$(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ IS A GELFAND PAIR FOR ANY LOCAL FIELD F

AVRAHAM AIZENBUD, DMITRY GOUREVITCH, AND EITAN SAYAG

Abstract. Let F be an arbitrary local field. Consider the standard embedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$ and the two-sided action of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$.

In this paper we show that any $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$-invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.

We show that this implies that the pair $(GL_{n+1}(F), GL_n(F))$ is a Gelfand pair. Namely, for any irreducible admissible representation (π, E) of $\mathrm{GL}_{n+1}(F)$,

$$\dim Hom_{\mathrm{GL}_n(F)}(E, \mathbb{C}) \leq 1.$$

For the proof in the archimedean case we develop several tools to study invariant distributions on smooth manifolds.

Contents

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1. Introduction

Let F be an arbitrary local field. Consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the two-sided action of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. In this paper we prove the following theorem:

Theorem (A). *Any $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.*

Theorem A has the following consequence in representation theory.

Theorem (B). *Let (π, E) be an irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$. Then*

$$\dim Hom_{\mathrm{GL}_n(F)}(E, \mathbb{C}) \leq 1. \tag{1}$$

Since any character of $\mathrm{GL}_n(F)$ can be extended to $\mathrm{GL}_{n+1}(F)$, we obtain

Corollary. *Let (π, E) be an irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$ and let χ be a character of $\mathrm{GL}_n(F)$. Then*

$$\dim Hom_{\mathrm{GL}_n(F)}(\pi, \chi) \leq 1.$$

In the non-archimedean case we use the standard notion of admissible representation (see [BZ]). In the archimedean case we consider admissible smooth Fréchet representations (see section 2).

Theorem B has some application to the theory of automorphic forms, more specifically to the factorizability of certain periods of automorphic forms on GL_n (see [Fli] and [FN]).

We deduce Theorem B from Theorem A using an argument due to Gelfand and Kazhdan adapted to the archimedean case. In our approach we use two deep results: the globalization theorem of Casselman-Wallach (see [Wal2]), and the regularity theorem of Harish-Chandra ([Wal1], chapter 8).

Clearly, Theorem B implies in particular that (1) holds for unitary irreducible representations of $\mathrm{GL}_{n+1}(F)$. That is, the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a generalized Gelfand pair in the sense of [vD] and [BvD].

The notion of Gelfand pair was studied extensively in the literature both in the setting of real groups and p-adic groups (e.g. [GK], [vD], [vDP], [BvD], [Gro], [Pra] and [JR] to mention a few). In [vD], the notion of generalized Gelfand pair is defined by requiring a condition of the form (1) for irreducible unitary representations. The definition suggested in [Gro] refers to the non-archimedean case and to a property satisfied by all irreducible admissible representations. In both cases, the verification of the said condition is achieved by means of a theorem on invariant distributions. However, the required statement on invariant distributions needed to verify condition (1) for unitary representation concerns only positive definite distributions. We elaborate on these issues in section 2.

1.1. **Related results.**

Several existing papers study related problems.

The case of non-archimedean fields of zero characteristic is covered in [AGRS] (see also [AG2]) where it is proven that the pair $(GL_{n+1}(F), GL_n(F))$ is a strong Gelfand pair i.e. $\dim_H(\pi, \sigma) \leq 1$ for any irreducible admissible representation π of G and *any* irreducible admissible representation σ of H. Here $H = \mathrm{GL}_n(F)$ and $G = \mathrm{GL}_{n+1}(F)$.

In [JR], it is proved that $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_1(F))$ is a Gelfand pair, where F is a local non-archimedean field of zero characteristic.

In [vDP] it is proved that for $n \geq 2$ the pair $(SL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$ is a generalized Gelfand pair and a similar result is obtained in [BvD] for the p-adic case, for $n \geq 3$. We emphasize that these results are proved in the realm of unitary representations. Another difference between these works and the present paper is that the embedding $GL_n(F) \subset GL_{n+1}(F)$ studied here does not factor through the embedding $GL_n(F) \hookrightarrow SL_{n+1}(F)$ of [vDP]. In particular, $(GL_2(\mathbb{R}), GL_1(\mathbb{R}))$ is a generalized Gelfand pair, and the pair $(SL_2(\mathbb{R}), GL_1(\mathbb{R}))$ is not a generalized Gelfand pair ([Mol],[vD]).

1.2. **Content of the Paper.**

We now briefly sketch the structure and content of the paper.

In section 2 we prove that Theorem A implies Theorem B. For this we clarify the relation between the theory of Gelfand pairs and the theory of invariant distributions both in the setting of [vD] and in the setting of [Gro].

In section 3 we present the proof of theorem A in the non-archimedean case. This section gives a good introduction to the rest of the paper since it contains many of the ideas but is technically simpler.

In section 4 we provide several tools to study invariant distributions on smooth manifolds. We believe that these results are of independent interest. In particular we introduce an adaption of a trick due to Bernstein which is very useful in the study of invariant distributions on vector spaces (proposition 4.3.2). These results partly relay on [AG1].

In section 5 we prove Theorem A in the archimedean case. This is the main result of the paper. The scheme of the proof is similar to the non-archimedean case. However, it is complicated by the fact that distributions on real manifolds do not behave as nicely as distributions on ℓ-spaces (see [BZ]).

We now explain briefly the main difference between the study of distributions on ℓ-spaces and distributions on real manifolds.

The space of distributions on an ℓ-space X supported on a closed subset $Z \subset X$ coincides with the space of distributions on Z. In the presence of group action on X, one can frequently use this property to reduce the study of distributions on X to distributions on orbits, that is on homogenous spaces. Although this property fails for distributions on real manifolds, one can still reduce problems to orbits. In the case of finitely many orbits this is studied in [Bru], [CHM], [AG1].

We mention that unlike the p-adic case, after the reduction to the orbits one needs to analyze generalized sections of symmetric powers of the normal bundles to the orbits, and not just distributions on those orbits. Here we employ a trick, proposition 4.3.1, which allows us to recover this information from a study of invariant distributions on a larger space.

In section A we provide the proof for the Frobenius reciprocity. The proof follows the proof in [Bar] (section 3).

In section B we prove the rest of the statements of section 4.

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2. Generalized Gelfand pairs and invariant distributions

In this section we show that Theorem A implies Theorem B. When F is non-archimedean this is a well known argument of Gelfand and Kazhdan (see [GK, Pra]). When F is archimedean and the representations in question are unitary such a reduction is due to [Tho]. We wish to consider representations which are not necessarily unitary and present here an argument which is valid in the generality of admissible smooth Fréchet representations. Our treatment is close in spirit to [Sha] (where multiplicity one result of Whittaker model is obtained for unitary representation) but at a crucial point we need to use the globalization theorem of Casselman-Wallach.

2.1. **Smooth Fréchet representations.**

The theory of representations in the context of Fréchet spaces is developed in [Cas2] and [Wal2]. We present here a well-known slightly modified version of that theory.

Definition 2.1.1. Let V be a complete locally convex topological vector space. A representation (π, V, G) is a continuous map $G \times V \to V$. A representation is called **Fréchet** if there exists a countable family of semi-norms ρ_i on V defining the topology of V and such that the action of G is continuous with respect to each ρ_i. We will say that V is **smooth Fréchet** representation if, for any $X \in \mathfrak{g}$ the differentiation map $v \mapsto \pi(X)v$ is a continuous linear map from V to V.

An important class of examples of smooth Fréchet representations is obtained from continuous Hilbert representations (π, H) by considering the subspace of smooth vectors H^∞ as a Fréchet space (see [Wal1] section 1.6 and [Wal2] 11.5).

We will consider mostly smooth Fréchet representations.

Remark 2.1.2. In the language of [Wal2] and [Cas] the representations above are called smooth Fréchet representations of moderate growth.

Recall that a smooth Fréchet representation is called *admissible* if it is finitely generated and its underlying $(\mathfrak{g}, K)$-module is admissible. In what follows *admissible representation* will always refer to admissible smooth Fréchet representation.

For a smooth admissible Fréchet representation (π, E) we denote by $(\widetilde{\pi}, \widetilde{E})$ the smooth contragredient of (π, E).

