The Weizmann Institute of Science
Department of Applied Mathematics and Computer Science
P.O.B. 26, Rehovot, 76100, Israel
FAX: +972-8-344-122
E-mail: zasl@wisdom.weizmann.ac.il

Multiscale Algorithm for Atmospheric Data Assimilation
Part I. Multiscale Iterative Process

Achi BRANDT and Leonid Yu. ZASLAVSKY

Department of Applied Mathematics and Computer Science,
The Weizmann Institute of Science, Rehovot, 76100, Israel

Technical Report CS95-09

March 1995

Rehovot, Israel
1995

Multiscale Algorithm for Atmospheric Data Assimilation

Part I. Multiscale Iterative Process [1]

ACHI BRANDT AND LEONID YU. ZASLAVSKY [2] [3]

Department of Applied Mathematics and Computer Science,
The Weizmann Institute of Science, Rehovot, 76100, Israel

March 8, 1995

Abstract

A multiscale algorithm for the problem of optimal statistical interpolation of the observed data has been developed. This problem includes the calculation of the vector of the "analyzed" (best estimated) atmosphere flow field w^a by the formula

$$w^a = w^f + P^f H^T y,$$

where the quantity y is defined by the equation

$$(HP^f H^T + R)y = w^o - Hw^f,$$

using the given model forecast first guess w^f and the vector of observations w^o. H is an interpolation operator from the regular grid to the observation network, P^f is the forecast error covariance matrix, and R is the observation error covariance matrix.

At this initial stage the case of univariate analysis of single level radiosonde height data is considered. The matrix R is assumed to be diagonal, and the matrix P^f to be given by the formula $P^f_{ij} = \sigma^f_i \mu_{ij} \sigma^f_j$, where μ_{ij} is a smooth decreasing function of the distance between the ith and the jth points.

Two different multiscale constructions can be used for efficient solving the problem of optimal statistical interpolation: a technique for fast evaluation of the discrete integral transform $\sum_i P^f_{ij} v_j$, and a fast iterative process which effectively works with a sequence of spatial scales. In this paper we describe a multiscale iterative process based on a multiresolution simultaneous displacement technique and a localized variational calculation of iteration parameters.

1 Introduction

The problem of optimal statistical interpolation of the observed data includes the calculation of the vector of the "analyzed" (best estimated) atmosphere flow field w^a by the formula

$$w^a = w^f + P^f H^T y,$$

[1] This report contains a preliminary version of the material which will be presented in [1].

[2] The second author has been partly supported by a Sir Charles Clore Post-Doctoral Fellowship.

[3] *Corresponding author address:* Dr. Leonid Zaslavsky, Department of Applied Mathematics and Computer Science, The Weizmann Institute of Science, P. O. B. 26, Rehovot, 76100, Israel.
E-mail: zasl@wisdom.weizmann.ac.il

where the quantity y is determined from the equation

$$(HP^fH^T + R)y = w^o - Hw^f, \tag{1}$$

using the given model forecast first guess w^f and the vector of observations w^o ([2], [3], [4]). Typically, w^f is defined on a regular spherical grid, while the set of observations w^o is defined on an irregular network of observation points. H is an interpolation operator from the regular grid to the observation network, P^f is the forecast error covariance matrix, and R is the observation error covariance matrix.

The observation error covariance matrix R is assumed to be diagonal, with

$$R_{ii} = (\sigma_i^o)^2.$$

The forecast error covariance function $P^f(x_1, x_2)$ is defined for any pair of points x_1 and x_2 on the sphere by the formula

$$P^f(x, z) = \sigma^f(x)\mu(x, z)\sigma^f(z),$$

where the forecast error correlation function $\mu(x, z)$ is described as a smooth decreasing function of the distance between the points x and z ([4]). The matrices P^f and μ are the restrictions of functions $P^f(x, z)$ and $\mu(x, z)$ on the regular latitude-longitude grid.

