HAUSDORFFNESS FOR LIE ALGEBRA HOMOLOGY OF SCHWARTZ SPACES

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Abstract. Let H be a Nash group acting equivariantly on a Nash manifold X and a Nash bundle $\mathcal{E}$ of X. Let $\mathfrak{h}$ be the Lie algebra of H. Let $\mathcal{S}(X,\mathcal{E})$ be the space of Schwartz sections of $\mathcal{E}$. In this paper, under some conditions, we prove a duality theorem concerning the Lie algebra homology $\mathrm{H}_i(\mathfrak{h},\mathcal{S}(X,\mathcal{E}))$. One equivalent formulation is that all the homological spaces $\mathrm{H}_i(\mathfrak{h},\mathcal{S}(X,\mathcal{E}))$ are separated, i.e Hausdorff. In particular, $\mathfrak{h}\mathcal{S}(X,\mathcal{E})$ is a closed subspace of $\mathcal{S}(X,\mathcal{E})$.

As an application we prove a certain comparison theorem between homologies of smooth representations and their Harish-Chandra modules.

1. Introduction

In this paper we start to develop the theoretical background for comparison theorems in homological representation theory attached to a real reductive group G.

The algebraic side of representation theory for G is encoded in the theory of Harish-Chandra modules V. These are modules for the Lie algebra $\mathfrak{g}$ of G with a compatible algebraic action of a fixed maximal compact subgroup K of G. According to Casselman-Wallach (see [Wal92, Chapter 11] or [Cas89] or, for a different approach, [BK]) Harish-Chandra modules V can be naturally completed to smooth moderate growth modules V^∞ for the group G. Somewhat loosely speaking one might think of V, resp. V^∞, as the regular, resp. smooth, functions on some real algebraic variety.

Fix a subalgebra $\mathfrak{h} < \mathfrak{g}$. As $V \subset V^\infty$ we obtain natural maps

$$\Phi_p : \mathrm{H}_p(\mathfrak{h}, V) \to \mathrm{H}_p(\mathfrak{h}, V^\infty)\,.$$

Conjecture A. (Comparison Conjecture) *If $\mathfrak{h}$ is a real spherical subalgebra, then Φ_p is an isomorphism for all p.*

Note that $\mathrm{H}_p(\mathfrak{h}, V)$ is finite dimensional (see Section 5). If $\mathfrak{h}$ is a maximal unipotent subalgebra then Conjecture A is the still not fully established Casselman comparison theorem (see [HT98] for G split).

According to the Casselman subrepresentation theorem every smooth completion V^∞ is the quotient of the section module of an equivariant vector bundle $\mathcal{E} \to X$ where $X = G/P$ is the minimal flag variety. A first step towards the Comparison Conjecture is to understand the topological nature of the modules $\mathrm{H}_p(\mathfrak{h}, C^\infty(X,\mathcal{E}))$. In particular one needs to know whether these topological vector spaces are separated (Hausdorff). Under some restrictions on $\mathfrak{h}$ and $\mathcal{E}$, this will be answered in this article.

Let now H' be a Nash group (not necessarily reductive) and H be a normal Nash sub-group of H'. Let X be a Nash manifold and $\mathcal{E}$ be a Nash vector bundle over X. Assume that H' acts equivariantly on X and $\mathcal{E}$. Let $\mathcal{S}(X,\mathcal{E})$ be the space of Schwartz sections with respect to $\mathcal{E} \longrightarrow X$. Then $\mathcal{S}(X,\mathcal{E})$ is a nuclear Fréchet space and H' acts smoothly on $\mathcal{S}(X,\mathcal{E})$. Let $\mathfrak{h},\mathfrak{h}'$ be the Lie algebras of H, H'.

Since the action of H' on $\mathcal{S}(X,\mathcal{E})$ is smooth, the space $\mathcal{S}(X,\mathcal{E})$ becomes a $\mathfrak{h}'$-module (then also a $\mathfrak{h}$-module). Equipped with the quotient topology, each homological space $\mathrm{H}_i(\mathfrak{h},\mathcal{S}(X,\mathcal{E}))$ becomes a topological vector space. The main theorem of this paper is:

Theorem B. *Suppose that the number of H'-orbits in X is finite, and H and all the stabilizers H_x ($x \in X$) are homologically trivial (e.g. contractible). Then $\mathrm{H}_i(\mathfrak{h},\mathcal{S}(X,\mathcal{E}))$ is separated and duality holds:*

$$\mathrm{H}_i(\mathfrak{h},\mathcal{S}(X,\mathcal{E})) \cong (\mathrm{H}^i(\mathfrak{h},\mathcal{S}(X,\mathcal{E})'))',$$

i.e., $\mathrm{H}_i(\mathfrak{h},\mathcal{S}(X,\mathcal{E}))$ is topologically isomorphic to the strong dual of $H^i(\mathfrak{h},\mathcal{S}(X,\mathcal{E})')$. In particular, the subspace $\mathfrak{h}\mathcal{S}(X,\mathcal{E}) \subset \mathcal{S}(X,\mathcal{E})$ is closed.

In Section 5 we deduce from this theorem a special case of Conjecture A (see Theorem 5.0.3).

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2. Complexes of topological vector spaces

2.1. **Preliminaries.** Let us begin with a brief recall on some standard facts about topological vector spaces. For more details we refer the reader to [CHM00, Appendix A].

A topological vector space V is called *separated* or *Hausdorff* if $\{0\} \subset V$ is closed. Non-separated topological vector spaces typically arise as quotients V/U where $U \subset V$ is a non-closed subspace of the topological vector space V.

If V is a topological vector space, then we denote by V' its topological dual, that is the space of continuous linear functionals $V \to \mathbb{C}$. We endow V' with the strong dual topology (i.e. the topology of uniform convergence on bounded sets) and note that V' is a separated topological vector space.