We will require the following corollary of the globalization theorem of Casselman and Wallach (see [Wal2] , chapter 11).

Theorem 2.1.3. *Let E be an admissible Fréchet representation, then there exists a continuous Hilbert space representation (π, H) such that $E = H^\infty$.*

This theorem follows easily from the embedding theorem of Casselman combined with Casselman-Wallach globalization theorem.

Fréchet representations of G can be lifted to representations of $\mathcal{S}(G)$, the Schwartz space of G. This is a space consisting of functions on G which, together with all their derivatives, are rapidly decreasing (see [Cas]. For an equivalent definition see section 4.1).

For a Fréchet representation (π, E) of G, the algebra $\mathcal{S}(G)$ acts on E through

$$\pi(\phi) = \int_G \phi(g)\pi(g)dg \tag{2}$$

(see [Wal1], section 8.1.1).
The following lemma is straitforward:

Lemma 2.1.4. *Let (π, E) be an admissible Fréchet representation of G and let $\lambda \in E^*$. Then $\phi \to \pi(\phi)\lambda$ is a continuous map $\mathcal{S}(G) \to \widetilde{E}$.*

The following proposition follows from Schur's lemma for $(\mathfrak{g}, K)$ modules (see [Wal1] page 80) in light of Casselman-Wallach theorem.

Proposition 2.1.5. *Let G be a real reductive group. Let W be a Fréchet representation of G and let E be an irreducible admissible representation of G. Let $T_1, T_2 : W \hookrightarrow E$ be two embeddings of W into E. Then T_1 and T_2 are proportional.*

We need to recall the basic properties of characters of representations.

Proposition 2.1.6. *Assume that (π, E) is admissible Fréchet representation. Then $\pi(\phi)$ is of trace class, and the assignment $\phi \to trace(\pi(\phi))$ defines a continuous functional on $\mathcal{S}(G)$ i.e. a Schwartz distribution. Moreover, the distribution $\chi_\pi(\phi) = trace(\pi(\phi))$ is given by a locally integrable function on G.*

The result is well known for continuous Hilbert representations (see [Wal1] chapter 8). The case of admissible Fréchet representation follows from the case of Hilbert space representation and theorem 2.1.3.

Another useful property of the character (see loc. cit.) is the following proposition:

Proposition 2.1.7. *If two irreducible admissible representations have the same character then they are isomorphic.*

Proposition 2.1.8. *Let (π, E) be an admissible representation. Then $\widetilde{\widetilde{E}} \cong E$.*

For proof see pages 937-938 in [GP].

2.2. **Three notions of Gelfand pair.**

Let G be a real reductive group and $H \subset G$ be a subgroup. Let (π, E) be an admissible Fréchet representation of G as in the previous section. We are interested in representations (π, E) which admit a continuous H-invariant linear functional. Such representations of G are called H-distinguished.

Put differently, let $Hom_H(E, \mathbb{C})$ be the space of continuous functionals $\lambda : E \to \mathbb{C}$ satisfying

$$\forall e \in E, \forall h \in H : \lambda(he) = \lambda(e)$$

The representation (π, E) is called **H-distinguished** if $Hom_H(E, \mathbb{C})$ is non-zero. We now introduce three notions of Gelfand pair and study their inter-relations.

Definition 2.2.1. Let $H \subset G$ be a pair of reductive groups.

- We say that (G, H) satisfy **GP1** if for any irreducible admissible representation (π, E) of G we have
$$\dim Hom_H(E, \mathbb{C}) \leq 1$$
- We say that (G, H) satisfy **GP2** if for any irreducible admissible representation (π, E) of G we have
$$\dim Hom_H(E, \mathbb{C}) \cdot \dim Hom_H(\widetilde{E}, \mathbb{C}) \leq 1$$
- We say that (G, H) satisfy **GP3** if for any irreducible **unitary** representation (π, W) of G on a Hilbert space W we have
$$\dim Hom_H(W^{\infty}, \mathbb{C}) \leq 1$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced by [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vDP], [BvD]).

We have the following straitforward proposition:

Proposition 2.2.2. $GP1 \Rightarrow GP2 \Rightarrow GP3$.

2.3. Gelfand pairs and invariant distributions.

The theory of generalized Gelfand pairs as developed in [vDP] and [Tho] provides the following criterion to verify GP3.

Theorem 2.3.1. *Let τ be an involutive anti-automorphism of G such that $\tau(H) = H$. Suppose $\tau(T) = T$ for all bi H-invariant positive definite distributions T on G. Then (G, H) satisfies GP3.*

This is a slight reformulation of Criterion 1.2 of [vD], page 583.

We now consider an analogous criterion which allows the verification of GP2. This is inspired by the famous Gelfand-Kazhdan method in the p-adic case.

Theorem 2.3.2. *Let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(T) = T$ for all bi H-invariant distributions [1] on G. Then (G, H) satisfies GP2.*

Proof. Let (π, E) be an irreducible admissible Fréchet representation. If E or $\widetilde{E}$ are not distinguished by H we are done. Thus we can assume that there exists a non-zero $\lambda : E \to \mathbb{C}$ which is H-invariant. Now let ℓ_1, ℓ_2 be two non-zero H-invariant functionals on $\widetilde{E}$. We wish to show that they are proportional. For this we define two distributions D_1, D_2 as follows

$$D_i(\phi) = \ell_i(\pi(\phi)\lambda)$$

[1] In fact it is enough to check this only for Schwartz distributions.

for $i = 1, 2$. Here $\phi \in \mathcal{S}(G)$. Note that D_i are also Schwartz distributions. Both distributions are bi-H-invariant and hence, by the assumption, both distributions are τ invariant. Now consider the bilinear forms on $\mathcal{S}(G)$ defined by

$$B_i(\phi_1, \phi_2) = D_i(\phi_1 * \phi_2).$$

Since E is irreducible, the right kernel of B_1 is equal to the right kernel of B_2. We now use the fact that D_i are τ invariant. Denote by J_i the left kernels of B_i. Then $J_1 = J_2$ which we denote by J. Consider the Fréchet representation $W = \mathcal{S}(G)/J$ and define the maps $T_i : \mathcal{S}(G) \to \widetilde{\widetilde{E}} \cong E$ by $T_i(\phi) = \pi(\phi)\ell_i$. These are well defined by Lemma 2.1.4 and we use the same letters to denote the induced maps $T_i : W \to E$. By proposition 2.1.5, T_1 and T_2 are proportional and hence ℓ_1 and ℓ_2 are proportional and the proof is complete. □

2.4. **Archimedean analogue of Gelfand-Kazhdan's theorem.**

To finish the proof that Theorem A implies Theorem B we will show that in certain cases, the property $GP1$ is equivalent to $GP2$.

Proposition 2.4.1. *Let $H < \mathrm{GL}_n(F)$ be a transposition invariant subgroup. Then $GP1$ is equivalent to $GP2$ for the pair $(\mathrm{GL}_n(F), H)$.*

For the proof we need the following notation. For a representation (π, E) of $GL_n(F)$ we let $(\widehat{\pi}, E)$ be the representation of $GL_n(F)$ defined by $\widehat{\pi} = \pi \circ \theta$, where θ is the (Cartan) involution $\theta(g) = {g^{-1}}^t$. Since

$$Hom_H(\pi, \mathbb{C}) = Hom_H(\widehat{\pi}, \mathbb{C})$$

the following analogue of Gelfand-Kazhdan theorem is enough.

Theorem 2.4.2. *Let (π, E) be an irreducible admissible representation of $GL_n(F)$. Then $\widehat{\pi}$ is isomorphic to $\widetilde{\pi}$.*

Remark 2.4.3. This theorem is due to Gelfand and Kazhdan in the p-adic case (they show that any distribution which is invariant to conjugation is transpose invariant, in particular this is valid for the character of an irreducible representation) and due to Shalika for unitary representations which are generic ([Sha]). We give a proof in complete generality based on Harish-Chandra regularity theorem (see chapter 8 of [Wal1]).

