DISJOINT PAIRS FOR $GL(n,\mathbb{R})$ AND $GL(n,\mathbb{C})$

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Abstract. We show the disjointness property of Klyachko for $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$.

1. Introduction

A finite family of subgroups, each endowed with a character, was introduced by Klyachko in [Kly83]. Over a finite field this family provides a model for GL_n (see [IS91]). In this note we consider the archimedean case and prove pairwise disjointness of Klyachko pairs in a sense we now explain.

Definition 1.1. Let G be a real reductive group, H_i a closed subgroup and χ_i a continuous character of H_i, $i = 1, 2$. We say that $(G, (H_1, \chi_1))$ and $(G, (H_2, \chi_2))$ are *disjoint pairs* if for every irreducible Fréchet representation π of G we have

$$\dim \mathrm{Hom}_{H_1}(\pi, \chi_1) \cdot \dim \mathrm{Hom}_{H_2}(\pi, \chi_2) = 0.$$

In order to formulate our main result we introduce some notation. In Section 3 we use this notation without further mention. Let F equal either $\mathbb{R}$ or $\mathbb{C}$ and let ψ be a non-trivial character of F. Set $X_n = GL_n(F)$, let U_n be the subgroup of upper uni-triangular matrices in X_n and let ψ_n be the character of U_n defined by

$$\psi_n(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}), \quad u \in U_n.$$

Let $w_n = (\delta_{i,n+1-j}) \in G_n$ and let

$$J_n = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix} \in G_{2n}.$$

Consider the symplectic group Sp_{2n} defined by

$$Sp_{2n} = \left\{ g \in G_{2n} : {}^t g J_n g = J_n \right\}.$$

Fix $n \in \mathbb{N}$. For $0 \leq r \leq n$ such that $n - r = 2k$ is even consider the Klyachko subgroup $H_{r,n}$ of G defined by

$$H_{r,n} = \left\{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} : u \in U_r,\ X \in M_{r\times 2k}(F),\ h \in Sp_{2k} \right\}$$

and let $\psi_{r,n}$ be the character of H_r defined by

$$\psi_{r,n} \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u).$$

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Theorem 1.1. *The pairs* $(G,(H_r,\psi_r))$, $0\le r\le n$, $r\equiv n \mod 2$ *are pairwise disjoint.*

The analogous result was obtained in [IS91] over a finite field and in [OS08] over a non-archimedean local field.

2. Generalities

We refer to [AG08] for the notions of Schwartz functions and Schwartz distributions in the following setting. For a Nash manifold X we denote by $\mathcal{S}(X)$ the Fréchet space of $\mathbb{C}$ valued Schwartz functions on X and by $\mathcal{S}^*(X)$ its topological dual, the space of Schwartz distributions.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. For every smooth topological representation (π,V) of G, i.e. a topological representation such that for any $v\in V$ the map

$$g\mapsto \pi(g)v : G\to\ V$$

is smooth, V is also naturally a $\mathfrak{g}$-module. For a character $\chi: G\to\mathbb{C}^*$ let

$$V^{G,\chi}=\{v\in V : g\cdot v=\chi(g)v \text{ for all } g\in G\}.$$

If χ is the trivial character we also denote $V^{G,\chi}$ by V^G. Denote by $V\otimes\chi$ the representation of G (or $\mathfrak{g}$-module structure) on V twisted by χ. Note that $V^{G,\chi}=(V\otimes\chi^{-1})^G$.

Let $V^{\mathfrak{g}}=\{v\in V:\mathfrak{g}v=0\}$ be the subspace annihilated by $\mathfrak{g}$ and $V_{\mathfrak{g}}=V/\mathfrak{g}V$ the space of co-invariants. Set also $V^{\mathfrak{g},\chi}=(V\otimes\chi^{-1})^{\mathfrak{g}}$. Then evidently

$$V^{G,\chi}\subseteq V^{\mathfrak{g},\chi}. \tag{2.1}$$

As an example let G be a Nash group with a Nash action on a Nash manifold X. It induces an action of G on $\mathcal{S}(X)$ and on $\mathcal{S}^*(X)$. In this case $\mathcal{S}^*(X)$ is a smooth topological representation of G.

