar scattering problem have been presented in [4, 3, 7].

In this paper, we propose an alternative iterative solution based on the general multilevel approach for fast evaluation of integral transforms with oscillatory kernels presented in [1]. We consider two-dimensional scattering by a

large but finite array of perfectly conducting strips. The problem requires the solution of a one-dimensional (1D) electric field integral equation. In the proposed approach, the integral equation is solved iteratively, and the main computational task is the repeated evaluation of the electric fields produced by the candidate solutions for the current distribution. Thus, we focus here on fast evaluation of the field. To this end, the 1D oscillatory kernel is represented as a linear combination of two "directional" kernels. Each such directional kernel is not oscillatory, but is asymptotically smooth: it is singular at short distances, but gets increasingly smoother at larger distances. As a result, it can be further decomposed into a local part (whose contribution to the total field is local and inexpensively computed), and a smooth part, which can be efficiently recovered from its values on a coarser grid. The task of evaluating the original field over N nodes is thus replaced by the task of evaluating the contribution of the smooth part of the kernel on a coarser resolution of about $N/2$ nodes, which may still be too large to compute directly. Consequently, further coarsening is applied recursively until a grid is reached on which the task can be computed directly in $O(N)$ operations. This implies that the original field evaluation can be carried out in only $O(N)$ computer operations, thereby reducing the $O(N^2)$ complexity required for a direct evaluation.

2 The Computational Task

Consider a problem of Three-Dimensional (3D) Transverse Magnetic (TM) scattering from a finite quasi-planar perfectly conducting structure, whose height is small compared with its length and width. The geometry and excitation are assumed uniform in the z direction. For clarity, the following formulation is presented for a strip of unit width depicted in Fig. **??**. Generalizing the proposed method to geometries comprising multiple strips is relatively straightforward and will be discussed in §4. We will assume the structure's landscape Γ to be a parametric curve

$$\Gamma := \{(s, \eta(s)), 0 \leq s \leq L_1\}, \qquad |\eta(s)| \leq \frac{L_2}{2}, \tag{1}$$

where L_1 is the width of the structure and $L_2 \ll L_1$ is its height (see Fig. **??**). Generalizations to other geometries are discussed in 4. The structure is illuminated by a z-polarized incident field $\mathrm{E}^{\mathrm{inc}}$ with a harmonic time dependence $e^{i\omega t}$, which is assumed and suppressed throughout the paper. The scattering from the strip is analyzed using the Method of Moments (MoM). An Electric Field Integral Equation (EFIE) is constructed in terms of a z-directed electric current J. The EFIE, which requires the total electric field along the perfect conductors to vanish, yields

$$\mathrm{E}^{\mathrm{inc}}(x_1, x_2) = \frac{\eta k}{4} \int_{(y_1,y_2)\in\Gamma} \mathrm{H}_0^{(2)}(k\sqrt{(x_1-y_1)^2+(x_2-y_2)^2})\mathrm{J}(y_1, y_2)ds, \qquad \forall (x_1, x_2) \in \Gamma, \tag{2}$$

or in terms of the parametrization (1) ($x_1(s) = y_1(s) = s, x_2(s) = y_2(s) = \eta(s)$),

$$\mathrm{E}^{\mathrm{inc}}(t) = \frac{\eta k}{4} \int_0^{L_1} \mathrm{H}_0^{(2)}(k\sqrt{(t-s)^2 + (\eta(t) - \eta(s))^2})\mathrm{J}(s)ds, \qquad \forall x \in [0, L_1], \tag{3}$$

where $\mathrm{H}_0^{(2)}$ denotes the zero-order Hankel function of the second kind, η is the free-space impedance, and $k = 2\pi/\lambda$, λ being the free-space wavelength. Iterative solution of the integral equation (3) for J calls for repeated evaluation of its (3), thus we will concentrate on the fast numerical evaluation of (3).

We will not explicitly impose any smoothness requirements on η, rather assume that a desirable discretization of (3) has been already formed; that is, the task (3) is replaced with the multi-summation

$$\mathrm{E}(t_j) = \sum_{m=1, m\neq j}^{N} \mathrm{H}_0^{(2)}(k\sqrt{(t_j - s_m)^2 + (\eta(t_j) - \eta(s_m))^2})\mathrm{J}(s_m), \qquad j = 1, \ldots, N, \tag{4}$$

where $\{t_m = s_m = (m - \frac{1}{2})h_1\}_{m=1}^N$ represents a uniform grid over $[0, L_1]$ with N segments of width h_1, which satisfies $h_1 < \lambda/10$, although J and E may be discretized in general on different grids $\{t_j\}_j, \{s_m\}_m$. The values $\{\mathrm{J}(s_m)\}_m$ may be general and depend on the grid point values of the continous J and the discretization weights. Since a direct summation of (4) requires $O(N^2)$ computer operations, our goal will be to evaluate it instead in only $O(N)$ operations.