Proof of theorem 2.4.2. Consider the characters $\chi_{\widetilde{\pi}}$ and $\chi_{\widehat{\pi}}$. These are locally integrable functions on G that are invariant with respect to conjugation. Clearly,

$$\chi_{\widehat{\pi}}(g) = \chi_\pi({g^{-1}}^t)$$

and

$$\chi_{\widetilde{\pi}}(g) = \chi_\pi(g^{-1}).$$

But for $g \in \mathrm{GL}_n(F)$, the elements g^{-1} and ${g^{-1}}^t$ are conjugate. Thus, the characters of $\widehat{\pi}$ and $\widetilde{\pi}$ are identical. Since both are irreducible, Theorem 8.1.5 in [Wal1], implies that $\widehat{\pi}$ is isomorphic to $\widetilde{\pi}$. □

Corollary 2.4.4. *Theorem A implies Theorem B.*

Remark 2.4.5. The above argument proves also that Theorem B follows from a weaker version of Theorem A, where only Schwartz distributions are considered (these are continuous functionals on the space $\mathcal{S}(G)$ of Schwartz functions).

Remark 2.4.6. The non-archimedean analogue of theorem 2.3.2 is a special case of Lemma 4.2 of [Pra]. The rest of the argument in the non-archimedean case is identical to the above.

3. Non-archimedean case

In this section F is a non-archimedean local field of arbitrary characteristic. We will use the standard terminology of l-spaces introduced in [BZ], section 1. We denote by $\mathcal{S}(X)$ the space of Schwartz functions on an l-space X, and by $\mathcal{S}^*(X)$ the space of distributions on X equipped with the weak topology.

We fix a nontrivial additive character ψ of F.

3.1. **Preliminaries.**

Definition 3.1.1. Let V be a finite dimensional vector space over F. A subset $C \subset V$ is called a **cone** if it is homothety invariant.

Definition 3.1.2. Let V be a finite dimensional vector space over F. Note that $F^\times$ acts on V by homothety. This gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$. Let α be a character of $F^\times$.

We call a distribution $\xi \in \mathcal{S}^*(V)$ **homogeneous of type** α if for any $t \in F^\times$, we have $\rho(t)(\xi) = \alpha^{-1}(t)\xi$. That is, for any function $f \in \mathcal{S}(V)$, $\xi(\rho(t^{-1})(f)) = \alpha(t)\xi(f)$, where $\rho(t^{-1})(f)(v) = f(tv)$.

Let $LsubsetF$ be a subfield. We will call a distribution $\xi \in \mathcal{S}^*(V)$ **L-homogeneous of type** α if for any $t \in L^\times$, we have $\rho(t)(\xi) = \alpha^{-1}(t)\xi$.

Example 3.1.3. A Haar measure on V is homogeneous of type $|\cdot|^{\dim V}$. The Dirac's δ-distribution is homogeneous of type 1.

The following proposition is straightforward.

Proposition 3.1.4. *Let a l-group G act on an l-space X. Let $X = \bigcup_{i=0}^{l} X_i$ be a G-invariant stratification of X. Let χ be a character of G. Suppose that for any $i = 1 \ldots l$, $\mathcal{S}^*(X_i)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.*

Proposition 3.1.5. *Let $H_i \subset G_i$ be l-groups acting on l-spaces X_i for $i = 1 \ldots n$. Suppose that $\mathcal{S}^*(X_i)^{H_i} = \mathcal{S}^*(X_i)^{G_i}$ for all i. Then $\mathcal{S}^*(\prod X_i)^{\prod H_i} = \mathcal{S}^*(\prod X_i)^{\prod G_i}$.*

Proof. It is enough to prove the proposition for the case $n = 2$. Let $\xi \in \mathcal{S}^*(X_1 \times X_1)^{H_1 \times H_2}$. Fix $f_1 \in \mathcal{S}(X_1)$ and $f_2 \in \mathcal{S}(X_1)$. It is enough to prove that for any $g_1 \in G_1$ and $g_2 \in G_2$, we have $\xi(g_1(f_1) \otimes g_2(f_2)) = \xi(f_1 \otimes f_2)$. Let $\xi_1 \in \mathcal{S}^*(X_1)$ be the distribution defined by $\xi_1(f) := \xi(f \otimes f_2)$. It is H_1-invariant. Hence also G_1-invariant. Thus $\xi(f_1 \otimes f_2) = \xi(g_1(f_1) \otimes f_2)$. By the same reasons $\xi(g_1(f_1) \otimes f_2) = \xi(g_1(f_1) \otimes g_2(f_2))$. □

We will use the following important theorem proven in [Ber1], section 1.5.

Theorem 3.1.6 (Frobenius reciprocity)**.** *Let a unimodular l-group G act transitively on an l-space Z. Let $\varphi : X \to Z$ be a G-equivariant continuous map. Let $z \in Z$. Suppose that its stabilizer $\mathrm{Stab}_G(z)$ is unimodular. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{\mathrm{Stab}_G(z),\chi}$.*

The next proposition formalizes an idea from [Ber2]. The key tool used in its proof is Fourier Transform.

Proposition 3.1.7. *Let G be an l-group. Let V be a finite dimensional representation of G over F. Suppose that the action of G preserves some non-degenerate bilinear form on V. Let $V = \bigcup_{i=1}^{n} C_i$ be a stratification of V by G-invariant cones.*

Let $\mathfrak{X}$ be a set of characters of $F^\times$ such that the set $\mathfrak{X} \cdot \mathfrak{X}$ does not contain the character $|\cdot|^{\dim V}$. Let χ be a character of G. Suppose that for any i, the space $\mathcal{S}^*(C_i)^{G,\chi}$ consists of homogeneous distributions of type α for some $\alpha \in \mathfrak{X}$. Then $\mathcal{S}^*(V)^{G,\chi} = 0$.

In section B.3 we prove an archimedean analog of this proposition, and the same proof is applicable in this case.

3.2. **Proof of Theorem A for non-archimedean F.**

We need some further notations.

Notation 3.2.1. Denote $H := H_n := \mathrm{GL}_n := \mathrm{GL}_n(F)$. Denote

$$G := G_n := \{(h_1, h_2) \in \mathrm{GL}_n \times \mathrm{GL}_n | \, det(h_1) = det(h_2)\}.$$

We consider H to be diagonally embedded to G.

Consider the action of the 2-element group S_2 on G given by the involution $(h_1, h_2) \mapsto (h_2^{-1^t}, h_1^{-1^t})$. It defines a semidirect product $\widetilde{G} := \widetilde{G}_n := G \rtimes S_2$. Denote also $\widetilde{H} := \widetilde{H}_n := H_n \rtimes S_2$.

Let $V = F^n$ and $X := X_n := \mathrm{gl}_n(F) \times V \times V^*$.

The group $\widetilde{G}$ acts on X by

$$(h_1, h_2)(A, v, \phi) := (h_1 A h_2^{-1}, h_1 v, h_2^{-1^t} \phi) \text{ and}$$
$$\sigma(A, v, \phi) := (A^t, \phi^t, v^t)$$

where $(h_1, h_2) \in G$ and σ is the generator of S_2. Note that $\widetilde{G}$ acts separately on gl_n and on $V \times V^*$. Define a character χ of $\widetilde{G}$ by $\chi(g, s) := sign(s)$.

We will show that the following theorem implies Theorem A.

Theorem 3.2.2. $\mathcal{S}^*(X)^{\widetilde{G},\chi} = 0$.

3.2.1. *Proof that theorem 3.2.2 implies theorem A.*

We will divide this reduction to several propositions.

Consider the action of $\widetilde{G}_n$ on GL_{n+1} and on gl_{n+1}, where G_n acts by the two-sided action and the generator of S_2 acts by transposition.

Proposition 3.2.3. *If $\mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi} = 0$ then theorem A holds.*

The proof is straightforward.

Proposition 3.2.4. *If $\mathcal{S}^*(\mathrm{gl}_{n+1})^{\widetilde{G}_n,\chi} = 0$ then $\mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi} = 0$.*

Proof.[2] Let $\xi \in \mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \mathrm{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in \mathcal{S}(F)$ be such that f vanishes in a neighborhood of zero and $f(t) \neq 0$. Consider the determinant map $\det : \mathrm{GL}_{n+1} \to F$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi}$ and $p \in \mathrm{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathrm{gl}_{n+1})^{\widetilde{G}_n,\chi}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. $\square$

[2] This proposition is an adaption of a statement in [Ber1], section 2.2.