The purpose of this paper is to develop a concept of a fast multiscale iterative process for solving y from equation (1) when the observation network is strongly inhomogeneous in space. At this initial stage the case of univariate analysis of single level radiosonde height data is considered.

In this paper we consider only convergence properties of the iterative process. Accordingly, in the computer experiments all summations have been performed straightforwardly. An effective procedure for the fast evaluation of the integral transform on the sphere, based on the Brandt & Lubrecht approach ([6]), will be presented separately.

Without loss of generality equation (1) can be replaced by the system of equations

$$\sum_j \tilde{P}_{ij}^f y_j + R_{ii} y_i = w_i^o - (Hw^f)_i, \tag{2}$$

where

$$\tilde{P}_{ij}^f = \tilde{\sigma}_i^f \mu_{ij} \tilde{\sigma}_j^f$$

for the ith and jth observation points x_i and x_j,

$$\mu_{ij} = \mu(x_i, x_j),$$

and

$$\tilde{\sigma}_i^f = (H\sigma^f)_i.$$

(While the matrix P is defined for the points of the regular grid and interpolated to the observation network using the operator H, the matrix $\tilde{P}$ is defined by the same formula directly on the observation network.)

Indeed, the difference

$$HPH^T y - \tilde{P}y$$

may be treated as an additional source in the right hand side. This small term is non-principal at all scales and can easily be taken into account in iterations.

Since we want to deal explicitly with the smoothness properties of the kernel μ_{ij}, we replace (2) by

$$\sum_j \mu_{ij} u_j + (\frac{\sigma^o}{\tilde{\sigma}^f})_i u_i = \frac{1}{\tilde{\sigma}^f}(w_i^o - (Hw^f)_i), \tag{3}$$

where $u_i = \tilde{\sigma}_i^f y_i$. The system of equations (3) can be written in matrix notation as

$$Au = f, \tag{4}$$

where the matrix A is symmetric and presumably positive definite.

2 General strategy

It is important to understand why a usual iterative process like Jacobi, Gauss-Seidel or the conjugate gradient method, when applied to the solution of (4) converges slowly. Let us consider, for example, the simplest iterative process

$$u^{(n+1)} = u^{(n)} + \omega r^{(n)}, \tag{5}$$

where the residual

$$r^{(n)} = f - Au^{(n)},$$

and parameter $\omega \approx (\rho(A))^{-1}$, with $\rho(A)$ being the spectral radius of the operator A.

The process (5) reduces effectively the error components corresponding to the large eigenvalues λ_l, such that

$$\omega\lambda_l \sim 1,$$

while the error components corresponding to the small eigenvalues λ_s, for which

$$\omega\lambda_s \ll 1,$$

are reduced slowly [5]. Since the summation $\sum_j \mu_{ij} u_j$ in (3) is made with a smooth kernel, eigenvectors of A corresponding to the large eigenvalues are (mostly) spatially smooth, and eigenvectors of A corresponding to small eigenvalues are oscillatory in space. Therefore, one cannot define one particular value of ω which would give an essential reduction of all spectral error components.

The effect described above is well-studied for the case when (4) is obtained as a grid approximation of the continuous integral equation. A few multiscale techniques based on multigrid ([6], [7]) and wavelet ([8]) approaches were developed in 1990's. Unfortunately, these techniques cannot be applied to the considered problem straightforwardly because of the strong inhomogeneity of the observation network.

The central idea of the approach developed below is to filter sequentially spectral components of $r^{(n)}$ and to choose for each of them a value of the iteration parameter which gives an essential reduction of the corresponding error component.

The major particular difficulty which has been overcome successfully in this work is how to define variable pass spatial filters $\mathcal{F}_h$ depending on the scale parameter h, for a field defined on a very inhomogeneous network. An appropriate filter will be described in §3.