3 The Evaluation Algorithm

Our fast evaluation of E is based on reducing the task (4) to a sum of one dimensional (1D) integrals, each may be evaluated in $O(N)$ operations using the algorithm presented in [6].

3.1 Derivation of the Algorithm

Unlike the 1D case [6], we cannot directly utilize any smoothness property of $\mathrm{H}_0^{(2)}(k\sqrt{(t-s)^2+(\eta(t)-\eta(s))^2})$ as a function of s. However, $H(x_1,y_1,x_2,y_2) = \mathrm{H}_0^{(2)}(k\sqrt{(x_1-y_1)^2+(x_2-y_2)^2})$ is smooth as a function of y_2 for sufficiently large $|x_1-y_1|$ (see App. **??**). Consequently, we can approximate H by a p_2th order polynomial interpolation in y_2 from its values on a uniform grid $\{Y_{2,\beta}\}_{\beta=1}^{N_2}$ with meshsize $h_2 := L_2/N_2$ and N_2 points over $[-L_2/2,-L_2/2]$, which may include $O(p_2)$ points to the left of $-L_2/2$ and to the right of $L_2/2$ to keep the interpolation central. Namely,

$$H(t_j,s_m,\eta(t_j),\eta(s_m)) = \sum_{\beta\in\sigma_m} \omega_\beta(\eta(s_m))H(t_j,s_m,\eta(t_j),Y_{2,\beta}) + O(\varepsilon_I), \quad (5)$$

where $\omega_\beta(\eta(s_m))$ are the weights of interpolation from the gridpoints $Y_{2,\beta}$ to y_2, and ε_I is a bound on the interpolation error. It follows that

$$\begin{aligned} \mathrm{E}(t_j) &= \textstyle\sum_{j=1,j\neq m}^{N}\sum_{\beta\in\sigma_m}\omega_\beta(\eta(s_m))H(t_j,s_m,\eta(t_j),Y_{2,\beta})\mathrm{J}(s_m) + O(\varepsilon_I) \\ &= \textstyle\sum_{\beta=1}^{N_2}\sum_{j=1,j\neq m}^{N} H(t_j,s_m,\eta(t_j),Y_{2,\beta})\omega_\beta(\eta(s_m))\mathrm{J}(s_m) + O(\varepsilon_I). \end{aligned} \quad (6)$$

Note that $\omega_\beta(y_{2,m}) = 0$ for all $\beta\notin\sigma_m$, for any $y_{2,m} := \eta(s_m), m = 1,\ldots,N$. Similarly, we can replace in (6) $H(t_j,s_m,\eta(t_j),Y_{2,\beta}))$ by a p_2th order poly-

nomial interpolation in x_2 from its values on the uniform grid $\{X_{2,\alpha}\}_{\alpha=1}^{N_2} = \{Y_{2,\alpha}\}_{\alpha=1}^{N_2}$. Thus we obtain up to an $O(\varepsilon_I)$ error

$$\mathrm{E}(t_j) = \sum_{\alpha=1}^{N_2} \omega_\alpha(\eta(t_j)) \sum_{\beta=1}^{N_2} E_{\alpha\beta}(t_j) := E_S(t_j), \qquad j = 1, \ldots, N, \tag{7}$$

where

$$E_{\alpha\beta}(t_j) := \sum_{j=1, j\neq m}^{N} g(|t_j - s_m|; C_{\alpha\beta}) J_\beta(s_m), \qquad \alpha, \beta = 1, \ldots, N_2, j = 1, \ldots, N, \tag{8}$$

$$J_\beta(s_m) := \omega_\beta(\eta(s_m))\mathrm{J}(s_m), \qquad \beta = 1, \ldots, N_2 \tag{9}$$

and

$$g(|t-s|; C) := \mathrm{H}_0^{(2)}(k\sqrt{(t-s)^2 + C^2}), \qquad C_{\alpha\beta} := X_{2,\alpha} - Y_{2,\beta}. \tag{10}$$