Proposition 3.2.5. *If* $\mathcal{S}^*(X_n)^{\widetilde{G}_n,\chi} = 0$ *then* $\mathcal{S}^*(\mathrm{gl}_{n+1})^{\widetilde{G}_n,\chi} = 0$.

Proof. Note that gl_{n+1} is isomorphic as a $\widetilde{G}_n$-equivariant l-space to $X_n \times F$ where the action on F is trivial. This isomorphism is given by

$$\begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & t \end{pmatrix} \mapsto ((A, v, \phi), t).$$

The proposition now follows from proposition 3.1.5. □

This finishes the proof that theorem 3.2.2 implies Theorem A.

3.2.2. *Proof of theorem 3.2.2.*
We will now stratify $X(= gl_n \times V \times V^*)$ and deal with each strata separately.

Notation 3.2.6. Denote $W := W_n := V_n \oplus V_n^*$. Denote by $Q^i := Q_n^i \subset \mathrm{gl}_n$ the set of all matrices of rank i. Denote $Z^i := Z_n^i := Q_n^i \times W_n$.

Note that $X = \bigcup Z^i$. Hence by proposition 3.1.4, it is enough to prove the following proposition.

Proposition 3.2.7. $\mathcal{S}^*(Z^i)^{\widetilde{G},\chi} = 0$ *for any* $i = 0, 1, ..., n$.

We will use the following key lemma.

Lemma 3.2.8 (Non-archimedean Key Lemma)**.** $\mathcal{S}^*(W)^{\widetilde{H},\chi} = 0$.

For proof see section 3.3 below.

Corollary 3.2.9. *Proposition 3.2.7 holds for* $i = n$.

Proof. Clearly, one can extend the actions of $\widetilde{G}$ on Q^n and on Z^n to actions of $\widetilde{GL_n \times GL_n} := (GL_n \times GL_n) \rtimes S_2$ in the obvious way.

Step 1. $\mathcal{S}^*(Z^n)^{\widetilde{GL_n \times GL_n},\chi} = 0$.
Consider the projection on the first coordinate from Z^n to the transitive $\widetilde{GL_n \times GL_n}$-space $Q^n = GL_n$. Choose the point $Id \in Q^n$. Its stabilizer is $\widetilde{H}$ and its fiber is W. Hence by Frobenius reciprocity (theorem 3.1.6), $\mathcal{S}^*(Z^n)^{\widetilde{GL_n \times GL_n},\chi} \cong \mathcal{S}^*(W)^{\widetilde{H},\chi}$ which is zero by the key lemma.

Step 2. $\mathcal{S}^*(Z^n)^{\widetilde{G},\chi} = 0$.
Consider the space $Y := Z^n \times F^\times$ and let the group $GL_n \times GL_n$ act on it by $(h_1, h_2)(z, \lambda) := ((h_1, h_2)z, \det h_1 \det h_2^{-1} \lambda)$. Extend this action to action of $\widetilde{GL_n \times GL_n}$ by $\sigma(z, \lambda) := (\sigma(z), \lambda)$. Consider the projection $Z^n \times F^\times \to F^\times$. By Frobenius reciprocity (theorem 3.1.6),

$$\mathcal{S}^*(Y)^{\widetilde{GL_n \times GL_n},\chi} \cong \mathcal{S}^*(Z^n)^{\widetilde{G},\chi}.$$

Let Y' be equal to Y as an l-space and let $\widetilde{GL_n \times GL_n}$ act on Y' by $(h_1, h_2)(z, \lambda) := ((h_1, h_2)z, \lambda)$ and $\sigma(z, \lambda) := (\sigma(z), \lambda)$. Now Y is isomorphic to Y' as a $\widetilde{GL_n \times GL_n}$ space by $((A, v, \phi), \lambda) \mapsto ((A, v, \phi), \lambda \det A^{-1})$.

Since $\mathcal{S}^*(Z^n)^{\widetilde{GL_n \times GL_n},\chi} = 0$, proposition 3.1.5 implies that $\mathcal{S}^*(Y')^{\widetilde{GL_n \times GL_n},\chi} = 0$ and hence $\mathcal{S}^*(Y)^{\widetilde{GL_n \times GL_n},\chi} = 0$ and thus $\mathcal{S}^*(Z^n)^{\widetilde{G}_n,\chi} = 0$. □

Corollary 3.2.10. *We have*

$$\mathcal{S}^*(W_i \times W_{n-i})^{H_i \times H_{n-i}} = \mathcal{S}^*(W_i \times W_{n-i})^{\widetilde{H}_i \times \widetilde{H}_{n-i}}.$$

Proof. It follows from the key lemma and proposition 3.1.5. □

Now we are ready to prove proposition 3.2.7.

Proof of proposition 3.2.7. Fix $i < n$. Consider the projection $pr_1 : Z^i \to Q^i$. It is easy to see that the action of $\widetilde{G}$ on Q^i is transitive. We are going to use Frobenius reciprocity.

Denote

$$A_i := \begin{pmatrix} Id_{i\times i} & 0 \\ 0 & 0 \end{pmatrix} \in Q^i.$$

Denote by $G_{A_i} := \mathrm{Stab}_G(A_i)$ and $\widetilde{G}_{A_i} := \mathrm{Stab}_{\widetilde{G}}(A_i)$.

It is easy to check by explicit computation that

- G_{A_i} and $\widetilde{G}_{A_i}$ are unimodular.
- $H_i \times G_{n-i}$ can be canonically embedded into G_{A_i}.
- W is isomorphic to $W_i \times W_{n-i}$ as $H_i \times G_{n-i}$-spaces.

By Frobenius reciprocity (theorem 3.1.6),

$$\mathcal{S}^*(Z^i)^{\widetilde{G},\chi} = \mathcal{S}^*(W)^{\widetilde{G}_{A_i},\chi}.$$

Hence it is enough to show that $\mathcal{S}^*(W)^{G_{A_i}} = \mathcal{S}^*(W)^{\widetilde{G}_{A_i}}$. Let $\xi \in \mathcal{S}^*(W)^{G_{A_i}}$. By the previous corollary, ξ is $\widetilde{H}_i \times \widetilde{H}_{n-i}$-invariant. Since ξ is also G_{A_i}-invariant, it is $\widetilde{G}_{A_i}$-invariant. □

3.3. **Proof of the key lemma (lemma 3.2.8).**

Our key lemma is proved in section 10.1 of [RS]. The proof below is slightly different and more convenient to adapt to the archimedean case.

Proposition 3.3.1. *It is enough to prove the key lemma for $n = 1$.*

Proof. Consider the subgroup $T_n \subset H_n$ consisting of diagonal matrices, and $\widetilde{T}_n := T_n \rtimes S_2 \subset \widetilde{H}_n$. It is enough to prove $\mathcal{S}^*(W_n)^{\widetilde{T}_n,\chi} = 0$.

Now, by proposition 3.1.5 it is enough to prove $\mathcal{S}^*(W_1)^{\widetilde{H}_1,\chi} = 0$. □

From now on we fix $n := 1$, $H := H_1$, $\widetilde{H} := \widetilde{H}_1$ and $W := W_1$. Note that $H = F^\times$ and $W = F^2$. The action of H is given by $\rho(\lambda)(x, y) := (\lambda x, \lambda^{-1} y)$ and extended to the action of $\widetilde{H}$ by the involution $\sigma(x, y) = (y, x)$.

Let $Y := \{(x, y) \in F^2 |\, xy = 0\} \subset W$ be the **cross** and $Y' := Y \setminus \{0\}$.

By proposition 3.1.7, it is enough to prove the following proposition.

Proposition 3.3.2.
(i) $\mathcal{S}^*(\{0\})^{\widetilde{H},\chi} = 0$.
(ii) Any distribution $\xi \in \mathcal{S}^*(Y')^{\widetilde{H},\chi}$ *is homogeneous of type 1.*
(iii) $\mathcal{S}^*(W \setminus Y)^{\widetilde{H},\chi} = 0$.