If $T : V \to W$ is a morphism of topological vector spaces, then we denote by $T' : V' \to W'$ the corresponding dual morphism. A morphism $T : V \to W$ is called *strict*, provided that T induces an isomorphism of topological vector spaces $V/\ker T \simeq \operatorname{im} T$. Strict morphisms typically arise as morphisms which have closed image and V, W are such that the open mapping theorem holds, e.g. V, W are Fréchet spaces or more generally if V is strictly bornological and W is an inductive limit of Fréchet spaces (see [G73, Ch. 4, §5, Th. 2]).

A topological vector space V is called *reflexive* if the canonical map

$$\iota : V \to V'' := (V')';\ \iota(v)(\lambda) := \lambda(v)$$

is an isomorphism.

Our concern in this section is with bounded complexes of topological vector spaces $(V_\bullet, d_\bullet)$. That is for all $p \in \mathbb{Z}$ we are given a topological vector space V_p and morphisms $d_p : V_p \to V_{p-1}$ which satisfy $d_{p+1} \circ d_p = 0$. We are interested in the homology of the complex $\mathrm{H}_\bullet(V_\bullet, d_\bullet)$ which is given by $\mathrm{H}_p(V_\bullet, d_\bullet) := \ker d_p / \operatorname{im} d_{p+1}$. Clearly $\mathrm{H}_\bullet(V_\bullet, d_\bullet)$ is separated if all differentials d_p have closed image.

Attached to $(V_\bullet, d_\bullet)$ is the dual complex $(V^\bullet, d^\bullet)$ where $V^p := (V_p)'$ and $d^p := d'_{p+1} : V^p \to V^{p+1}$. We denote by $\mathrm{H}^\bullet(V^\bullet, d^\bullet)$ the corresponding cohomology spaces, that is $\mathrm{H}^p(V^\bullet, d^\bullet) := \ker d^p / \operatorname{im} d^{p-1}$. Note that there is a natural mapping:

$$\iota_p : \mathrm{H}_p(V_\bullet, d_\bullet) \to H^p(V^\bullet, d^\bullet)', \quad v + \operatorname{im} d_{p+1} \mapsto (\lambda + \ker d^{p-1} \mapsto \lambda(v)) .$$

We introduce an auxiliary notion and call complex of topological vector spaces $(V_\bullet, d_\bullet)$ *good* if all maps ι_p are isomorphisms.

In the sequel we will work with a special class of topological vector spaces which are suited for topological homology. Let us denote by NF the category of nuclear Fréchet spaces. Note that:

- Closed subspaces of an NF-space are NF.
- Quotients of NF-spaces by closed subspaces are NF.
- NF-spaces are reflexive.

The spaces dual to NF spaces are called DNF-spaces. Note that DNF-spaces satisfy the assumptions of the open mapping theorem. Furthermore, reflexivity implies that a morphism $T : V \to W$ between NF-spaces is strict if and only if the dual morphism $T' : W' \to V'$ is strict. We summarize the discussion (see also [CHM00, Lemma A.2]):

Lemma 2.1.1. *Let $(V_\bullet, d_\bullet)$ be a complex of NF-spaces (or DNF-spaces). Then the following assertions hold:*

(1) $(V_\bullet, d_\bullet)$ is good if and only if all $\mathrm{H}_p(V_\bullet, d_\bullet)$ are separated.
(2) $\mathrm{H}_p(V_\bullet, d_\bullet)$ is separated if and only if $\mathrm{H}^{p+1}(V^\bullet, d^\bullet)$ is separated.

2.2. **Extensions and limits of good complexes.**

Now we prove the following lemma which is important in this paper.

Lemma 2.2.1. (Extension property) *Let*

$$0 \longrightarrow U_\bullet \longrightarrow V_\bullet \longrightarrow W_\bullet \longrightarrow 0$$

be a short exact sequence of bounded NF-complexes. Then if two of them are good, the third is also good.

Proof. We only treat the case where $U_\bullet$ and $W_\bullet$ are good, as the other two cases have a similar proof. Our objective is to show that $V_\bullet$ is good, that is $\mathrm{H}_\bullet(V_\bullet)$ is naturally isomorphic to $\mathrm{H}^\bullet(V^\bullet)'$.

Firstly, according to Lemma 2.1.1, the dual sequence

$$0 \longrightarrow W^\bullet \longrightarrow V^\bullet \longrightarrow U^\bullet \longrightarrow 0$$

is also exact.

We look at the sequence of homologies for the complex and the corresponding dual complexes:

$$\mathrm{H}_{i+1}(W_\bullet) \longrightarrow \mathrm{H}_i(U_\bullet) \longrightarrow \mathrm{H}_i(V_\bullet) \longrightarrow \mathrm{H}_i(W_\bullet) \longrightarrow \mathrm{H}_{i-1}(U_\bullet) \tag{1}$$

and

$$\mathrm{H}^{i-1}(U^\bullet) \longrightarrow \mathrm{H}^i(W^\bullet) \longrightarrow \mathrm{H}^i(V^\bullet) \longrightarrow \mathrm{H}^i(U^\bullet) \longrightarrow \mathrm{H}^{i+1}(W^\bullet). \tag{2}$$

We prove by induction on i that $\mathrm{H}_i(V_\bullet)$ are separated. Since the complex is bounded, the base is trivial. Fix $i \in \mathbb{Z}$ and assume that $\mathrm{H}_{i-1}(V_\bullet)$ is separated. Lemma 2.1.1(2) implies that $\mathrm{H}^i(V^\bullet)$ is separated. Then according to Lemma 2.1.1(1), we dualize (2) and arrive at the exact sequence:

$$\mathrm{H}^{i+1}(W^\bullet)' \longrightarrow \mathrm{H}^i(U^\bullet)' \longrightarrow \mathrm{H}^i(V^\bullet)' \longrightarrow \mathrm{H}^i(W^\bullet)' \longrightarrow \mathrm{H}^{i-1}(U^\bullet)' \,.$$

Since $U_\bullet$ and $W_\bullet$ are good we have natural isomorphism $\mathrm{H}_\bullet(W_\bullet) \simeq \mathrm{H}^\bullet(W^\bullet)'$ and likewise for $U_\bullet$. The Five Lemma implies that $\mathrm{H}_i(V_\bullet) \to \mathrm{H}^i(V^\bullet)'$ is a linear isomorphism. In particular $\mathrm{H}_i(V_\bullet)$ is separated. □

Corollary 2.2.2. *Let $V^i_\bullet$ ($i \in \mathbb{N}$) be a projective system of uniformly bounded NF-complexes, such that $T_i : V^{i+1}_\bullet \longrightarrow V^i_\bullet$ is onto. Let $V_\bullet = \varprojlim V^i_\bullet$ (the topological projective limit). Suppose that all $V^i_\bullet$ are good. Then $V_\bullet$ is good.*

Proof. Let $V^{pro}_\bullet = \prod V^i_\bullet$. Then it is known that $V^{pro}_\bullet$ is also an NF-complex (see [Tre67, Proposition 50.1]) and $V_\bullet$ is a closed sub-complex of $V^{pro}_\bullet$. Moreover it is easy to check that $V^{pro}_\bullet$ is good.

