HOLONOMICITY OF SPHERICAL CHARACTERS AND APPLICATIONS TO MULTIPLICITY BOUNDS

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ABSTRACT. In this paper, we prove that any spherical character of any admissible representation of a real reductive group G with respect to any pair of spherical subgroups is a holonomic distribution on G. This implies that the restriction of the spherical character to an open dense subset is given by an analytic function. The proof is based on an argument from algebraic geometry and thus implies also analogous results in the p-adic case.

As an application we give a short proof of the recent results of [KO13; KS] on boundedness and finiteness of multiplicities of irreducible representations in the space of functions on a spherical space.

In order to deduce this application we prove relative and quantitative analogs of the Bernstein-Kashiwara theorem, which states that the space of solutions of a holonomic system of differential equations in the space of distributions is finite-dimensional. We also deduce that for every algebraic group G, the space of G-equivariant distributions on any algebraic G-manifold X is finite-dimensional if G has finitely many orbits on X.

CONTENTS

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1. Introduction

1.1. **The spherical character.** In this paper we prove that a spherical character of a smooth admissible Fréchet representation of moderate growth of a real reductive group is holonomic. The spherical character is a basic notion of relative representation theory that generalizes the notion of a character of a representation. By a real reductive group we mean a connected algebraic reductive group defined over $\mathbb{R}$. Unless confusion is possible, we will not distinguish between such a group and the group of its real points. Let us now recall the notions of spherical pair, spherical character and holonomic distribution. For the notion of smooth admissible Fréchet representation of moderate growth we refer the reader to [Cas89] or [Wal88, Chapter 11].

Definition 1.1.1. Let G be a real reductive group and let $H \subset G$ be its (algebraic) subgroup. Let P denote a minimal parabolic subgroup of G and B denote a Borel subgroup of the complexification $G_{\mathbb{C}}$. The subgroup H is called *real spherical* if it has finitely many orbits on G/P and *spherical* if its complexification has finitely many orbits on $G_{\mathbb{C}}/B$.

Definition 1.1.2. Let G be a real reductive group and let $H_1, H_2 \subset G$ be its (algebraic) spherical subgroups. Let χ_i be characters of H_i. Let π be a smooth admissible Fréchet representation of moderate growth of G and $\hat{\pi}$ be its smooth contragredient. Let $\phi_1 \in (\pi^*)^{H_1,\chi_1}$ and $\phi_2 \in (\hat{\pi}^*)^{H_2,\chi_2}$ be equivariant functionals. Fix a Haar measure on G. It gives rise to an action of the space of Schwartz functions $\mathcal{S}(G)$ on π^* and $\hat{\pi}^*$, and this action maps elements of π^* and $\hat{\pi}^*$ to elements of $\hat{\pi}$ and $\hat{\hat{\pi}} = \pi$ respectively. For the definition of the space of Schwartz functions $\mathcal{S}(G)$ see e.g. [Cas89; Wal88; AG08].

The spherical character ξ_{ϕ_1,ϕ_2} of π, with respect to ϕ_1 and ϕ_2, is the tempered distribution on G (i.e. a continuous functional on $\mathcal{S}(G)$) defined by $\langle \xi_{\phi_1,\phi_2}, f\rangle = \langle \phi_2, \pi^*(f)\cdot\phi_1\rangle$.

Definition 1.1.3. The singular support[1] $\mathrm{SS}(\xi)$ of a distribution ξ on a real algebraic manifold X is the joint zero set in T^*X of all the symbols of (algebraic) differential operators that annihilate ξ. The distribution ξ is called holonomic if $\dim \mathrm{SS}(\xi) = \dim X$.

In this paper we prove the following theorem.

Theorem A (See §4.2)**.** *In the situation of Definition 1.1.2, the spherical character ξ_{ϕ_1,ϕ_2} is holonomic.*

We prove Theorem A using the following well-known statement.

Proposition 1.1.4 (See §4.2)**.** *Let $\mathfrak{g}, \mathfrak{h}_i$ be the Lie algebras of G and H_i Let*

$$S := \{(g,\alpha) \in G \times \mathfrak{g}^* | \alpha \text{ is nilpotent, } \langle \alpha, \mathfrak{h}_1\rangle = 0,\ \langle \alpha, Ad^*(g)(\mathfrak{h}_2)\rangle = 0\}.$$

Then $\mathrm{SS}(\xi_{\phi_1,\phi_2}) \subset S$.

Note that the Bernstein inequality states that the dimension of the singular support of any non-zero distribution is at least the dimension of the underlying manifold. Thus Theorem A follows from the following more precise version, which is the core of this paper.