Proof. (i) and (ii) are trivial.
(iii) Denote $U := W \setminus Y$. We have to show $\mathcal{S}^*(U)^{\widetilde{H},\chi} = 0$. Consider the coordinate change $U \cong F^\times \times F^\times$ given by $(x, y) \mapsto (xy, x/y)$. It is an isomorphism of $\widetilde{H}$-spaces where the action of $\widetilde{H}$ on $F^\times \times F^\times$ is only on the second coordinate, and given by $\lambda(w) = \lambda^2 w$ and $\sigma(w) = w^{-1}$. Clearly, $\mathcal{S}^*(F^\times)^{\widetilde{H},\chi} = 0$ and hence by proposition 3.1.5 $\mathcal{S}^*(F^\times \times F^\times)^{\widetilde{H},\chi} = 0$. □

4. Preliminaries on equivariant distributions in the archimedean case

From now till the end of the paper F denotes an archimedean local field, that is $\mathbb{R}$ or $\mathbb{C}$. Also, the word smooth means infinitely differentiable.

4.1. **Notations.**

4.1.1. *Distributions on smooth manifolds.*
Here we present basic notations on smooth manifolds and distributions on them.

Definition 4.1.1. Let X be a smooth manifold. Denote by $C_c^\infty(X)$ the space of complex-valued test functions on X, that is smooth compactly supported functions, with the standard topology, i.e. the topology of inductive limit of Fréchet spaces.

Denote $\mathcal{D}(X) := C_c^\infty(X)^*$ equipped with the weak topology.

For any vector bundle E over X we denote by $C_c^\infty(X, E)$ the complexification of space of smooth compactly supported sections of E and by $\mathcal{D}(X, E)$ its dual space. Also, for any finite dimensional real vector space V we denote $C_c^\infty(X, V) := C_c^\infty(X, X \times V)$ and $\mathcal{D}(X, V) := \mathcal{D}(X, X \times V)$, where $X \times V$ is a trivial bundle.

Definition 4.1.2. Let X be a smooth manifold and let $Z \subset X$ be a closed subset. We denote $\mathcal{D}_X(Z) := \{\xi \in \mathcal{D}(X) | \mathrm{Supp}(\xi) \subset Z\}$.

For locally closed subset $Y \subset X$ we denote $\mathcal{D}_X(Y) := \mathcal{D}_{X \setminus (\overline{Y} \setminus Y)}(Y)$. In the same way, for any bundle E on X we define $\mathcal{D}_X(Y, E)$.

Notation 4.1.3. Let X be a smooth manifold and Y be a smooth submanifold. We denote by $N_Y^X := (T_X|_Y)/T_Y$ the normal bundle to Y in X. We also denote by $CN_Y^X := (N_Y^X)^*$ the conormal bundle. For a point $y \in Y$ we denote by $N_{Y,y}^X$ the normal space to Y in X at the point y and by $CN_{Y,y}^X$ the conormal space.

We will also use notions of a cone in a vector space and of homogeneity type of a distribution defined in the same way as in non-archimedean case (definitions 3.1.1 and 3.1.2).

4.1.2. *Schwartz distributions on Nash manifolds.*
Our proof of Theorem A uses a trick (proposition 4.3.2) involving Fourier Transform which cannot be directly applied to distributions. For this we require a theory of Schwartz functions and distributions as developed in [AG1]. This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered (section B is a minor exception). Therefore the reader can safely replace the word *Nash* by *smooth real algebraic*.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

Notation 4.1.4. Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.

Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the space of Schwartz distributions on X.

For any Nash vector bundle E over X we denote by $\mathcal{S}(X, E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X, E)$ its dual space.

Definition 4.1.5. Let X be a smooth manifold, and let $Y \subset X$ be a locally closed (semi-)algebraic subset. Let E be a Nash bundle over X. We define $\mathcal{S}_X^*(Y)$ and $\mathcal{S}_X^*(Y, E)$ in the same way as $\mathcal{D}_X(Y)$ and $\mathcal{D}_X(Y, E)$.

Remark 4.1.6. All the classical bundles on a Nash manifold are Nash bundles. In particular the normal and conormal bundle to a Nash submanifold of a Nash manifold are Nash bundles. For proof see e.g. [AG1], section 6.1.

Remark 4.1.7. For any Nash manifold X, we have $C_c^\infty(X) \subset \mathcal{S}(X)$ and $\mathcal{S}^*(X) \subset \mathcal{D}(X)$.

Remark 4.1.8. Schwartz distributions have the following two advantages over general distributions:
(i) For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence

$$0 \to \mathcal{S}_X^*(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

(see Theorem B.2.2 in Appendix B).
(ii) Fourier transform defines an isomorphism $\mathcal{F} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.

4.2. **Basic tools.**

We present here basic tools on equivariant distributions that we will use in this paper. All the proofs are given in the appendices.

Theorem 4.2.1. *Let a real reductive group G act on a smooth affine real algebraic variety X. Let $X = \bigcup_{i=0}^{l} X_i$ be a smooth G-invariant stratification of X. Let χ be an algebraic character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and any $0 \leq i \leq l$ we have $\mathcal{D}(X_i, Sym^k(CN_{X_i}^X))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

For proof see appendix B.2.

Proposition 4.2.2. *Let $H_i \subset G_i$ be Lie groups acting on smooth manifolds X_i for $i = 1 \ldots n$. Let $E_i \to X_i$ be (finite dimensional) G_i-equivariant vector bundles. Suppose that $\mathcal{D}(X_i, E_i)^{H_i} = \mathcal{D}(X_i, E_i)^{G_i}$ for all i. Then $\mathcal{D}(\prod X_i, \boxtimes E_i)^{\prod H_i} = \mathcal{D}(\prod X_i, \boxtimes E_i)^{\prod G_i}$, where $\boxtimes$ denotes the external product of vector bundles.*

The proof of this proposition is the same as of its non-archimedean analog (proposition 3.1.5).

Theorem 4.2.3 (Frobenius reciprocity)**.** *Let a unimodular Lie group G act transitively on a smooth manifold Z. Let $\varphi : X \to Z$ be a G-equivariant smooth map. Let $z_0 \in Z$. Suppose that its stabilizer $\mathrm{Stab}_G(z_0)$ is unimodular. Let X_{z_0} be the fiber of z_0. Let χ be a character of G. Then $\mathcal{D}(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_{z_0})^{\mathrm{Stab}_G(z_0),\chi}$. Moreover, for any G-equivariant bundle E on X and a closed $\mathrm{Stab}_G(z_0)$-invariant subset $Y \subset X_{z_0}$, the space $\mathcal{D}_X(GY, E)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}_{X_{z_0}}(Y, E|_{X_{z_0}})^{\mathrm{Stab}_G(z_0),\chi}$.*

In section A we formulate and prove a more general version of this theorem.

The next theorem shows that in certain cases it is enough to show that there are no equivariant Schwartz distributions. This will allow us to use Fourier transform.

We will need the following theorem from [AG3], Theorem 4.0.2.

Theorem 4.2.4. *Let a real reductive group G act on a smooth affine real algebraic variety X. Let V be a finite-dimensional algebraic representation of G. Suppose that*

$$\mathcal{S}^*(X, V)^G = 0.$$

Then

$$\mathcal{D}(X,V)^G = 0.$$

For proof see [AG3], Theorem 4.0.2.

4.3. **Specific tools.**

We present here tools on equivariant distributions which are more specific to our problem. All the proofs are given in Appendix B.

Proposition 4.3.1. *Let a Lie group G act on a smooth manifold X. Let V be a real finite dimensional representation of G. Suppose that G preserves the Haar measure on V. Let $U \subset V$ be an open non-empty G-invariant subset. Let χ be a character of G. Suppose that $\mathcal{D}(X \times U)^{G,\chi} = 0$. Then $\mathcal{D}(X, Sym^k(V))^{G,\chi} = 0$.*

For proof see section B.4.

Proposition 4.3.2. *Let G be a Nash group. Let V be a finite dimensional representation of G over F. Suppose that the action of G preserves some non-degenerate bilinear form B on V. Let $V = \bigcup_{i=1}^{n} S_i$ be a stratification of V by G-invariant Nash cones.*

Let $\mathfrak{X}$ be a set of characters of $F^\times$ such that the set $\mathfrak{X} \cdot \mathfrak{X}$ does not contain the character $|\cdot|^{\dim_{\mathbb{R}} V}$. Let χ be a character of G. Suppose that for any i and k, the space $\mathcal{S}^*(S_i, Sym^k(CN_{S_i}^V))^{G,\chi}$ consists of homogeneous distributions of type α for some $\alpha \in \mathfrak{X}$. Then $\mathcal{S}^*(V)^{G,\chi} = 0$.