For every $x\in X$ denote by Gx the G-orbit of x, by G_x the stabilizer of x in G and by $\mathfrak{g}_x$ the Lie algebra of G_x. Let $T(X)$ be the tangent bundle of X. For a Nash submanifold Y of X let $N_Y^X=(T(X)|_Y)/T(Y)$ be the normal bundle to Y in X and let $\mathrm{CN}_Y^X=(N_Y^X)^*$ be the conormal bundle. For a point $y\in Y$ we denote by $N_{Y,y}^X$ (resp. $\mathrm{CN}_{Y,y}^X$) the fiber over y in N_Y^X (resp. CN_Y^X), i.e. the normal (resp. conormal) space to Y in X at the point y.

If X is itself a Nash group and H_i is a closed subgroup $i=1,2$ then we shall always consider the left action of $H_1\times H_2$ on X defined by $((h_1,h_2),x)\mapsto h_1xh_2^{-1}$ for $h_1\in H_1$, $h_2\in H_2$ and $x\in X$.

The following is an immediate consequence of [SZ, Theorem 2.3 (b)]. The statement in [loc. cit.] is in terms of tempered generalized functions rather then Schwartz distributions. The translation is straightforward.

Theorem 2.1 (Sun-Zhu)**.** *Let G be a real reductive group, H_i a closed subgroup and χ_i a continuous character of H_i, $i=1,2$. If $\mathcal{S}^*(G)^{H_1\times H_2,\chi_1^{-1}\times\chi_2^{-1}}=0$ then for every irredcible Fréchet representation π of G we have*

$$\dim \mathrm{Hom}_{H_1}(\pi,\chi_1)\cdot\dim\mathrm{Hom}_{H_2}(\tilde{\pi},\chi_2)=0$$

where $\tilde{\pi}$ is the contragredient of π.

Next we provide a sufficient condition for vanishing of the space of equivariant distributions in an algebraic context.

Lemma 2.2. *Let $G = \mathbb{G}_a(F)(= F)$ and let $\mathfrak{g}(= F)$ be the Lie algebra of G. Let $\chi : G \to \mathbb{C}^*$ be a non-trivial character and let π be a finite dimensional algebraic representation of G. Then $(\pi \otimes \chi)_{\mathfrak{g}} = 0$.*

Proof. Since π is algebraic and G unipotent, the only eigenvalue of $\pi \otimes \chi$ on G is χ. The derivative of χ at zero is not zero and therefore every non-zero element of $\mathfrak{g}$ acts on $\pi \otimes \chi$ by an invertible linear transformation. Hence $\mathfrak{g}(\pi \otimes \chi) = \pi \otimes \chi$ and there are no non-zero coinvariants. □

Proposition 2.3. *Let G be an F-linear algebraic group acting on a smooth algebraic variety X. Let $\chi : G \to \mathbb{C}^*$ be a unitary character and assume that for every $x \in X$ there exists a unipotent $u \in G_x$ such that $\chi(u) \neq 1$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.*

Proof. By (2.1) we have $T|_{\mathfrak{g}(\mathcal{S}(X)\otimes\chi)} \equiv 0$ for every $T \in \mathcal{S}^*(X)^{G,\chi}$. It is therefore enough to show that $\mathcal{S}(X) \otimes \chi = \mathfrak{g}(\mathcal{S}(X) \otimes \chi)$. By [AG, Theorem 2.2.15] it is enough to show that $(\mathrm{Sym}^k(CN^X_{Gx,x}) \otimes \chi')_{\mathfrak{g}_x} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$ where $\chi' = \chi|_{G_x} \cdot ((\Delta_G)|_{G_x}/\Delta_{G_x})$ and Δ_H denotes the modulus function of a locally compact group H.

Since χ is unitary and $(\Delta_G)|_{G_x}/\Delta_{G_x}$ positive we have $\chi'(u) \neq 1$. Since u is unipotent it lies in the image of some algebraic homomorphism $\varphi : F \to G_x$ (see e.g. [Fog69, Proposition 5.29]). Let $\mathfrak{u}$ be the Lie algebra of $\varphi(F)$. It follows from Lemma 2.2 that $(\mathrm{Sym}^k(CN^X_{Gx,x}) \otimes \chi')_{\mathfrak{u}} = 0$ and since $\mathfrak{u} \subseteq \mathfrak{g}_x$ also that $(\mathrm{Sym}^k(CN^X_{x,Gx}) \otimes \chi')_{\mathfrak{g}_x} = 0$. The Theorem follows. □

Let ψ be a unitary character of F and G an F-linear algebraic group. A character χ of G is *ψ-algebraic* if there exists an F-algebraic homomorphism $\phi : G \to \mathbb{G}_a(F)$ such that $\chi = \psi \circ \phi$.