On the other hand, since all T_i are onto, it is not difficult to check that the following short sequence is exact:

$$0 \longrightarrow V_\bullet \longrightarrow V^{pro}_\bullet \xrightarrow{T} V^{pro}_\bullet \longrightarrow 0,$$

with $T(v_0, ..., v_i, ...) = (v_0 - T_0(v_1), ..., v_i - T_i(v_{i+1}), ...)$, is exact. The assertion follows now from the previous Lemma 2.2.1 □

Later we will apply the results of this section to the Koszul complex attached to a NF-module V for a finite dimensional Lie algebra $\mathfrak{h}$: $V_p := \bigwedge^p \mathfrak{h} \otimes V$ for $p \geq 0$ and $V_p = \{0\}$ for $p < 0$. The homology of the the Koszul-complex is denoted by $\mathrm{H}_\bullet(\mathfrak{h}, V)$.

We will call an NF-$\mathfrak{h}$-module V *good* if the associated Koszul complex is good.

3. Schwartz spaces on Nash manifolds

3.1. **Nash manifolds.** Nash manifolds are smooth semi-algebraic manifolds. Nash manifolds are equipped with the *restricted topology*, in which open sets are open semi-algebraic sets. This is not a topology in the usual sense of the word as infinite unions of open sets are not necessarily open sets in the restricted topology. However, finite unions of open sets are open and therefore in the restricted topology we consider only finite covers. In particular, if $\mathcal{E}$ over X is a Nash vector bundle it means that there exists a finite open cover U_i of X such that $\mathcal{E}|_{U_i}$ is trivial. For more details on Nash manifolds we refer the reader to [BCR98, Shi87, Sun].

Theorem 3.1.1 (Local triviality of Nash manifolds; [Shi87, Theorem I.5.12])**.** *Any Nash manifold can be covered by a finite number of open submanifolds Nash diffeomorphic to* $\mathbb{R}^n$.

Theorem 3.1.2 (see e.g. [AG10, Theorem 2.4.16])**.** *Let* $s : X \to Y$ *be a surjective submersive Nash map. Then locally it has a Nash section, i.e. there exists a finite open cover* $Y = \bigcup_{i=1}^{k} U_i$ *such that* s *has a Nash section on each* U_i.

A Lie group G is called a *Nash group* provided that G is a Nash manifold and all group operations being Nash maps.

By a *(Nash) stratification* of a Nash-manifold X we understand a finite union $X = \bigcup_{i=1}^k X_i$ such that $\bigcup_{i=j}^k X_i$ is an open Nash subset of X for any j.

Lemma 3.1.3. *Let* X *be a Nash manifold, let* $\mathcal{E}, \mathcal{F}$ *be a Nash bundle over* X *and* $\phi : \mathcal{E} \to \mathcal{F}$ *be a morphism of Nash bundles. Then there exists a stratification of* X *by Nash submanifolds such that* ϕ *has constant rank on each stratum.*

For the proof of this lemma we will need another Lemma:

Lemma 3.1.4. *Let* X *be a Nash manifold and* Z *be a closed subset (in the restricted topology). Then there exists a stratification of* Z *by (locally closed) Nash submanifolds of* X.

Proof. Let $\Delta(Z)$ denote the singular locus of Z. The lemma is equivalent to the fact that $\Delta(...\Delta(Z)) = \emptyset$ for enough iterations of Δ. This statement is local, so by Theorem 3.1.1 it is enough to prove it for $X = \mathbb{R}^n$. In this case the lemma is [BCR98, Theorem 9.1.8]. □

Proof of Lemma 3.1.3. It is easy that $X_k := \{x \in X \text{ s.t. } \mathrm{rk}\phi_x \leq k\}$ is closed. The assertion follows now from Lemma 3.1.4. □

3.2. **Schwartz functions on Nash manifolds.** Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. We also need the notion of tempered functions, i.e. smooth functions that grow not faster than a polynomial, and so do all their derivatives. For precise definitions of those notions we refer the reader to [AG08]. In this section we summarize some elements of the theory of Schwartz functions.

Fix a Nash manifold X and a Nash bundle $\mathcal{E}$ over X. We denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X and by $\mathcal{S}(X,\mathcal{E})$ the space of Schwartz sections of $\mathcal{E}$. We collect a few central facts which will be used implicitly in the sequel:

- $\mathcal{S}(\mathbb{R}^n)$ = Classical Schwartz functions on $\mathbb{R}^n$. See [AG08, Theorem 4.1.3].
- The space $\mathcal{S}(X,\mathcal{E})$ is a nuclear Fréchet space. See [AG10, Corollary 2.6.2].
- Let $Z \subset X$ be a closed Nash submanifold. Then the restriction maps $\mathcal{S}(X,\mathcal{E})$ onto $\mathcal{S}(Z,\mathcal{E}|_Z)$. See [AG08, §1.5].