Each of the N_2^2 integrals $\{E_{\alpha\beta}\}_{\alpha\beta}$ is a "1D multi-summation task" similar to the discretized integral transform considered in [6], with the kernel $g(|t-s|; C)$ replacing $\mathrm{H}_0^{(2)}(k|t-s|)$ and J_β replacing J. The function $g(\cdot; C)$ is asymptotically smooth in the sense of [1, §4] for any C, as shown in App. **??**. Thus we can directly apply the fast multilevel evaluation algorithm of [6] to obtain each of the $E_{\alpha\beta}$'s to accuracy ε in $O(N \log(1/\varepsilon))$ computer operations. The algorithm's parameters (interpolation order p and softening distance $S = sh_1$) should be tuned as explained in §3.2. The integrals are then summed up using (7) to obtain E_S to accuracy ε.

The last neccessary computational stage consists of "local corrections" in the manner of [6, (18) and §2.4] for all the points x_1, y_1, x_2, y_2 that satisfy

$$r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \geq S_2 := s_2 h_1, \tag{11}$$

for some softening distance S_2. s_2 should be chosen as explained in §3.2. The corrections ensure that the interpolation error $\varepsilon_I = O(h_2^p(\partial^p/\partial y_2)^p H)$ is annihilated whenever exceeding $O(\varepsilon)$. To sum up, the multi-summation (4) is evaluated to accuracy $O(\varepsilon)$ using the following steps:

1. *Splitting:* calculate the "splitted" $\{J_\beta\}_\beta$ from (9).
2. *1D summations:* carry out the tasks (8) for all $\alpha, \beta = 1, \ldots, N$ using the multilevel algorithm of [6].
3. *Aggregation:* compute E_S defined by (7).
4. *Local corrections:* add the local corrections as in cite[(18) and §2.4]Amir to E_S, for all x_1, y_1, x_2, y_2 satisfying (11).

Following this basic description of the evaluation algorithm, the next section provides more details on how to choose the various parameters involved.

3.2 Complexity and Precision

The values of S, S_2, p, p_2, h, h_2 should be determined to minimize the computational work W under the constraint of a desired evaluation error ε. In this section we do not opt for an exact optimization and expressions for the work and error, rather present the orders of magnitude main terms in them.
In step 1 of the algorithm we perform $O(N_2 N)$ operations. At step 2 , (**??**) (see App. **??**) implies that the main terms in the evaluation error of $E_{\alpha\beta}$ is

$$O(h^p|(\frac{\partial}{\partial r})^p g(r; C_{\alpha\beta})|) = O((\frac{ph}{r})^p + r^{-p}). \tag{12}$$

Thus, in order to obtain a desirably small interpolation error (say, ε_1) in the 1D algorithm of [6], the softening distance $S = sh$ should satisfy $p/S = C_1$ where C_1 is a constant of order 1 and somewhat smaller than 1 (e.g. $C_1 = \frac{1}{4}$), i.e. $s = O(p)$. The Thus, the argument of [6, §2.5–2.6] on optimal values of p, s, h holds for $g(\cdot; C)$, and we obtain $p = s = O(\log(1/\varepsilon_1))$ and $h = O(1)$ at all levels of the multilevel evaluation of $\{E_{\alpha\beta}\}_{\alpha\beta}$ (plus $O(\log\log(1/\varepsilon_1))$ negligible contributions). By choosing $\varepsilon_1 = \varepsilon/N_2^2$, the total work in computing $\{E_{\alpha\beta}\}_{\alpha\beta}$ is $O(NN_2^2\log(N_2^2/\varepsilon))$ for an accuracy ε.

Step 3 (computing (7)) involves $O(N_2^2 N)$ operations. The interpolation error introduced by (5) is bounded by

$$\varepsilon_I = O(\left(\frac{kh_2^2}{r}\right)^p + \left(\frac{p_2 h_2}{r}\right)), \tag{13}$$

where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, as explained in App. **??**. Consequently, we obtain a controlled ε_I by choosing

$$S_2 = C_2 h_2 \max\{kh_2, p_2\}, \qquad C_2 = O(1) < 1. \tag{14}$$

Finally, we determine p_2 and h_2, assuming p, S are determined by the previous argument and S_2 is given by (14). For $kh_2 \le 1$, the computational work per finest grid node in step 4 is $O(p_2 L_2/h_1)$, since the local region consists of $O(N_2 s_2) = O(N_2 p_2 h_2/h_1)$ and $N_2 = L_2/h_2$. The total work per finest grid node is therefore