Corollary 2.4. *With the above notation assume that G acts on a smooth algebraic variety X. Let χ be a ψ-algebraic character of G such that $\chi|_{G_x} \not\equiv 1$ for every $x \in X$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.*

Proof. Let $\phi : G \to \mathbb{G}_a(F)$ be as above. For $x \in X$ the stabilizer G_x is an F-linear algebraic group and therefore each of its elements has a Jordan decomposition in G_x (see e.g. [Hum75, §34.2]). If $\chi(s) \neq 1$ for some semi-simple $s \in G_x$ then let S be an F-torus in G_x containing s. Then $\phi|_S$ is a non-trivial algebraic homomorphism from a non trivial F-torus to the additive group $\mathbb{G}_a(F)$, which is a contradiction. Thus $\chi(u) \neq 1$ for some unipotent element $u \in G_x$. The Corollary therefore follows from Theorem 2.3. □

Theorem 2.5. *Let X be an F-reductive group, H_i an algebraic subgroup and χ_i a ψ-algebraic character of H_i, $i = 1, 2$. Set $G = H_1 \times H_2$ and $\chi = \chi_1 \times \chi_2$ and assume that $\chi|_{G_x} \not\equiv 1$ for all $x \in X$.*

(1) *For every irreducible Fréchet representation π of X we have*

$$\dim \mathrm{Hom}_{H_1}(\pi, \chi_1) \cdot \dim \mathrm{Hom}_{H_2}(\tilde{\pi}, \chi_2) = 0.$$

(2) *If* $X = GL_n(F)$ *and* ι *is the Cartan involution on* X *defined by* $g^\iota = {}^t g^{-1}$ *then* $(X, (H_1, \chi_1))$ *and* $(X, (H_2^\iota, \chi_2^\iota))$ *are disjoint pairs.*

Proof. The first part is immediate from Theorem 2.1 and Corollary 2.4. (Note that χ^{-1} is ψ^{-1}-algebraic and $\chi^{-1}|_{G_x} \not\equiv 1$, $x \in X$.) For $X = GL_n(F)$ it follows from [AGS08, Theorem 2.4.2]) that for every irreducible Fréchet representation π of X we have $\pi^\iota \simeq \tilde{\pi}$. Thus,

$$\mathrm{Hom}_{H_2}(\tilde{\pi}, \chi_2) \simeq \mathrm{Hom}_{H_2}(\pi^\iota, \chi_2) \simeq \mathrm{Hom}_{H_2^\iota}(\pi, \chi_2^\iota).$$

The second part therefore follows from the first. □

3. Disjointness

Fix $n \in \mathbb{N}$ and $0 \le r \ne r' \le n$ such that $r \equiv n \equiv r' \mod 2$. Set $X = G_n$, $H = H_{r,n}$ and $H' = H_{r',n}$. Let ι be the Cartan involution on X defined by $g^\iota = {}^t g^{-1}$, $G = H^\iota \times H'$ and $\theta = \psi_{r,n}^\iota \times \psi_{r',n}$ a character of G. Clearly θ is ψ-algebraic.

Theorem 3.1. *With the above notation* $\theta|_{G_x} \not\equiv 1$ *for all* $x \in X$*.*

Proof. Let $G' = H_{r'} \times H_r^\iota$ and let $\theta' = \psi_{r'} \times \psi_r^\iota$. It follows from [OS08, Proposition 2] (see Remark 2 of [ibid.]) that

$$\theta'|_{G'_x} \not\equiv 1, \quad x \in X. \tag{3.1}$$

Note that the map $\xi : G \to G'$ defined by $\xi(h_1, h_2) = (h_2^\iota, h_1^\iota)$ is an isomorphism and $\theta' \circ \xi|_G = \theta$. Since we further have

$${}^t(g \cdot x) = \xi(g) \cdot ({}^t x), \ g \in G, \, x \in X$$

it follows that $\xi(G_x) = G'_{{}^t x}$ and therefore (3.1) implies that $\theta|_{G_x} \not\equiv 1, \quad x \in X$ as required. □

Proof of Theorem 1.1. The Theorem follows from Theorems 2.5(2) and 3.1. □

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