For proof see section B.3.

In order to prove homogeneity of invariant distributions we will use the following corollary of Frobenius reciprocity.

Proposition 4.3.3 (Homogeneity criterion)**.** *Let G be a Lie group. Let V be a finite dimensional representation of G over F. Let $C \subset V$ be a G-invariant G-transitive smooth cone. Consider the actions of $G \times F^\times$ on V, C and CN_C^V, where $F^\times$ acts by homotheties. Let χ be a character of G. Let α be a character of $F^\times$. Consider the character $\chi' := \chi \times \alpha^{-1}$ of $G \times F^\times$. Let $x_0 \in C$ and denote $H := Stab_G(x_0)$ and $H' := Stab_{G\times F^\times}(x_0)$. Suppose that G, H, H' are unimodular. Fix $k \in Z_{\geq 0}$.*

Then the space $\mathcal{D}(C, Sym^k(CN_C^V))^{G,\chi}$ consists of homogeneous distributions of type α if and only if

$$(Sym^k(N_{C,x_0}^V) \otimes_{\mathbb{R}} \mathbb{C})^{H,\chi} = (Sym^k(N_{C,x_0}^V) \otimes_{\mathbb{R}} \mathbb{C})^{H',\chi'}.$$

5. Proof of Theorem A for archimedean F

We will use the same notations as in the non-archimedean case (see notation 3.2.1). Again, the following theorem implies Theorem A.

Theorem 5.0.1. $\mathcal{D}(X)^{\widetilde{G},\chi} = 0$.

The implication is proven exactly in the same way as in the non-archimedean case (subsection 3.2.1).

5.1. **Proof of theorem 5.0.1.**

We will now stratify $X(= gl_n \times V \times V^*)$ and deal with each strata separately.

Notation 5.1.1. Denote $W := W_n := V_n \oplus V_n^*$. Denote by $Q^i := Q^i_n \subset \mathrm{gl}_n$ the set of all matrices of rank i. Denote $Z^i := Z^i_n := Q^i_n \times W_n$.

Note that $X = \bigcup Z^i$. Hence by theorem 4.2.1, it is enough to prove the following proposition.

Proposition 5.1.2. *$\mathcal{D}(Z^i, Sym^k(CN^X_{Z^i}))^{\widetilde{G},\chi} = 0$ for any k and i.*

We will use the following key lemma.

Lemma 5.1.3 (Key Lemma)**.** $\mathcal{D}(W)^{\widetilde{H},\chi} = 0$.

For proof see subsection 5.2 below.

Corollary 5.1.4. *Proposition 5.1.2 holds for $i = n$.*

The proof is the same as in the non-archimedean case (corollary 3.2.9).

Corollary 5.1.5. $\mathcal{D}(W_n, Sym^k(\mathrm{gl}_n^*))^{\widetilde{G},\chi} = 0$.

Proof. Consider the Killing form $K : \mathrm{gl}_n^* \to \mathrm{gl}_n$. Let $U := K^{-1}(Q^n_n)$. In the same way as in the previous corollary one can show that $\mathcal{D}(W_n \times U)^{\widetilde{G},\chi} = 0$. Hence by proposition 4.3.1, $\mathcal{D}(W_n, Sym^k(\mathrm{gl}_n^*))^{\widetilde{G},\chi} = 0$. ∎

Corollary 5.1.6. *We have*

$$\mathcal{D}(W_i \times W_{n-i}, Sym^k(0 \times \mathrm{gl}_{n-i}^*))^{H_i \times G_{n-i}} = \mathcal{D}(W_i \times W_{n-i}, Sym^k(0 \times \mathrm{gl}_{n-i}^*))^{\widetilde{H}_i \times \widetilde{G}_{n-i}}.$$

Proof. It follows from the key lemma, the last corollary and proposition 4.2.2. ∎

Now we are ready to prove proposition 5.1.2.

Proof of proposition 5.1.2. Fix $i < n$. Consider the projection $pr_1 : Z^i \to Q^i$. It is easy to see that the action of $\widetilde{G}$ on Q^i is transitive. Denote

$$A_i := \begin{pmatrix} Id_{i \times i} & 0 \\ 0 & 0 \end{pmatrix} \in Q^i.$$

Denote by $G_{A_i} := \mathrm{Stab}_G(A_i)$ and $\widetilde{G}_{A_i} := \mathrm{Stab}_{\widetilde{G}}(A_i)$. Note that they are unimodular. By Frobenius reciprocity (theorem 4.2.3),

$$\mathcal{D}(Z^i, Sym^k(CN^X_{Z^i}))^{\widetilde{G},\chi} = \mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{\widetilde{G}_{A_i},\chi}.$$

Hence it is enough to show that

$$\mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{G_{A_i}} = \mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{\widetilde{G}_{A_i}}.$$

It is easy to check by explicit computation that

- $H_i \times G_{n-i}$ is canonically embedded into G_{A_i},
- W is isomorphic to $W_i \times W_{n-i}$ as $H_i \times G_{n-i}$-spaces
- $CN^{\mathrm{gl}_n}_{Q^i,A_i}$ is isomorphic to $0 \times \mathrm{gl}_{n-i}^*$ as $H_i \times G_{n-i}$ representations.

Let $\xi \in \mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{G_{A_i}}$. By the previous corollary, ξ is $\widetilde{H}_i \times \widetilde{G}_{n-i}$-invariant. Since ξ is also G_{A_i}-invariant, it is $\widetilde{G}_{A_i}$-invariant. ∎

5.2. **Proof of the key lemma (lemma 5.1.3).**

As in the non-archimedean case, it is enough to prove the key lemma for $n = 1$ (see proposition 3.3.1).

From now on we fix $n := 1$, $H := H_1$, $\widetilde{H} := \widetilde{H}_1$ and $W := W_1$. Note that $H = F^\times$ and $W = F^2$. The action of H is given by $\rho(\lambda)(x, y) := (\lambda x, \lambda^{-1} y)$ and extended to the action of $\widetilde{H}$ by the involution $\sigma(x, y) = (y, x)$.

Let $Y := \{(x, y) \in F^2 | \, xy = 0\} \subset W$ be the **cross** and $Y' := Y \setminus \{0\}$.

Lemma 5.2.1. *Every $(\widetilde{H}, \chi)$-equivariant distribution on W is supported inside the cross Y.*

The proof of this lemma is identical to the proof of proposition 3.3.2, (iii).

To apply proposition 4.3.2 (which uses Fourier transform) we need to restrict our consideration to Schwartz distributions. By theorem 4.2.4, in order to show that $\mathcal{D}_W(Y)^{\widetilde{H},\chi} = 0$ it is enough to show that $\mathcal{S}^*(W)^{\widetilde{H},\chi} = 0$ [3]. By proposition 4.3.2, it is enough to prove the following proposition.

Proposition 5.2.2.
(i) $\mathcal{S}^*(W \setminus Y)^{\widetilde{H},\chi} = 0$.
(ii) For all $k \in \mathbb{Z}_{\geq 0}$, any distribution $\xi \in \mathcal{S}^*(Y', Sym^k(CN_{Y'}^W))^{\widetilde{H},\chi}$ is $\mathbb{R}$-homogeneous of type α_k where $\alpha_k(\lambda) := \lambda^{-2k}$.
(iii) $\mathcal{S}^*(\{0\}, Sym^k(CN_{\{0\}}^W))^{\widetilde{H},\chi} = 0$.