Proposition 3.2.1 (Partition of unity, [AG08, §5])**.** *Let $X = \bigcup U_i$ be a finite open cover of X. Then there exists a tempered partition of unity $1 = \sum_{i=1}^n \lambda_i$ such that for any Schwartz section $f \in \mathcal{S}(X,\mathcal{E})$ the section $\lambda_i f$ is a Schwartz section of $\mathcal{E}$ on U_i (extended by zero to X).*

Proposition 3.2.2 ([AG08, Theorem 5.4.3])**.** *Let $U \subset X$ be a (semi-algebraic) open subset, then*

$$\mathcal{S}(U,\mathcal{E}) \cong \{\phi \in \mathcal{S}(X,\mathcal{E})| \quad \phi \textit{ is 0 on } X \setminus U \textit{ with all derivatives}\}.$$

In particular, extension by zero defines a closed imbedding $\mathcal{S}(U,\mathcal{E}) \hookrightarrow \mathcal{S}(X,\mathcal{E})$.

Let Z be a locally closed semi-algebraic subset of X. Denote

$$\mathcal{S}_X(Z,\mathcal{E}) := \mathcal{S}(X - (\overline{Z} - Z),\mathcal{E})/\mathcal{S}(X - \overline{Z},\mathcal{E})\,.$$

Here we identify $\mathcal{S}(X - \overline{Z},\mathcal{E})$ with a closed subspace of $\mathcal{S}(X - (\overline{Z} - Z),\mathcal{E})$ using the description of Schwartz functions on an open set (Proposition 3.2.2).

To obtain a feeling for the objects $\mathcal{S}_X(Z,\mathcal{E})$ let us consider the case of the trivial bundle and $Z = \{pt\}$ a point. Then $\mathcal{S}_{\{pt\}}(X) = \mathcal{S}(X)/\mathcal{S}(X - \{pt\})$ and Proposition 3.2.2 implies that there is a well defined injective map (the Taylor series map at the point $\{pt\}$) into the ring of power series in $n = \dim X$ variables:

$$\mathcal{S}_{\{pt\}}(X) \to \mathbb{C}[[x_1,\ldots,x_n]]\,.$$

The contents of Borel's Lemma is that this map is surjective. Note that the formal power series have a natural structure as projective limit. The generalization of Borel's lemma now reads as (see [AG13, Lemmas B.0.8 and B.0.9]):

Lemma 3.2.3. *Let $Z \subset X$ be a Nash submanifold.*

Then $\mathcal{S}_X(Z,\mathcal{E})$ has a canonical countable decreasing filtration by closed subspaces $\mathcal{S}_X(Z,\mathcal{E})^i$ satisfying:

(1) $\bigcap \mathcal{S}_X(Z,\mathcal{E})^i = \{0\}$.
(2) $gr_i(\mathcal{S}_X(Z,\mathcal{E})) \cong \mathcal{S}(Z, \mathrm{Sym}^i(CN_Z^X) \otimes \mathcal{E})$, *where* $CN_Z^X = (TX|_Z/TZ)^*$ *denotes the conormal bundle to* Z *in* X.
(3) The natural map

$$\mathcal{S}_X(Z,\mathcal{E}) \to \lim_{\leftarrow}(\mathcal{S}_X(Z,\mathcal{E})/\mathcal{S}_X(Z,\mathcal{E})^i)$$

is an isomorphism.

The spaces $\mathcal{S}_Z(X,\mathcal{E})$ naturally appear in the context of stratifications.

Lemma 3.2.4. *Let* $X = \bigcup_{i=1}^k X_i$ *be a Nash stratification of* X. *Then* $\mathcal{S}(X,\mathcal{E})$ *has a natural filtration of length* k *such that* $Gr^i(\mathcal{S}(X,\mathcal{E})) = \mathcal{S}_X(X_i,\mathcal{E})$. *Moreover, if* Y *is a Nash manifold and* $X \subset Y$ *is a (locally closed) Nash submanifold then* $\mathcal{S}_Y(X,\mathcal{E})$ *has a natural filtration of length* k *such that* $Gr^i(\mathcal{S}_Y(X,\mathcal{E})) = \mathcal{S}_Y(X_i,\mathcal{E})$.

Proof. Straightforward from the definitions. □

Finally we record:

Lemma 3.2.5 ([AGS, Lemma 5.1.1]). *Suppose*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is an exact sequence of Nash bundles on X. *Then*

$$0 \to \mathcal{S}(X,\mathcal{E}_1) \to \mathcal{S}(X,\mathcal{E}_2) \to \mathcal{S}(X,\mathcal{E}_3) \to 0.$$

is an exact sequence of Fréchet spaces.

3.3. **Relative Shapiro lemma.** We will use upper case Latin letters, e.g. H, G etc., to denote real Lie groups and lower case Gothic letters for their corresponding Lie algebras, e.g. $\mathfrak{h}$, $\mathfrak{g}$ etc.

Theorem 3.3.1 ([AGS, Theorem 6.2.1]). *Let* H *be an affine Nash group and* X *be a transitive Nash* H*-manifold. Let* Y *be a Nash manifold. Let* $x \in X$ *and denote* $L := H_x$. *Let* $\mathcal{E} \to X \times Y$ *be a* H *equivariant Nash bundle. Suppose that* H *and* L *are homologically trivial (i.e. all their homology except* H_0 *vanish and* $\mathrm{H}_0 = \mathbb{R}$*).*

$$\mathrm{H}_i(\mathfrak{h}, \mathcal{S}(X \times Y, \mathcal{E})) \cong \mathrm{H}_i(\mathfrak{l}, \mathcal{S}(\{x\} \times Y, \mathcal{E}_{\{x\}\times Y} \otimes \Delta_L \cdot (\Delta_H^{-1})|_L)),$$

where Δ_L *and* Δ_H *denote the modular characters of the groups* L *and* H.

The proof in [AGS, Theorem 6.2.1] gives formally only an isomorphism of vector spaces. However in view of the following lemma the same proof gives isomorphism of topological vector spaces.

Lemma 3.3.2. *Let*

$$\cdots \to V_i \xrightarrow{d_i} \cdots$$

and

$$\cdots \to W_i \xrightarrow{e_i} \cdots$$

be complexes of Fréchet spaces. Let $\phi_i : V_i \to W_i$ be a continuous morphism between them. Assume that the co-homologies of one of those complexes are separated. Assume also that the ϕ_i induce linear isomorphisms on the co-homologies.

