Remarks on the wave-ray multigrid solvers for Helmholtz equations

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Abstract

Helmholtz equations form a family of equations arising in acoustic, electromagnetic, and other applied fields. The processes that they describe are often considered on large computational domains and accompanied by non-local boundary conditions. In this paper we describe a "wave-ray" algorithm for solving Helmholtz equations with radiation boundary conditions, and discuss the measures needed to be taken to make the solver efficient and accurate. In addition, we introduce an algebraic multigrid (AMG) solver for Helmholtz eigenvalue problems with variable potentials.

Keywords: Multigrid; Helmholtz; Wave-ray; Eigenvalue; Radiation

1. Introduction

Helmholtz equations are well known as being hard to solve, both analytically and numerically. The source of the inefficiency of traditional numerical techniques, including standard multigrid methods, is an existence of some erroneous frequencies that have poor convergence. In this paper we describe a multigrid approach that overcomes the difficulties that complicate the process of finding accurate numerical solutions. This approach is beneficial if the Helmholtz equations are considered on large (infinite) domain and accompanied by the radiation boundary conditions. We also introduce an algorithm that solves Helmholtz eigenvalue problems for arbitrary potentials. As the model problems, we will use two-dimensional Helmholtz equations with a constant potential, $V(x) = k^2$,

$$\Delta u(x,y) + k^2 u(x,y) = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2, \tag{1}$$

and the one-dimensional eigenvalue problem with a variable positive potential

$$u_{xx}(x) + V(x)u(x) = \lambda u(x), \quad x \in \Omega \subset \mathbb{R}, \tag{2}$$

where d, the size of Ω, is much bigger than $2\pi/\sqrt{V}$.

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2. Wave-ray algorithm

Let us consider model problem (1). Fourier components of the form

$$e^{i(w_1 x + w_2 y)}, \quad w_1^2 + w_2^2 \approx k^2, \tag{3}$$

that almost satisfy homogeneous Helmholtz equations cause the main slowness in applying standard multigrid techniques to Helmholtz equations. They are almost invisible on the fine grids and poorly approximated on the coarse grids. Simple Fourier analysis of the erroneous components left unreduced by the multigrid cycle can be represented as

$$e(x,y) = \sum_{l=1}^{L} a_l(x,y)\, e^{i(k_1^l x + k_2^l y)}, \tag{4}$$

where $e^{i(k_1^l x + k_2^l y)}$, $l = 1, \ldots, L$ are *lattice* components with frequencies uniformly distributed on the circle of radius k. The wave-ray approach suggests approximating smooth *ray* envelope functions $a_l(x,y)$, $l = 1, \ldots, L$ rather than the error (4). To enable such approximations, the solver employs ray coordinate systems with one axis being aligned with the component propagation direction (k_1, k_2), *ray* coarse grids (due to the smoothness of ray functions), *ray* equations, derived from Helmholtz equations and (4). As input from the fine wave grids the algorithm uses the envelope functions of the wave residual that can be represented in the same form as (4). In the Full Approximation Scheme (FAS) formulation, this approach also allows a natural introduction

of the radiation boundary conditions, a frequent companion of Helmholtz equations.

3. Solution accuracy

The quality of the numerical solution in most multigrid solvers is controlled by the properties of the finest, target grid, e.g. its mesh-size and discretization order; while coarse grids are used to provide a smooth correction to the fine-grid solution. In the wave-ray algorithm, however, the coarse ray grids have an extended role. They are used to approximate all components (3). In addition, these are the grids where the boundary conditions are introduced. As a result, an accurate resolution of some solution components on the coarse grids becomes important.

3.1. Truncation errors

Analysis of the truncation errors shows that in the case of Helmholtz equations the most significant impact on the solution accuracy is a so-called *phase* error, the error that occurs because of the accumulated differences between the wavelengths of differential and corresponded discrete principal components. Both wave and ray representations may introduce significant phase errors. The wave phase error is given by [3]:

$$E = O\big(kd(k^2h^2)\big). \tag{5}$$

The ray phase errors are estimated by

$$E_1 = O(kd/L^2), \qquad E_2 = O(kd/L^3). \tag{6}$$

in propagation and perpendicular directions, correspondingly. A refinement of the finest wave grid leads to reduction of (5). To reduce (6), however, it is lattice discretization that should be refined (by increasing L) along with coarsening of ray grids.

3.2. Other errors

Two other types of errors may decrease solution accuracy: errors introduced by interpolation of boundary conditions from coarse ray grids, and errors caused by reflection from artificial boundaries that also occur on the coarsest ray grids. To diminish their influence on the solution quality at the target domain, it is sufficient just to extend ray computational domains.

4. Eigenvalue problem

The next step in extending applicability of the wave-ray approach is to develop a technique that efficiently finds an accurate approximation for an arbitrary potential. In the more general formulation, this task can be considered as an eigenvalue problem for Helmholtz operators. As initial approximations to solution of (2) on the finest, *wave* grid

$$u_1^0 = e^{-i\sqrt{V(x)-\lambda}} \quad \text{and} \quad u_2^0 = e^{i\sqrt{V(x)-\lambda}}$$

are chosen. Exact solutions, u_1 and u_2 are represented as

$$u_1(x) = v_1(x)u_1^0(x) + v_2(x)u_2^0(x)$$

and

$$u_2(x) = v_3(x)u_1^0(x) + v_4(x)u_2^0(x),$$

where $v_i(x)$ are unknown ray functions. The AMG approach, based on Galerkin approximation, is used to define coarse grid equations for v_i. The interpolation operators, I_1 and I_2, are used to interpolate a constant unit function defined on the finest ray grid into u_1^0 and u_2^0, correspondingly. Simple linear interpolation is used on the coarser ray grids. The solver is a regular V cycle; λ is updated on the coarsest ray grid. Interpolation from coarse grids to the next fine grids is performed operators I_1 and I_2, correspondingly, creating eventually a new and better, approximation on the finest grid. The procedure can then be repeated with the current fine grid approximation used as u_1^0 and u_2^0.

5. Conclusions

The approach described in Section 4 is currently under development; preliminary one-dimensional results for (2) with periodic boundary conditions has been obtained so far. The ultimate goal of this research is to develop a solver that will calculate a multiscale eigenbasis (see, for example, [1]) for (2), and, eventually, will be applicable to two-dimensional eigenvalue problems.

References

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