When some component $\mathcal{F}_h r^{(n)}$ of the residual $r^{(n)}$ has been filtered, one should next calculate the correction vector. A possible simple way to do it is to use a *scalar* iteration

parameter ω_h, i. e. to calculate the correction as $\omega_h \mathcal{F}_h r^{(n)}$. Then the modified iterative process (5) can be written as

$$u^{(n+1)} = u^{(n)} + \omega_h^{(n)} \mathcal{F}_h r^{(n)}, \tag{6}$$

where the iteration parameter depends on the scale h in some way.

An intrinsic disadvantage of schemes like (6) is that one *global* iteration parameter $\omega_h^{(n)}$ is determined for the entire domain. The optimal correction at a spatial point x_i should however depend only on the residual values at points located at most few h from x_i. Therefore, in §4, we construct a procedure for calculating an iteration parameter $\omega_{h,i}^{(n)}$ for each point x_i *locally*, using only the values of the residual components in some area around it. This means that the iterative process which we construct, can be written as

$$u^{(n+1)} = u^{(n)} + \Omega_h^{(n)} \mathcal{F}_h r^{(n)}, \tag{7}$$

where

$$\Omega_h = \mathrm{diag}(\omega_{h,i}^{(n)}).$$

We discuss the structure of the multilevel iterative cycle in §5.

3 Spatial Filter

We now define a filter applicable to functions defined on a very irregular discrete network. Obviously, we want our filter to work like a usual spectral high pass filter in the data dense regions.

What do we want to get in data sparse regions? Suppose we have an observation point s which is separated from other points by distance

$$d_s = \min_{p \neq s} \mathrm{dist}(s, p).$$

We would like to take into account the residual component r_s only on the scales h large enough:

$$h \simeq d_s \text{ and } h > d_s,$$

and neglect it on the small scales h:

$$h \ll d_s.$$

We define the filter $\mathcal{F}_h$ which satisfies these requirements by the formula:

$$(\mathcal{F}_h r)_i = r_i - \gamma_i \sum_j r_j \exp(-\frac{1}{2}\frac{\mathrm{dist}(i,j)}{h}), \tag{8}$$

where h is the current scale, and the parameter γ_i is defined by the formula

$$\gamma_i = (\sum_j \exp(-\frac{1}{2}\frac{\mathrm{dist}(i,j)}{h}))^{-1}.$$

Note that the filter can be calculated efficiently using the fast summation procedure.

4 Calculation of Ω_h

The *scalar* iteration parameter $\omega_h^{(n)}$ in (6) can be determined from the variational condition of minimizing the Euclidean norm of the scale h component of the new residual $r_h^{(n+1)}$:

$$\min_{\omega_h^{(n)}} ||\mathcal{F}_h(r^{(n)} - \omega_h^{(n)} A\mathcal{F}_h r^{(n)})||^2,$$

where $||\cdot||$ is the Euclidean norm on the observation network:

$$||u||^2 = (u,u),$$

and

$$(u,v) = \sum_i u_i v_i,$$

the summation being made over the observation points. This condition leads to the formula

$$\omega_h^{(n)} = \frac{(p,q)}{(q,q)}, \tag{9}$$

where

$$p = \mathcal{F}_h r^{(n)},$$
$$q = \mathcal{F}_h A p.$$

As mentioned above, some disadvantage of this formula is its globality. In order to localize it, we use a family of weighted Euclidean norms. Let us introduce

$$(u,v)_{l,i} = \sum_j u_j v_j \exp(-\frac{1}{2}\frac{\mathrm{dist}(i,j)}{l}),$$

where the summation is made over the observation points, and

$$||u||_{l,i}^2 = (u,u)_{l,i}.$$

Now we can define the matrix Ω_h in (7). We choose Ω_h as follows:

$$\Omega_h^{(n)} = \mathrm{diag}(\omega_{h,i}^{(n)}),$$

where

$$\omega_{h,i}^{(n)} = \frac{(p^{(n)}, q^{(n)})_{3h,i}}{(q^{(n)}, q^{(n)})_{3h,i}},$$
$$p^{(n)} = \mathcal{F}_h r^{(n)},$$
$$q^{(n)} = \mathcal{F}_h A p^{(n)}.$$

Note again that the fast summation procedure can be used to calculate efficiently $\omega_{h,i}^{(n)}$.