Then the co-homologies of both complexes are separated and ϕ_i induces topological vector spaces' isomorphism on the co-homologies.

Proof. By the Banach open map theorem, it is enough to show that the co-homologies of both complexes are separated. If the co-homologies of

$$\cdots \to W_i \to \cdots$$

are separated then the statement is obvious, since $\operatorname{Im} d_{i-1} = \phi_i^{-1}(\operatorname{Im} e_{i-1})$. Thus we can assume that the co-homologies of

$$\cdots \to V_i \to \cdots$$

are separated, and we want to prove that the co-homologies of

$$\cdots \to W_i \to \cdots$$

are separated. Fix i. We need to prove that $\operatorname{Im} e_{i-1}$ is closed. Without loss of generality we can assume that $V_{i+1} = W_{i+1} = 0$. So we know that

(1) $\operatorname{Im} d_{i-1}$ is closed
(2) $\operatorname{Im} e_{i-1} + \operatorname{Im} \phi_i = W_i$
(3) $\phi_i^{-1}(\operatorname{Im} e_{i-1}) = \operatorname{Im} d_{i-1}$.

Consider the map $e_{i-1} \oplus \phi_i : W_{i-1} \oplus V_i \to W_i$. We know that this map is onto. It is easy to see that

$$(e_{i-1} \oplus \phi_i)^{-1}(\operatorname{Im} e_{i-1}) = W_{i-1} \oplus \phi_i^{-1}(\operatorname{Im} e_{i-1}) = W_{i-1} \oplus \operatorname{Im} d_{i-1}.$$

Thus $(e_{i-1} \oplus \phi_i)^{-1}(\operatorname{Im} e_{i-1})$ is closed. By the Banach open map theorem this implies that $\operatorname{Im} e_{i-1}$ is closed. □

Let a Nash group H act transitively on a Nash manifold Y and let Z be a Nash manifold. Let Z "act" on H i.e. let H' be a Nash group acting on H by automorphisms and $a : Z \to H'$ be a Nash map. This defines a twisted action of H on $Z \times Y$. More precisely

$$\rho_1(g)(x, y) = (x, a(x)(g)(y)) \,.$$

Let ρ_2 denote the non-twisted action of H on $Z \times Y$, i.e. $\rho_2(g)(x, y) = (x, gy)$. Let $\mathcal{E} \longrightarrow Z \times Y$ be a H-equivariant Nash-bundle, with respect to the action ρ_1. We want to construct a H-equivariant structure on $\mathcal{E}$ with respect to the action ρ_2, such that the representations of $\mathfrak{g}$ on the global Schwartz sections of the two bundles will have isomorphic homologies.

Let $\mathrm{p} : H \times Z \times Y \longrightarrow Z \times Y$ be the natural projection. Define $\widetilde{a} : H \times Z \times Y \longrightarrow H \times Z \times Y$ by $\widetilde{a}(g, x, y) = (a(x)(g), x, y)$, and note that $\mathrm{p} = \mathrm{p} \circ \widetilde{a}$ and $\rho_1 = \rho_2 \circ \widetilde{a}$. The equivariant structure on $\mathcal{E}$ gives an isomorphism $p^*(\mathcal{E}) \simeq \rho_1^*(\mathcal{E})$, and thus $\widetilde{a}^*(\mathrm{p}^*(\mathcal{E})) \simeq \widetilde{a}^*(\rho_2^*(\mathcal{E}))$. Applying $(\widetilde{a^{-1}})^*$ we get an isomorphism $p^*(\mathcal{E}) \simeq \rho_2^*(\mathcal{E})$, which defines a ρ_2 equivariant structure. Let π_i denote the representation of $\mathfrak{h}$ on $\mathcal{S}(Z \times Y, \mathcal{E})$ given by the action ρ_i.

Proposition 3.3.3. *In the setting as above we have*

$$\mathrm{H}_*(\mathfrak{h}, \pi_1) = \mathrm{H}_*(\mathfrak{h}, \pi_2).$$

This proposition is similar to [AGS, Proposition 6.2.5], which concerns only line bundles, but in greater generality. However since we care here also about the topology we will have to make the proof more explicit.

Proof. Consider the Koszul complexes

$$\mathcal{C}_i : \cdots \to \Lambda^k(\mathfrak{h}) \otimes \mathcal{S}(Z \times Y, \mathcal{E}) \xrightarrow{d_k^i} \Lambda^{k-1}(\mathfrak{h}) \otimes \mathcal{S}(Z \times Y, \mathcal{E}) \to \cdots .$$

that compute the homologies of π_i. Identify $\Lambda^k(\mathfrak{h}) \otimes \mathcal{S}(Z \times Y, \mathcal{E})$ with $\mathcal{S}(Z \times Y, \Lambda^k(\mathfrak{h}) \otimes \mathcal{E})$. For any $x \in Z$ we have an isomorphism

$$\phi_x := \Lambda^k(da(x)) : \Lambda^k(\mathfrak{h}) \to \Lambda^k(\mathfrak{h}).$$

This gives us an isomorphism of vector bundles

$$\Phi_k(x) : \Lambda^k(\mathfrak{h}) \otimes \mathcal{E} \to \Lambda^k(\mathfrak{h}) \otimes \mathcal{E},$$

which in turn gives us an isomorphism

$$\mathcal{F}_k : \mathcal{S}(Z \times Y, \Lambda^k(\mathfrak{h}) \otimes \mathcal{E}) \to \mathcal{S}(Z \times Y, \Lambda^k(\mathfrak{h}) \otimes \mathcal{E}).$$