Theorem B (See §2)**.** *We have* $\dim S = \dim G$.

This theorem immediately implies the following corollary.

Corollary C. *Let*

$$U := \left\{g \in G \mid S \cap T^*_g G = \{(g,0)\}\right\}.$$

Then U is a Zariski open dense subset of G.

[1] a.k.a. characteristic variety

This corollary is useful in view of the next proposition, which follows from Proposition 1.1.4 and Corollary 3.1.3 below.

Proposition 1.1.5. *The restriction $\xi_{\phi_1,\phi_2}|_U$ is an analytic function.*

1.2. **Bounds on the dimension of the space of solutions.** Next we apply our results to representation theory. For this we use the following theorem.

Theorem 1.2.1 (Bernstein-Kashiwara)**.** *Let X be a real algebraic manifold. Let*

$$\{D_i\xi = 0\}_{i=1...n}$$

be a system of linear PDE on X with algebraic coefficients. Suppose that the joint zero set of the symbols of D_i is $\dim X$-dimensional. Then the space of solutions of this system in $\mathcal{S}^*(X)$ is finite dimensional.

It seems that this theorem is not found in the literature in this formulation, however it has two proofs, one due to Kashiwara (see [Kas74; KK76] for similar statements) and another due to Bernstein (unpublished).

In order to make our applications in representation theory more precise, we need an effective version of this theorem. We prove such a version (see Theorem 3.2.2 below) following Bernstein's approach, as it more appropriate for effective bounds. We use this effective version to derive a relative version. Namely we show that if the system depends on a parameter in an algebraic way, then the dimension of the space of solutions is bounded (see §3.3 below).

This relative version allows us to deduce the following theorem.

Theorem D (See §3.3)**.** *Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X. Then there exists $C \in \mathbb{N}$ such that for any character χ of $\mathfrak{g}$, $\dim \mathcal{S}^*(X,\mathcal{E})^{(\mathfrak{g},\chi)} < C$.*

1.3. **Applications to representation theory.** Using §3 we give a short proof of the following recent result of [KO13; KS].

Theorem E (See §4)**.** *Let G be a real reductive group and H be an algebraic subgroup.*

(i) If H is a real spherical subgroup then $\dim(\pi^*)^H$ is finite for any $\pi \in Irr(G)$.
(ii) If H is a spherical subgroup then $\dim(\pi^*)^H$ is universally bounded, i. e. there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^H \leq C$ for any $\pi \in Irr(G)$.

The inverse implications for Theorem E are proven in [KO13].

1.4. **The non-Archimedean case.** Theorem B and Corollary C hold over arbitrary fields of characteristic zero. They are useful also for p-adic local fields F, since the analogs of Propositions 1.1.4 and 1.1.5 hold in this case, see [AGS, Theorem A and Corollary F]. Namely, we have the following theorem.

Theorem 1.4.1 ([AGS])**.** *Let G be a reductive group defined over a non-Archimedean field F of characteristic 0 and let ξ be a spherical character of a smooth admissible representation with respect to two spherical subgroups $H_1, H_2 \subset G$. Let S and U be the sets defined in Proposition 1.1.4 and Corollary C. Then*

(i) The wave front set of ξ lies in S.
(ii) The restriction of ξ to U is given by a locally constant function.

1.5. **Related results.** In the group case, i.e. the case when $G = H \times H$ and $H_1 = H_2 = \Delta H \subset H \times H$, Theorem A essentially becomes the well-known fact that characters of admissible representations are holonomic distributions.

As we mentioned above, Theorem E was proven earlier in [KO13; KS], using different methods. An analog of Theorem E(i) over non-Archimedean fields is proven in [Del10] and [SV, Theorem 5.1.5] for many spherical pairs, including arbitrary symmetric pairs.

The group case of Corollary C, Proposition 1.1.5, and Theorem 1.4.1(ii) is (the easy part of) the Harish-Chandra regularity theorem (see [HC63; HC65]). Another known special case of these results is the regularity of Bessel functions, see [LM; AGK; AG].

1.6. **Future applications.** Our proof of Theorem E(ii) does not use the Casselman embedding theorem, unlike [KO13; KS]. This gives us hope that it can be extended to the non-Archimedean case. The main difficulty is the fact that our proof heavily relies on the theory of modules over the ring of differential operators, which does not act on distributions in the non-Archimedean case. However in view of Theorem 1.4.1 we believe that this difficulty can be overcome. Namely one can deduce an analog of Theorem E(ii) for many spherical pairs from the following conjecture .

