Brandt, Achi, Weizmann Institute of Science, Israel, Multi-level approaches to large scale problems.

Most massive computational tasks facing us today have one feature in common: they are mainly governed by *local* relations in some low (eg 2 or 3) dimensional space or grid. Such are all differential problems, including flows, electromagnetism, magnetohydrodynamics, quantum mechanics, structural mechanics, tectonics, tribology, general relativity, etc., etc., as well as non-differential problems like those in statistical physics (critical phenomena, lattice gauge theory, etc.), geodesy, multivariate interpolation, image reconstruction, pattern recognition, many design, optimization and mathematical programming problems, and most integral equations, including various tomography problems. This common feature can be exploited very effectively by multi-level (multigrid) solvers, which combine local processing on different scales with various types of inter-scale transfers of residuals and corrections.

The computational cost of such solvers is essentially as low as cost can ever be. For example, if $Lhuh = fh$ is a grid approximation to a steady-state differential boundary-value problem $Lu = f$, it can be solved to truncation level (i.e., to errors smaller than $|u - uh|$, in any desired norm) in just few Lh-work-units, where an Lh-work-unit is the amount of work involved in just *expressing* Lh and fh at all gridpoints. The 5-point approximation to the Poisson equation, for instance, is being solved on the CDC CYBER 205 at the rate of 5 million equations per second.

Moreover, these multigrid solvers can fully exploit very high degree of parallel processing, and for very small extra work can incorporate local grid adaptation or provide a sequence of extra solutions to a sequence of similar problems (eg for design and optimization purposes, in continuation processes, etc.). The same efficiency is obtained whether L is linear or not (no linearization is required), elliptic or non-elliptic, and can be maintained even when L or u are strongly discontinuous, or when the problem include free surfaces, shocks, boundary shape singularities, etc., or when eigenproblems, or some inverse (eg system identification) problems, are solved instead.

A similar efficiency is also obtained for time-dependent problems. Given any initial conditions at $t = 0$, the multi-level solution to $\partial uh/\partial t = Lhuh - fh$ at *any* target time (finite or infinite), can still be calculated, to within truncation errors, in just few Lh-work-units (hence equivalent to just few explicit time steps, in case Lh and fh are time independent).

At the limit of large grids, the exact number of required work units can *rigorously* be predicted by local mode analysis. It can rigorously be shown to be independent of the boundary shape and the boundary conditions. For L which is a general *system* of equations, procedures have been devised to derive this number, and the type of local processing that should be used, directly from the factors of the h-principal part of the determinant of L.

The multilevel apparatus can also be used to obtained better discretizations; for example by employing discrete operators with good *local* properties (eg stability and admittance of discontinuities) in the local processing, while *other* operators, excelling in *global* attributes (eg higher accuracy, conservation, etc.) are used in the fine-to-coarse residual transfers; or, by adding progressively finer levels over *local* subdomains, each possibly with its own, *locally* adapted, coordinate systems

solution representations on different grids: in case of wave problems, for example, the coarser the grid the more resolution should be introduced in terms of "rays" (thus obtaining a hybrid of wave equations and geometric optics, efficiently treating problems which neither of them can). In case of singular integral equations, each integration should be performed partly on finer grids (near the singularity) and partly on coarser grids. And so on. The storage needed for the discretization may also be reduced very much – by using finer grids only piecewise and temporarily, only to supply defect corrections to coarser grids.

Fast multi-level techniques have been designed for solving fairly general sparse algebraic systems and for calculating *determinants* of linear systems defined on large grids. This can for example be applied in calculating the fermions effect in quantum chromo-dynamics. Such and other statistical physics calculations may also very much benefit from multi-level Monte-Carlo techniques, speeding up statistical convergence, especially of long range correlations.

In case of highly nonlinear minimization and constrained minimization problems, including discrete-state (eg spin systems) and combinatorial (eg traveling salesman) problems, multi-levelling not only greatly accelerates convergence, but is also essential for escaping local minima with large attraction basins.

Some very large linear programming problems have been solved by multi-level interaction of simplex processes, reducing simplex solution times by several orders of magnitude.