It is enough to show that

$$d^2_{k+1} \circ \mathcal{F}_k = \mathcal{F}_k \circ d^1_{k+1} \tag{3}$$

For this let m be an element of the total space of $\mathcal{E}$ and denote by (x_m, y_m) its projection to $Z \times Y$. Then according to our construction, for any $h \in H$ we have $\rho_2(h)(m) = \rho_1(a^{-1}(x_m)h)(m)$. Thus for any $s \in \mathcal{S}(Z \times Y, \mathcal{E})$ we have

$$(\pi_1(h).s)(x, y) = \rho_1(h)(s(x, (a(x)h^{-1}).y)) \tag{4}$$

$$(\pi_2(h).s)(x, y) = \rho_1(a^{-1}(x)h)(s(x, h^{-1}.y)) \tag{5}$$

Further let $X \otimes s \in \mathcal{S}(Z \times Y, \Lambda^k(\mathfrak{h}) \otimes \mathcal{E})$, where $X \in \Lambda^k(\mathfrak{h})$ and $s \in \mathcal{S}(Z \times Y, \mathcal{E})$. Then for any $(x, y) \in Z \times Y$ we have

$$(\mathcal{F}_k(X \otimes s))(x, y) = \phi_x(X) \otimes s(x, y) = da(x)(X) \otimes s(x, y). \tag{6}$$

Now let $X_j \in \mathfrak{h}$ $(1 \leq j \leq k+1)$ and $s \in \mathcal{S}(Z \times Y, \mathcal{E})$. Then the differentials in the Kozhul complexes are given by the following formula:

(7)

$$d^i_{k+1}(X_1 \wedge ... \wedge X_{k+1} \otimes s) = \sum_{l=1}^{k+1} (-1)^l (X_1 \wedge ... \wedge \widehat{X_l} \wedge ... \wedge X_{k+1} \otimes d\pi_i(X_l).s) +$$
$$+ \sum_{r<s} (-1)^{r+s} ([X_r, X_s] \wedge X_1 \wedge ... \wedge \widehat{X_r} \wedge ... \widehat{X_s} \wedge ... \wedge X_{k+1} \otimes s).$$

Now note that the equality (3) follows now from (4,5,6,7) and implies the assertion. □

4. Proof of Theorem B

Recall our general setting. H' is a Nash group and H is a normal Nash subgroup of H'. X is a Nash manifold and E is a Nash vector bundle of X. $\mathcal{S}(X, \mathcal{E})$ denotes the space of Schwartz sections of $\mathcal{E}$. Let H' act equivariantly on X and $\mathcal{E}$. Our assumptions are

1. The number of H'-orbits in X is finite.
2. H and all the stabilizers H_x $(x \in X)$ are homologically trivial (e.g. contractible).

Our goal is to prove that as a $\mathfrak{h}$-module, $\mathcal{S}(X, \mathcal{E})$ is good.

Let us begin with two lemmas that we deduce from sections 2 and 3.2.

Lemma 4.0.1. *Let $X = \bigcup_{i=1}^n U_i$ be a finite open covering. Assume that all the spaces $\mathcal{S}(U_i, \mathcal{E})$ and for any finite set of indices $\{i_1, \ldots i_k\} \subset \{1, \ldots, n\}$, the $\mathfrak{h}$-module $\mathcal{S}(\bigcap_{j=1}^k U_{i_j}, \mathcal{E})$ is good. Then $\mathcal{S}(X, \mathcal{E})$ is also good.*

Proof. Firstly, the proof can be reduced to the case where $X = U_1 \bigcup U_2$. Indeed, by induction on n, we can assume $\mathcal{S}(U, \mathcal{E})$ and $\mathcal{S}(U_n, \mathcal{E})$ are good, where $U = \bigcup_{i=1}^{n-1} U_i$. On the other hand, we have $X = U \bigcup U_n$ and $U \bigcap U_n = \bigcup_{i=1}^{n-1} U_i \cap U_n$ and thus the induction hypothesis implies that $\mathcal{S}(U_n, \mathcal{E})$ is good.

Now it remains to deal with the situation where $X = U_1 \bigcup U_2$. By Proposition 3.2.1, we have the following exact sequence of $\mathfrak{h}$-modules:

$$0 \longrightarrow \mathcal{S}(U_1 \bigcap U_2, \mathcal{E}) \xrightarrow{T_1} \mathcal{S}(U_1, \mathcal{E}) \bigoplus \mathcal{S}(U_2, \mathcal{E}) \xrightarrow{T_2} \mathcal{S}(X, \mathcal{E}) \longrightarrow 0.$$

with $T_1(f) = (\widetilde{f_1}, -\widetilde{f_2})$ and $T_2(f, g) = \widetilde{f} + \widetilde{g}$. Here for $U \subset V$ (U, V are two open subsets) and $f \in \mathcal{S}(U, \mathcal{E})$, we use $\widetilde{f} \in \mathcal{S}(V, \mathcal{E})$ to denote the extension by zero of f.

Since by assumption the first two $\mathfrak{h}$-modules in the sequence are good, the goodness of $\mathcal{S}(X, \mathcal{E})$ follows from Lemma 2.2.1. □

Lemma 4.0.2. *Let $X = \bigcup X_i$ be a finite stratification of sub-manifolds. We denote by $CN_{X_i}^X$, the co-normal bundle of X_i with respect to X. Assume that all $\mathcal{S}(X_i, Sym^j(CN_{X_i}^X) \otimes \mathcal{E})$ are good, where "Sym^j" means the j-th symmetric power. Then $\mathcal{S}(X, \mathcal{E})$ is also good.*

Proof. To begin with it is not difficult to see that the proof can be reduced to the case where $X = U \bigcup Z$, with U open and Z closed.

Now consider the short exact sequence:

$$0 \longrightarrow \mathcal{S}(U, \mathcal{E}) \longrightarrow \mathcal{S}(X, \mathcal{E}) \longrightarrow \mathcal{S}(X, \mathcal{E})/\mathcal{S}(U, \mathcal{E}) \longrightarrow 0.$$

By Lemma 3.2.3 there exists a (onto) projective system F_i, such that

$$F_i \cong \mathcal{S}(Z, \mathrm{Sym}^i(CN_Z^X) \otimes \mathcal{E}) \text{ and } \mathcal{S}(X, E)/\mathcal{S}(U, \mathcal{E}) \cong \varprojlim F_i.$$

By the assumption, all $\mathcal{S}(Z, \mathrm{Sym}^i(CN_Z^X) \otimes \mathcal{E})$ are good. Then the goodness of $\mathcal{S}(X, \mathcal{E})$ follows from Lemma 2.2.1 and Corollary 2.2.2. □

Proof of the main theorem.