Conjecture 1.6.1. Let G be a reductive group defined over a non-Archimedean field F of characteristic 0 and let $H_1, H_2 \subset G$ be its (algebraic) spherical subgroups. Let χ_i be characters of H_i. Fix a character λ of the Bernstein center $\mathfrak{z}(G)$.

Then the space of distributions which are:

(1) left (H_1, χ_1)-equivariant,
(2) right (H_2, χ_2)-equivariant,
(3) $(\mathfrak{z}(G), \lambda)$-eigen,

is finite dimensional. Moreover, this dimension is uniformly bounded when λ varies.

Note that Theorem B and Theorem 1.4.1(i) imply that the dimension of (the Zariski closure of) the wave front set of a distribution that satisfies (1-3) does not exceed $\dim G$. In many ways the wave front set replaces the singular support, in absence of the theory of differential operators (see e.g [Aiz13; AD; AGS; AGK]). Thus in order to prove Conjecture 1.6.1 it is left to prove analogs of Theorems 1.2.1 and 3.2.2 for the integral system of equations (1-3).

1.7. **Structure of the paper.** In §2 we prove Theorem B, using a theorem of Steinberg [Ste76] concerning the Springer resolution.

In §3 we prove an effective version of Theorem 1.2.1, and then adapt it to algebraic families. We also derive Theorem D.

In §4 we derive Theorem E from Theorem B and §3. We do that by embedding the multiplicity space into a certain space of spherical characters.

In Appendix A we prove Lemma 3.1.1 which computes the pullback of the D-module of distributions with respect to a closed embedding. We use this lemma in §3.

In Appendix B we give a short proof for the result of [Ada] on the isomorphism between the contragredient of an irreducible representation π and the twist of π by an involution. We use this result in §4.

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2. Proof of Theorem B

It is enough to prove the theorem for a reductive group G defined over an algebraically closed field k of characteristic 0. Since S includes the zero section of $T^*G \cong G \times \mathfrak{g}^*$, we have $\dim S \geq \dim G$. Thus, it is enough to prove that $\dim S \leq \dim G$. Let $\mathcal{B}$ denote the flag variety of G and $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone. Since G is reductive, we can identify

$$T^*\mathcal{B} \cong \{(B, X) \in \mathcal{B} \times \mathfrak{g}^* \mid X \in (\mathrm{Lie}(B))^{\perp}\}.$$

Recall the Springer resolution $\mu : T^*\mathcal{B} \to \mathcal{N}$ defined by $\mu(B, X) = X$ and consider the following diagram.

(1)
$$\begin{tikzcd} T^*\mathcal{B} \times T^*\mathcal{B} \arrow[dr, two heads, "\mu\times\mu"] & & G \times \mathcal{N} \arrow[dl, "\alpha"'] \\ & \mathcal{N} \times \mathcal{N} \arrow[d, "res"] & \\ & \mathfrak{h}_1^* \times \mathfrak{h}_2^* & \end{tikzcd}$$

Here, α is defined by $\alpha(g, X) = (X, ad^*(g^{-1})X)$, and res is the restriction. Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$, we obtain the following diagram.

(2)
$$\begin{tikzcd} L_1 \times L_2 \arrow[dr, two heads, "\mu'"] & & S \arrow[dl, "\alpha'"'] \\ & \mathcal{N}_{\mathfrak{h}_1} \times \mathcal{N}_{\mathfrak{h}_2} & \end{tikzcd}$$

Here, $\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^{\perp}$ and $L_i := \{(B, X) \in T^*\mathcal{B} \,|\, X \in \mathfrak{h}_i^{\perp}\}$.We need to estimate $\dim S$. We do it using the following lemma.

Lemma 2.0.1 (See §2.1 below)**.** *Let $\varphi_i : X_i \to Y$, $i = 1, 2$, be morphisms of algebraic varieties. Suppose that φ_2 is surjective. Then, there exists $y \in Y$ such that*

$$\dim X_1 \leq \dim X_2 + \dim \varphi_1^{-1}(y) - \dim \varphi_2^{-1}(y).$$

By this lemma, applied to $\phi_1 = \alpha'$ and $\phi_2 = \mu'$, it is enough to estimate the dimensions of L_i and of the fibers of μ' and α'.

Lemma 2.0.2. *We have* $\dim L_1 = \dim L_2 = \dim \mathcal{B}$.