5 Structure of the multiscale iterative cycle

In order to define the order of the multiresolution iterations, we have to prescribe some spatial scale to each iteration. The current spatial scale is determined by the formula

$$h = H \cdot 2^{1-level(n)},$$

where H is the largest scale, and $level(n)$ is the level prescribed for the nth iteration. If $level(n) = 0$, the filter is not used. The multiscale iterative algorithm can be written as follows:

DO n = 1, NITER

Residual calculation

$r^{(n)} = f - Au^{(n)}$

IF $level(n) > 0$ THEN

Definition of the current scale

$h = H \cdot 2^{1-level(n)}$

Filtering

$p^{(n)} = \mathcal{F}_h r^{(n)}$

$q^{(n)} = \mathcal{F}_h A p^{(n)}$

Calculation of the iteration parameters

$\omega_{h,i}^{(n)} = \frac{(p^{(n)}, q^{(n)})_{3h,i}}{(q^{(n)}, q^{(n)})_{3h,i}}$

$\Omega_h^{(n)} = \mathrm{diag}(\omega_{h,i}^{(n)})$

Calculation of the new approximation to the solution

$u^{(n+1)} = u^{(n)} + \Omega_h^{(n)} p^{(n)}$

ELSE

Filtering is not used

$p^{(n)} = r^{(n)}$

$q^{(n)} = Ap^{(n)}$

Calculation of the iteration parameter

$\omega_h^{(n)} = \frac{(p^{(n)}, q^{(n)})}{(q^{(n)}, q^{(n)})}$

Calculation of the new approximation to the solution

$u^{(n+1)} = u^{(n)} + \omega_h^{(n)} p^{(n)}$

ENDIF

ENDDO

We have used for our initial tests the standard V(2,2) multilevel cycle with 3 iterations at the *0*th level ([9], [10]). This means that the function $level(n)$ is periodic:

$$level(LC + k) = level(k) \quad \text{for any } k > 0,$$

and

$$level(n) = \begin{cases} NLVL - k + 1, & \text{if } n = 2 \cdot k - 1 + l, \ k = 1, 2, \ldots, NLVL, \ l = 0, 1; \\ 0, & \text{if } n = 2 \cdot NLVL + l, \ l = 1, 2, 3; \\ k, & \text{if } n = 2 \cdot NLVL + 2 + 2k + l, \ k = 1, 2, \ldots, NLVL, \ l = 0, 1; \end{cases}$$

where $NLVL$ is the finest level number, and $LC = 4 \cdot NLVL + 3$.

6 Numerical results

At this initial stage of the work we made all the numerical tests with radiosonde height data only. The forecast correlation function is modeled by the formula

$$\mu(x_1, x_2) = \left(1 + \frac{(\text{dist}(x_1, x_2))^2}{L^2}\right)^{-1.208},$$

where $\text{dist}(x_1, x_2)$ is the three-dimensional distance between points x_1 and x_2, and L is the correlation distance, $L = 951$ km.

The radiosonde station locations, values of σ^o, σ^f, and $w^o - Hw^f$ have been obtained from the Data Assimilation Office of NASA/Goddard Space Flight Center. The data file contains model parameters and radiosonde height observations from 715 stations. Observation error variances σ_i^o were taken to be equal to 14.6 m for all radiosonde stations. Forecast error variances σ_i^f vary from point to point, ranging from 18 m to 35 m.

We made our experiments with $NLVL = 5$. The scales which were used are shown in Table 1. The results of our experiments are shown in Table 2.