Proof. We have proven (i) in the proof of the previous lemma.
(ii) Fix $x_0 := (1, 0) \in Y'$. Now we want to use the homogeneity criterion (proposition 4.3.3). Note that $Stab_{\widetilde{H}}(x_0)$ is trivial and $Stab_{\widetilde{H}\times\mathbb{R}^\times}(x_0) \cong \mathbb{R}^\times$. Note that $N_{Y',x_0}^W \cong F$ and $Stab_{\widetilde{H}\times\mathbb{R}^\times}(x_0)$ acts on it by $\rho(\lambda)a = \lambda^2 a$. So we have

$$Sym^k(N_{Y',x_0}^W) = Sym^k(N_{Y',x_0}^W)^{\mathbb{R}^\times,\alpha_k^{-1}}.$$

So by the homogeneity criterion any distribution $\xi \in \mathcal{S}^*(Y', Sym^k(CN_{Y'}^W))^{\widetilde{H},\chi}$ is $\mathbb{R}$-homogeneous of type α_k.
(iii) is a simple computation. Also, it can be deduced from (i) using proposition 4.3.1. ∎

Appendix A. Frobenius reciprocity

In this section we obtain a slight generalization of Frobenius reciprocity proven in [Bar] (section 3). The proof will go along the same lines and is included for the benefit of the reader. To simplify the formulation and proof of Frobenius reciprocity we pass from distributions to generalized functions. Note that the space of smooth functions embeds canonically into the space of generalized functions but there is no canonical embedding of smooth functions to the space of distributions.

Notation A.0.1. Let X be a smooth manifold. We denote by D_X the bundle of densities on X. For a point $x \in X$ we denote by $D_{X,x}$ its fiber in the point x. If X is a Nash manifold then the bundle D_X has a natural structure of a Nash bundle. For its description see [AG1], section 6.1.1.

[3] Alternatively, one can show that any H-invariant distribution on W supported at Y is a Schwartz distribution since Y has finite number of orbits.

Notation A.0.2. Let X be a smooth manifold. We denote by $C^{-\infty}(X)$ the space $C^{-\infty}(X) := \mathcal{D}(X, D_X)$ of **generalized functions** on X. Let E be a vector bundle on X. We also denote by $C^{-\infty}(X, E)$ the space $C^{-\infty}(X, E) := \mathcal{D}(X, D_X \otimes E^*)$ of generalized sections of E. For a locally closed subset $Y \subset X$ we denote $C_X^{-\infty}(Y) := \mathcal{D}_X(Y, D_X)$ and $C_X^{-\infty}(Y, E) := \mathcal{D}_X(Y, D_X \otimes E^*)$.

We will prove the following version of Frobenius reciprocity.

Theorem A.0.3 (Frobenius reciprocity)**.** *Let a Lie group G act transitively on a smooth manifold Z. Let $\varphi : X \to Z$ be a G-equivariant smooth map. Let $z_0 \in Z$. Denote by G_{z_0} the stabilizer of z_0 in G and by X_{z_0} the fiber of z_0. Let E be a G-equivariant vector bundle on X. Then there exists a canonical isomorphism* $\mathrm{Fr} : C^{-\infty}(X_{z_0}, E|_{X_{z_0}})^{G_{z_0}} \to C^{-\infty}(X, E)^G$*. Moreover, for any closed G_z-invariant subset $Y \subset X_{z_0}$, Fr maps $C^{-\infty}_{X_{z_0}}(Y, E|_{X_{z_0}})^{G_{z_0}}$ to $C_X^{-\infty}(GY, E)^G$.*

First we will need the following version of Harish-Chandra's submersion principle.

Theorem A.0.4 (Harish-Chandra's submersion principle)**.** *Let X, Y be smooth manifolds. Let $E \to X$ be a vector bundle. Let $\varphi : Y \to X$ be a submersion. Then the map $\varphi^* : C^{\infty}(X, E) \to C^{\infty}(Y, \varphi^*(E))$ extends to a continuous map $\varphi^* : C^{-\infty}(X, E) \to C^{-\infty}(Y, \varphi^*(E))$.*

Proof. By partition of unity it is enough to show for the case of trivial E. In this case it can be easily deduced from [Wal1], 8.A.2.5. $\square$

Also we will need the following fact that can be easily deduced from [Wal1], 8.A.2.9.

Proposition A.0.5. *Let $E \to Z$ be a vector bundle and G be a Lie group. Then there is a canonical isomorphism $C^{-\infty}(Z, E) \to C^{-\infty}(Z \times G, \mathrm{pr}^*(E))^G$, where $pr : Z \times G \to Z$ is the standard projection and the action of G on $Z\times$ G is the left action on the G coordinate.*

The last two statements give us the following corollary.

Corollary A.0.6. *Let a Lie group G act on a smooth manifold X. Let E be a G-equivariant bundle over X. Let $Z \subset X$ be a submanifold such that the action map $G \times Z \to X$ is submersive. Then there exists a canonical map $HC : C^{-\infty}(X, E)^G \to C^{-\infty}(Z, E|_Z)$.*

Now we can prove Frobenius reciprocity (Theorem A.0.3).

Proof of Frobenius reciprocity. We construct the map $\mathrm{Fr} : C^{-\infty}(X_{z_0}, E|_{X_{z_0}})^{G_{z_0}} \to C^{-\infty}(X, E)^G$ in the same way like in [Ber1] (1.5). Namely, fix a set-theoretic section $\nu : Z \to G$. It gives us in any point $z \in Z$ an identification between X_z and X_{z_0}. Hence we can interpret a generalized function $\xi \in C^{-\infty}(X_{z_0}, E|_{X_{z_0}})$ as a functional $\xi_z : C_c^{\infty}(X_z, E^*|_{X_z} \otimes D_{X_z}) \to \mathbb{C}$, or as a map $\xi_z : C_c^{\infty}(X_z, (E^* \otimes D_X)|_{X_z}) \to D_{Z,z}$. Now define

$$\mathrm{Fr}(\xi)(f) := \int_{z \in Z} \xi_z(f|_{X_z}).$$

It is easy to see that Fr is well-defined.

It is easy to see that the map $HC : C^{-\infty}(X, E)^G \to C^{-\infty}(X_{z_0}, E|_{X_{z_0}})$ described in the last corollary gives the inverse map.

The fact that for any closed G_z-invariant subset $Y \subset X_{z_0}$, Fr maps $C^{-\infty}_{X_{z_0}}(Y, E|_{X_{z_0}})^{G_{z_0}}$ to $C^{-\infty}_X(GY, E)^G$ follows from the fact that Fr commutes with restrictions to open sets. $\square$

Corollary A.0.7. *Theorem 4.2.3 holds.*

Proof. Without loss of generality we can assume that χ is trivial, since we can twist E by χ^{-1}. We have

$$\mathcal{D}(X,E)^G \cong C^{-\infty}(X, E^* \otimes D_X)^G \cong C^{-\infty}(X_{z_0}, (E^* \otimes D_X)|_{X_{z_0}})^{G_{z_0}} \cong (\mathcal{D}(X_{z_0}, E^*|_{X_{z_0}}) \otimes D_{Z,z_0})^{G_{z_0}}.$$

It is easy to see that in case that G and G_{z_0} are unimodular, the action of G_{z_0} on D_{Z,z_0} is trivial. $\square$

Remark A.0.8. For a Nash manifold X one can introduce the space of **generalized Schwartz functions** by $\mathcal{G}(X) := \mathcal{S}^*(X, D_X)$. Given a Nash bundle E one may consider the generalized Schwartz sections $\mathcal{G}(X,E) := \mathcal{S}^*(X, D_X \otimes E^*)$. Frobenius reciprocity in the Nash setting is obtained by restricting Fr and yields

$$Fr : \mathcal{G}(X,E)^G \cong \mathcal{G}(X_z, E|_{X_z})^{G_z}.$$

The proof goes along the same lines, but one has to prove that the corresponding integrals converge. We will not give the proof here since we will not use this fact.

Appendix B. Filtrations on spaces of distributions

B.1. **Filtrations on linear spaces.**

In what follows, a filtration on a vector space is always increasing and exhaustive. We make the following definition:

Definition B.1.1. Let V be a vector space. Let I be a well ordered set. Let F^i be a filtration on V indexed by $i \in I$. We denote $\mathrm{Gr}^i(V) := F^i/(\bigcup_{j<i} F^j)$.

The following lemma is obvious.