Case 1 $X = X_1 \times Y$, where H acts transitively on X_1 and the H-action on Y is trivial. We will derive the goodness of $\mathcal{S}(X_1 \times Y, \mathcal{E})$ from Theorem 3.3.1 and lemma 4.0.1 as follows: Fix a $x \in X_1$ and denote its stabilizer H_x by L. Then by Theorem 3.3.1 we have

$$\mathrm{H}_i(\mathfrak{h}, \mathcal{S}(X_1 \times Y, \mathcal{E})) \cong \mathrm{H}_i(\mathfrak{l}, \mathcal{S}(\{x\} \times Y, \mathcal{E}|_{\{x\} \times Y} \otimes \Delta_L.(\Delta_H^{-1})|_L)).$$

It is clear that $\{x\} \times Y \cong Y$ and $\mathfrak{l}$ acts trivially on it. Let $\mathcal{E}' := \mathcal{E}|_{\{x\} \times Y} \otimes \Delta_L.(\Delta_H^{-1})|_L)$ and note that $\mathcal{E}' \to Y$ is a Nash bundle. Observe that the action of $\mathfrak{h}$, although trivial on the base , is not necessarily trivial on $\mathcal{E}'$. The objective is now to show that $\mathrm{H}_i(\mathfrak{l}, \mathcal{S}(Y, \mathcal{E}'))$ is separated. To compute the homology we use the Koszul complex $\bigwedge^\bullet \mathfrak{l} \bigotimes \mathcal{S}(Y, \mathcal{E}')$ with differentials $d_\bullet$. As $\mathfrak{l}$ acts trivially on the base we get that $\bigwedge^i \mathfrak{l} \bigotimes \mathcal{S}(Y, \mathcal{E}') = \mathcal{S}(Y, \bigwedge^i \mathfrak{l} \bigotimes E')$, with "$\bigwedge^i \mathfrak{l}$" on the right being considered as a trivial vector bundle over Y (of fibre $\bigwedge^i \mathfrak{l}$). We abbreviate $\bigwedge^i \mathfrak{l} \bigotimes \mathcal{E}' =: \mathcal{E}'_i$.

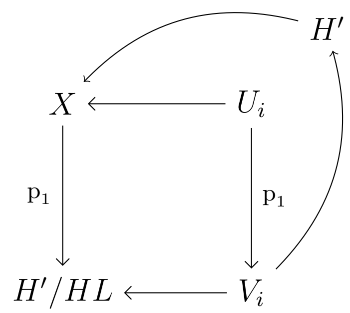
Now if all bundle maps $d_i : \mathcal{E}'_i \longrightarrow \mathcal{E}'_{i-1}$ are of *constant rank*, then we obtain the desired result. Because in this case, all the images and kernels (of d_i) are sub-bundles over Y. Thus each $\operatorname{im} d_i$ becomes the space of Schwartz sections of a sub-bundle of $\mathcal{E}'_{i-1}$ and thus it is closed. In the general case, by Lemma 3.1.3 we can divide Y into a finite stratification of Y_j with each (Nash sub-manifold) Y_j verifying the "constant rank" condition. Then the goodness of $\mathcal{S}(Y, \mathcal{E}')$ follows from Lemma 4.0.2.

Case 2 $X = Z \times Y$, where the H-action on $Z \times Y$ is "twisted" by Z (with the same setting as in Proposition 3.3.3).

This case follows from Proposition 3.3.3 and the previous case. More precisely, in this case we have $\mathrm{H}_*(\mathfrak{h}, \pi_1) = \mathrm{H}_*(\mathfrak{h}, \pi_2)$ (Proposition 3.3.3). On the other hand, π_2 fits the setting of the previous case, so it is good.

Case 3 $X = H'/L$ for some Nash subgroup $L < H'$.
We want to prove $\mathcal{S}(H'/L, \mathcal{E})$ is good. In this case, the goodness of $\mathcal{S}(H'/L, \mathcal{E})$ follows essentially from the previous case and Lemma 4.0.1 in the following way.

Let $\mathrm{p} : H' \longrightarrow X$ and $\mathrm{p}_1 : X \longrightarrow H'/HL$ be the natural projections. By Theorem 3.1.2 we can find a finite open cover $\{V_i\}$ of H'/HL, such that (1) $U_i := \mathrm{p}^{-1}(V_i) \cong HL/L \times V_i$. (2) There exists a (smooth) map $s_i : V_i \longrightarrow H'$ and the following diagram commutes:



Then $\mathcal{S}(U_i, \mathcal{E})$ and $\mathcal{S}(p_1^{-1}(W_i), \mathcal{E})$ (for any open subset W_i of V_i) fit the setting of the previous step. Thus, $\mathcal{S}(U_i, \mathcal{E})$ and $\mathcal{S}(p_1^{-1}(W_i), \mathcal{E})$ are good. Consequently, $X = \bigcup U_i$ fits the setting of Lemma 4.0.1. Thus we conclude that $\mathcal{S}(H'/L, \mathcal{E})$ is good.

Case 4 General case.
Follows from Lemma 4.0.2 and the previous case. Indeed, X has a stratification of H'-orbits, and each H'-orbit fits the setting of the previous case. Then Lemma 4.0.2 implies that $\mathcal{S}(X, \mathcal{E})$ is good.

□

5. Relation to Comparison Theorems

In this section we let G be an algebraic real reductive group. We fix a maximal compact subgroup K and write θ for the corresponding Cartan involution on G which fixes K. Let $P < G$ be a minimal parabolic subgroup and $P = MAN$ be a Langlands decomposition for P.

An algebraic subgroup $H < G$ is called *real spherical* provided that the action of P on G/H admits open orbits. We recall that real sphericity implies that the double coset space $P\backslash G/H$ is finite [KS1]. Typical examples for real spherical subgroups are $H = N$ or H symmetric, e.g. $H = K$.