$$\frac{W}{N} = O(\frac{L_2^2}{h_2^2}\log(\frac{L_2^2}{h_2^2\varepsilon})\frac{p_2 L_2}{h_1}), \tag{15}$$

which corresponds to the contributions of steps 2 and 4. The work in steps 1 and 3 is smaller and may be neglected hereafter. The total evaluation error

is $O(\varepsilon + \varepsilon_I)$, hence W should be minimized subject to the constraint

$$\frac{p_2 {h_2}^{p_2}}{S_0} = \varepsilon \tag{16}$$

In fact, the right hand side in (16) should be $O(e)$, but constants were omitted. Since S_2 was chosen by (14), (16) immediately implies $p_2 = O(\log(1/\varepsilon))$. Substituting back into (15) implies

$$\frac{W}{N} = O(\frac{L_2^2}{h_2^2}\log(\frac{L_2^2}{h_2^2\varepsilon}) + \frac{L_2\log(\varepsilon)}{h_1}).$$

W is minimized if and only if

$$\frac{d}{d\tilde{h}_2}(\frac{W}{N}) \propto \frac{d}{d\tilde{h}_2}(\frac{\log(\tilde{h}_2)}{\tilde{h}_2^2}) = 0, \qquad \tilde{h}_2 := \frac{L_2 h_2}{\varepsilon}$$

. The latter is satisfied when $\tilde{h}_2 = \sqrt{e}$, and the optimal h_2 is thus of order $L_2/\sqrt{\varepsilon}$, implying $N_2 = O(\sqrt{\varepsilon})$. However, for resonable ε (e.g. less than 1) the integer N_2 would be too small to allow a $p_2 th$ order interpolation, thus is constrained on the lower bound $N_2 = q = O(\log(1/\varepsilon))$. The total work in this case is

$$W = O(N((\log(\frac{1}{\varepsilon}))^3 + \frac{L_2}{h_1}\log(\frac{1}{\varepsilon}))). \tag{17}$$

4 Concluding remarks

In the previous section we described a fast multilevel field evaluation algorithm, for a quasi-planar strip scattering problem. The algorithm scales linearly with the number of spatial gridpoints resolving the structure. The algorithm can be extended in various directions such as the following.

The multiple strip problem can be addressed by the same algorithm of §3, where J is defined to be zero outside the strips. Since the coarse grids usually extend beyond the edges of the physical strips, at some coarsening stage they cover the gaps between the strips, thereby reducing the computational task to a single-strip-type task. Complicated geometries can be efficiently addressed by local refinements (see for example [5]).

The presented approach can be adapted to the fast *multilevel solution* of the integral equation (3) for the current, basically at the cost of *one* evaluation of its right-hand side (see [2]). Moreover, The softening [6, §2.2] can be used to design discretization schemes whose resolution (to a given accuracy) *does not depend on* k. The multilevel evaluation complexity presented in this paper scales $O(k^{\frac{2}{3}})$ for large k, thus should be then also modified to get rid of that computational factor.

The extension to d-dimensional scattering problems can be effected by following the approach of [1, §5] and is also discussed in [6, §3].

References

[1] A. Brandt, *Multilevel computations of integral transforms and particle interactions with oscillatory kernels*, Comp. Phys. Comm. **65** (1991), 471–476.

[2] A. Brandt and A. A. Lubrecht, *Multilevel matrix multiplication and the fast solution of integral equations*, J. Comput. Phys. **90** (1990), 348–370.

[3] W. C. Chew and C. C. Lu, *A fast recursive algorithm to compute the wave scattering solution of a large strip*, J. Comput. Phys. **107** (1993), 378–387.

[4] J. Jandhyala, B. Shanker, and E. Michielssen, *Fast algorithm for the analysis of scattering by dielectric rough surfaces*, J. Optical Soc. Am. A **15 (7)** (1998), 1877–1885.

[5] O. E. Livne and A. Brandt, *N roots of secular equation in $O(N)$ operations*, submitted to SIAM J. Matrix Anal. Appl.

[6] O.E. Livne, A. Brandt, and A. Boag, *Multigrid analysis of scattering by large quasi-planar structures*, IEE Micro. Opt. Tech. Let. **?** (2002), ??–??

[7] E. Michielssen, A. Boag, and W. C. Chew, *Scattering from elongated objects: direct solution in $O(N \log^2 N)$ operations*, IEE Proc. Microwaves and Antennas **143 (4)** (1996), 277–283.

[8] A. F. Peterson, S. L. Ray, and R. Mittra, *Computational methods for electromagetics*, IEEE Press, New York, 1998.