Lemma B.1.2. *Let V be a representation of an abstract group G. Let I be a well ordered set. Let F^i be a filtration of V by G invariant subspaces indexed by $i \in I$. Suppose that for any $i \in I$ we have $\mathrm{Gr}^i(V)^G = 0$. Then $V^G = 0$. An analogous statement also holds if we replace the group G by a Lie algebra $\mathfrak{g}$.*

B.2. **Filtrations on spaces of distributions.**

Theorem B.2.1. *Let X be a Nash manifold. Let E be a Nash bundle on X. Let $Z \subset X$ be a Nash submanifold. Then the space $\mathcal{S}^*_X(Z,E)$ has a natural filtration $F^k := F^k(\mathcal{S}^*_X(Z,E))$ such that $F^k/F^{k-1} \cong \mathcal{S}^*(Z, E|_Z \otimes Sym^k(CN^X_Z))$.*

For proof see [AG1], corollary 5.5.4.

We will also need the following important theorem

Theorem B.2.2. *Let X be a Nash manifold, $U \subset X$ be an open Nash submanifold and E be a Nash bundle over X. Then we have the following exact sequence*

$$0 \to \mathcal{S}^*_X(X \setminus U, E) \to \mathcal{S}^*(X,E) \to \mathcal{S}^*(U, E|_U) \to 0.$$

Proof. The only non-trivial part is to show that the restriction map $\mathcal{S}^*(X,E) \to \mathcal{S}^*(U, E|_U) \to 0$ is onto. It is done in [AG1], corollary 5.4.4. $\square$

Now we obtain the following corollary of theorem B.2.1 using the exact sequence from theorem B.2.2.

Corollary B.2.3. *Let X be a Nash manifold. Let E be Nash bundle over X. Let $Y \subset X$ be locally closed subset. Let $Y = \bigcup_{i=0}^{l} Y_i$ be a Nash stratification of Y.*

Then the space $\mathcal{S}_X^*(Y,E)$ has a natural filtration $F^{ik}(\mathcal{S}_X^*(Y,E))$ such that

$$\mathrm{Gr}^{ik}(\mathcal{S}_X^*(Y,E)) \cong \mathcal{S}^*(Y_i, E|_{Y_i} \otimes Sym^k(CN_{Y_i}^X))$$

for all $i \in \{1...l\}$ and $k \in \mathbb{Z}_{\geq 0}$.

Corollary B.2.4. *Let X be a Nash manifold. Let E be Nash bundle over X. Let $Y \subset X$ be locally closed subset. Let $Y = \bigcup_{i=0}^{l} Y_i$ be a Nash stratification of Y.*

Suppose that for any $0 \leq i \leq l$ and any $k \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{S}^*(Y_i, E|_{Y_i} \otimes Sym^k(CN_{Y_i}^X))^G = 0.$$

Then $\mathcal{S}_X^*(Y,E)^G = 0$.

By theorem 4.2.4, this corollary implies theorem 4.2.1.

B.3. **Fourier transform and proof of proposition 4.3.2.**

Notation B.3.1 (Fourier transform). Let V be a finite dimensional vector space over F. Let B be a non-degenerate bilinear form on V. We denote by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ the Fourier transform defined using B and the self-dual measure on V.

We will use the following well known fact.

Proposition B.3.2. *Let V be a finite dimensional vector space over F. Let B be a non-degenerate bilinear form on V. Consider the homothety action ρ of $F^\times$ on $\mathcal{S}^*(V)$. Then for any $\lambda \in F^\times$ we have*

$$\rho(\lambda) \circ \mathcal{F}_B = |\lambda|^{-\dim_{\mathbb{R}} V} \mathcal{F}_B \circ \rho(\lambda^{-1}).$$

Notation B.3.3. Let $(\rho, \mathcal{E})$ be a complex representation of $F^\times$. We denote by $JH(\rho, \mathcal{E})$ the subset of characters of $F^\times$ which are subquotients of $(\rho, \mathcal{E})$.

We will use the following straightforward lemma.

Lemma B.3.4. *Let $(\rho, \mathcal{E})$ be a complex representation of $F^\times$. Let χ be a character of $F^\times$. Suppose that there exists an invertible linear operator $A : \mathcal{E} \to \mathcal{E}$ such that for any $\lambda \in F^\times$, $\rho(\lambda) \circ A = \chi(\lambda) A \circ \rho(\lambda^{-1})$. Then $JH(\mathcal{E}) = \frac{\chi}{JH(\mathcal{E})}$.*

We will also use the following standard lemma.

Lemma B.3.5. *Let $(\rho, \mathcal{E})$ be a complex representation of $F^\times$ of countable dimension.*
(i) If $JH(\mathcal{E}) = \emptyset$ then $\mathcal{E} = 0$.
(ii) Let I be a well ordered set and F^i be a filtration on $\mathcal{E}$ indexed by $i \in I$ by subrepresentations. Then $JH(\mathcal{E}) = \bigcup_{i \in I} JH(\mathrm{Gr}^i(\mathcal{E}))$.

Now we will prove proposition 4.3.2. First we remind its formulation.

Proposition B.3.6. *Let G be a Nash group. Let V be a finite dimensional representation of G over F. Suppose that the action of G preserves some non-degenerate bilinear form B on V. Let $V = \bigcup_{i=1}^{n} S_i$ be a stratification of V by G-invariant Nash cones.*

Let $\mathfrak{X}$ be a set of characters of $F^\times$ such that the set $\mathfrak{X}\cdot\mathfrak{X}$ does not contain the character $|\cdot|^{\dim_{\mathbb{R}} V}$. Let χ be a character of G. Suppose that for any i and k, the space $\mathcal{S}^*(S_i, Sym^k(CN_{S_i}^V))^{G,\chi}$ consists of homogeneous distributions of type α for some $\alpha\in\mathfrak{X}$. Then $\mathcal{S}^*(V)^{G,\chi}=0$.

Proof. Consider $\mathcal{S}^*(V)^{G,\chi}$ as a representation of $F^\times$. It has a canonical filtration given by corollary B.2.3. It is easy to see that $\mathrm{Gr}^{ik}(\mathcal{S}^*(V)^{G,\chi})$ is canonically imbedded into $(\mathrm{Gr}^{ik}(\mathcal{S}^*(V))^{G,\chi}$. Therefore by the previous lemma $JH(\mathcal{S}^*(V)^{G,\chi})\subset\mathfrak{X}^{-1}$. On the other hand G preserves B and hence we have $\mathcal{F}_B:\mathcal{S}^*(V)^{G,\chi}\to\mathcal{S}^*(V)^{G,\chi}$. Therefore by lemma B.3.4 we have

$$JH(\mathcal{S}^*(V)^{G,\chi})\subset|\cdot|^{-\dim_{\mathbb{R}} V}\mathfrak{X}.$$

Hence $JH(\mathcal{S}^*(V)^{G,\chi})=\emptyset$. Thus $\mathcal{S}^*(V)^{G,\chi}=0$. □

B.4. **Proof of proposition 4.3.1.**

The following proposition clearly implies proposition 4.3.1.

Proposition B.4.1. *Let X be a smooth manifold. Let V be a real finite dimensional vector space. Let $U\subset V$ be an open non-empty subset. Let E be a vector bundle over X. Then for any $k\geq 0$ there exists a canonical embedding $\mathcal{D}(X,E\otimes Sym^k(V))\hookrightarrow\mathcal{D}(X\times U,E\boxtimes D_V)$.*

Proof. It is enough to construct a continuous linear epimorphism

$$\pi: C_c^\infty(X\times U,E\boxtimes D_V)\twoheadrightarrow C_c^\infty(X,E\otimes Sym^k(V)).$$

By partition of unity it is enough to do it for trivial E. Let $w\in C_c^\infty(X\times U,D_V)$ and $x\in X$ we have to define $\pi(w)(x)\in Sym^k(V)$. Consider the space $Sym^k(V)$ as the space of linear functionals on the space of homogeneous polynomials on V of degree k. Define

$$\pi(w)(x)(p):=\int_{y\in V}p(y)w(x,y).$$

It is easy to check that $\pi(w)\in C_c^\infty(X,Sym^k(V))$ and π is continuous linear epimorphism. □

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Avraham Aizenbud and Dmitry Gourevitch, Faculty of Mathematics and Computer Science, The Weizmann Institute of Science POB 26, Rehovot 76100, Israel

E-mail address: aizenr@yahoo.com

E-mail address: guredim@yahoo.com

Eitan Sayag, Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel

E-mail address: eitan.sayag@gmail.com