We assume now that H is a spherical subgroup of G. It is no loss of generality to assume that PH is open, that is $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. Observe once H is fixed that there is no canonical choice of K. After replacing K by $\mathrm{Ad}(a)K$ for generic $a \in A$ we may and will assume in the sequel that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{a} + \mathfrak{k}$$

holds true (see [KS2], Section 5).

We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $\mathcal{Z}(\mathfrak{g})$ its center. The following is an analogue of the Casselman-Osborne-Lemma (see [HS83], Lemma 2.2.2) for spherical subalgebras (cf. [KS2], Lemma 5.5):

Lemma 5.0.1. *There exists a finite subset $\mathcal{Y} \subset \mathcal{U}(\mathfrak{g})$ such that*

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h})\mathcal{Y}\mathcal{Z}(\mathfrak{g})\mathcal{U}(\mathfrak{k}) .$$

In this section we use V to denote a Harish-Chandra module for the pair $(\mathfrak{g}, K)$. The unique smooth moderate growth globalization of V is denoted by V^∞ (see [BK]). Note that V^∞ is an NF-module for G.

Lemma 5.0.2. *Let V be a Harish-Chandra module for $(\mathfrak{g}, K)$. Then the following assertions hold true:*

(1) V is finitely generated as an $\mathfrak{h}$-module.
(2) $\mathrm{H}_p(\mathfrak{h}, V)$ is finite dimensional for all p.

Proof. We recall that every Harish-Chandra module is $\mathcal{Z}(\mathfrak{g})$-finite, i.e. V is annihilated by an ideal $\mathcal{I} \triangleleft \mathcal{Z}(\mathfrak{g})$ of finite co-dimension. As V is finitely generated as an $(\mathfrak{g}, K)$-module there is a finite dimensional K-invariant subspace $W \subset V$ which generates V as $\mathfrak{g}$-module. In view of Lemma 5.0.1, this implies (1). Finally (2) is a consequence of (1) and the fact that $\mathcal{U}(\mathfrak{h})$ is Noetherian. □

The Casselman subrepresentation theorem asserts that every Harish-Chandra module has a $(\mathfrak{g}, K)$-realization in the space of smooth sections of a finite dimensional homogeneous vector bundle $\mathcal{E} := G \times_P U \to X := G/P$ where U is a a finite dimensional P-module . We use the dual version: for every Harish-Chandra module V we obtain V^∞ as a quotient of $C^\infty(X, \mathcal{E})$.

If $\mathcal{E}$ is a Nash-vector bundle of X, then it is clear that $C^\infty(X, \mathcal{E}) = \mathcal{S}(X, \mathcal{E})$. In this case, we call the induced generalized principal series representation a *Nash generalized principal series representation.* It is noted that $\mathcal{E}$ is Nash if and only if U is a Nash representation of P. Moreover if U is irreducible, then U is a Nash representation of P if and only if the A-character $\lambda := U|_A$ is *rational* (see [Sun] for more details).

Note that the inclusion mapping $V \to V^\infty$ yields morphism in homology $\mathrm{H}_\bullet(\mathfrak{h}, V) \to \mathrm{H}_\bullet(\mathfrak{h}, V^\infty)$. For instance if $\mathfrak{h} = \overline{\mathfrak{n}}$, then the Casselman comparison theorem asserts that these two homology theories coincide. This is conjectured to hold for any spherical subgroup. A first step in this direction is:

Theorem 5.0.3. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra such that $\overline{\mathfrak{n}} \subset \mathfrak{h} \subset \mathfrak{a} + \overline{\mathfrak{n}}$. Let V be a Harish-Chandra module, such that V^∞ is a quotient of a Nash generalized principal series representation. Then the natural homomorphism*

$$\Phi_0 : \mathrm{H}_0(\mathfrak{h}, V) \to \mathrm{H}_0(\mathfrak{h}, V^\infty)$$

is an isomorphism.

Proof. We first show that Φ_0 is injective. A slight modification of the automatic continuity theorem of Casselman (see [BK], Theorem 11.4) implies that for all Harish-Chandra modules V, we have

$$\mathrm{Hom}_{\mathfrak{h}}(V, \mathbb{C}) \simeq \mathrm{Hom}_{\mathfrak{h}}(V^\infty, \mathbb{C})\,.$$

The isomorphism is given by the restriction of an $\mathfrak{h}$-invariant functional fom V^∞ to V. Note that $\mathrm{Hom}_{\mathfrak{h}}(V, \mathbb{C}) = (V/\mathfrak{h}V)^*$ and $\mathrm{Hom}_{\mathfrak{h}}(V^\infty, \mathbb{C}) = (V^\infty/\overline{\mathfrak{h}V^\infty})'$. We conclude that

$$V/\mathfrak{h}V \simeq V^\infty/\overline{\mathfrak{h}V^\infty} \tag{8}$$

and in particular that Φ_0 is injective.

To establish the surjectivity of Φ_0 we first assume that $V^\infty = \mathcal{S}(X, \mathcal{E})$. Then Theorem B implies that $\mathfrak{h}V^\infty = \overline{\mathfrak{h}V^\infty}$. The surjectivity is thus immediate from (8). Note that by Lemma 5.0.2, this implies especially $V^\infty/\mathfrak{h}V^\infty$ is finite dimensional.

The general case now follows from the realization of V^∞ as a quotient of $\mathcal{S}(X, \mathcal{E})$ in the following way. Firstly, since $\mathcal{S}(X, \mathcal{E})/\mathfrak{h}\mathcal{S}(X, \mathcal{E})$ is finite dimensional, it follows that $V^\infty/\mathfrak{h}V^\infty$ is finite dimensional. Further, consider the map $T : \mathfrak{h} \bigotimes V^\infty \longrightarrow V^\infty$ by $T(X \otimes v) = X.v$ for $X \in \mathfrak{h}$ and $v \in V^\infty$. Then the cokernel of T is exactly $V^\infty/\mathfrak{h}V^\infty$ which is finite dimensional. Thus according to [CHM00, Lemma A.1], T is a strict morphism, which means in our case that $\mathfrak{h}V^\infty = \overline{\mathfrak{h}V^\infty}$. □

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