Table 1. Scale structure

Level number	1	2	3	4	5
Scale h, km	10 000	5 000	2 500	1 250	625

Table 2. Convergence of the iterative procedure

Multiscale cycle	L_2 norm of the residual	Rate of the norm decreasing
Initial	2.5510^{+1}	
1	6.8610^{-1}	0.027
2	3.7610^{-2}	0.054
3	2.0510^{-3}	0.054
4	8.8710^{-5}	0.043

7 Discussion: Further Improvements

The algorithm described above represents the first step towards fast and efficient solver for the atmospheric data assimilation problem. It has been shown that the multiresolution algorithm can provide a fast solver. As long as the number of measurements is moderate (less than few thousands, say), this algorithm by itself is already effective enough. However, for larger sets of measurements, a major part of the work per cycle can be saved by a more advanced multiscale algorithm, featuring the following improvements.

- First, as already mentioned, a fast evaluation of the operator P^f, i. e., of the multisummation in Eq. (3), can be based on the method of [6] (see also [7]). That method can also be used for fast filtering.

- Secondly, at each scale h, in any region where the number of of measurements per $O(h \times h)$ cell is large, the multitude of residuals can be replaced by their proper local averages on a grid with mesh size $O(h)$. Similarly in such regions, the correction $u^{(n+1)} - u^{(n)}$ will also be calculated on such a grid, and will only later be interpolated to the measurement points. Actually, the residual averagings and the correction interpolation will not be done directly between the finest (measurements) level and each scale-h level, but will be transferred sequentially through all the intermediate levels.

- Thirdly, the residual filtering can be replaced by distributive relaxation (as in [6]). The latter is simple in the grid regions mentioned above. In other regions the filtering techniques may be easier to apply.

All these improvements will reduce the work of a cycle to a few operations per measurements, hopefully retaining the same convergence rates.

8 Acknowledgements

This paper presents part of a joint work with Drs. Steve Cohn, Greg Gaspari and Arlindo da Silva of the Data Assimilation Office, NASA/Goddard Space Flight Center. The authors thank them for raising their interest in the problem of atmospheric data assimilation, providing the observation data and the model parameters.

The authors are indebted to Drs. Alexander Kheifets, Ilya Rivin, Mikhail Shapiro and Kees Venner for helpful discussions.

References

[1] Brandt, A. and L. Yu. Zaslavsky, 1995: Multilevel Algorithm for Atmospheric Data Assimilation. *Copper Mountain Conference on Multigrid Methods Copper Mountain, Colorado. April 2-7, 1995.*

[2] Daley, R., 1991: *Atmospheric Data Analysis.* Cambridge University Press, 457 pp.

[3] Baker, W. E., S. C. Bloom, J. S. Woollen, M. S. Nestler, E. Brin, T. W. Schlatter, and G. W. Branstator, 1987: Experiments with a three-dimensional statistical objective analysis scheme using FGGE data. *Mon. Wea. Rev.*, **115**, 273-296.

[4] Pfaendter, J., S. Bloom, D. Lamich, M. Seabloom, M. Sienkiewicz, J. Stobie, and A. da Silva, 1995: Documentation on the Goddard Earth Observing System (GEOS) Data Assimilation System - Version 1, NASA Tech. Memorandum 104606, Vol. 4, NASA, Goddard Space Flight Center, Greenbelt, MD.

[5] Varga, R. S., 1962: *Matrix Iterative Analysis.* Prentice-Hall, Inc., N. J. 322 pp.

[6] Brandt, A. and A. A. Lubrecht, 1990: Multilevel Multi-Integration and Fast Solution of Integral Equations. *J. Comput. Phys.*, **90**, 348-370.

[7] Brandt, A., 1991: Multilevel algorithms of integral transforms and particle integrations with oscillatory kernels. *Comp. Phys. Commun.*, **65**, 24-38.

[8] Alpert, B., G. Beylkin, R. Coifman, and V. Rokhlin, 1993: Wavelet-like bases for the fast solution of second-kind integral equations. *SIAM J. Sci. Comput.*, **14**, 159-184.

[9] Brandt, A., 1977: Multi-level adaptive solutions to boundary-value problems. *Math. Comp.*, **31**, 333-390.

[10] Hackbusch, W., 1985: *Multi-Grid Methods and Applications.* Springer Verlag, Berlin, 377 pp.