Proof. Since H_i has finitely many orbits in $\mathcal{B}$, it is enough to show that L_i is the union of the conormal bundles to the orbits of H_i in $\mathcal{B}$. Let $B \in \mathcal{B}$, let $\mathfrak{b}$ be its Lie algebra and identify $T_B\mathcal{B} \cong \mathfrak{g}/\mathfrak{b}$. Then $T_B(H_i \cdot B) \cong \mathfrak{h}_i/(\mathfrak{b} \cap \mathfrak{h}_i)$ and the conormal space at B to the H_i-orbit of B is identified with $\mathfrak{b}^{\perp} \cap \mathfrak{h}_i^{\perp}$. $\square$

Let $(\eta, ad^*(g)\eta) \in \mathrm{Im}(\alpha')$. The fiber $(\alpha')^{-1}(\eta, ad^*(g)\eta)$ is isomorphic to the stabilizer G_η, and the dimension of the fiber $(\mu')^{-1}(\eta, ad^*(g)\eta)$ is twice the dimension of the Springer fiber $\mu^{-1}(\eta)$. Recall the following theorem of Steinberg (conjectured by Grothendieck):

Theorem 2.0.3 ([Ste76, Theorem 4.6])**.**

$$\dim G_\eta - 2 \dim \mu^{-1}(\eta) = \mathrm{rk}\, G.$$

Using Lemma 2.0.1 we obtain for some $(\eta, ad^*(g)\eta)$:

$$\dim S \leq \dim(L_1 \times L_2) + \dim(a')^{-1}(\eta, ad^*(g)\eta) - \dim(\mu')^{-1}(\eta, ad^*(g)\eta) =$$
$$= 2\dim\mathcal{B} + \dim G_\eta - 2\dim\mu^{-1}(\eta) = 2\dim\mathcal{B} + \operatorname{rk} G = \dim G.$$

□

2.1. **Proof of Lemma 2.0.1.** Recall that, for a dominant morphism $\varphi : X \to Y$ of irreducible varieties, there exists an open dense $U \subset Y$ such that $\dim X = \dim Y + \dim \varphi^{-1}(y)$ for all $y \in U$ (see, e. g., [Mum99, Theorem 1.8.3]). Let Z be an irreducible component of X_1 of maximal dimension and $W \subset Y$ be the Zariski closure of $\varphi_1(Z)$. Since W is irreducuble, there exists an open dense $U \subset W$ such that

$$\dim X_1 = \dim Z \leq \dim W + \dim \varphi_1^{-1}(y) \tag{3}$$

for all $y \in U$. Let $V \subset U$ be an open dense subset such that $\varphi_2^{-1}(V)$ intersects those and only those irreducible components $C_1, \ldots, C_j$ of $\varphi_2^{-1}(W)$ that map dominantly to W. Note that $j > 0$ since φ_2 is surjective. Moreover, without loss of generality, we may assume that, for every $1 \leq i \leq j$, all fibers over V of the restriction of φ_2 to C_i are of the same dimension. Since one of these dimensions has to be equal to $\dim \varphi_2^{-1}(V) - \dim V$, we have, that there is an $1 \leq i \leq j$ such that, for all $y \in V$,

(4)
$$\dim V = \dim C_i - \dim(\varphi_2|_{C_i})^{-1}(y) \leq \dim \varphi_2^{-1}(V) - \dim \varphi_2^{-1}(y) \leq \dim X_2 - \dim \varphi_2^{-1}(y).$$

Thanks to $\dim V = \dim W$, taking any $y \in V$, formulas (3) and (4) imply the statement. □

3. Dimension of the space of solutions of a holonomic system

In this section, we prove an effective version of Theorem 1.2.1, and then adapt it to algebraic families. We also derive Theorem D.

3.1. Preliminaries.

3.1.1. *D-modules.* In this section, we will use the theory of D-modules on algebraic varieties over an arbitrary field k of characteristic zero. We will now recall some facts and notions that we will use. For a good introduction to the algebraic theory of D-modules, we refer the reader to [Ber] and [Bor87]. For a short overview, see [AG09a, Appendix B].

By a *D-module* on a smooth algebraic variety X we mean a coherent sheaf of right modules over the sheaf D_X of algebras of algebraic differential operators. Denote the category of D_X-modules by $\mathcal{M}(D_X)$.

For a smooth affine variety V, we denote $D(V) := D_V(V)$. Note that the category $\mathcal{M}(D_V)$ of D-modules V is equivalent to the category of $D(V)$-modules. We will thus identify these categories.

The algebra $D(V)$ is equipped with a filtration which is called the geometric filtration and defined by the degree of differential operators. The associated graded algebra with respect to this fitration is the algebra $\mathcal{O}(T^*V)$ of regular functions on the total space of the cotangent bundle of V. This allows us to define the *singular support* of a D-module M on V in the following way. Choose a good filtration on M, i.e. a filtration such that the associated graded module is a finitely-generated module over $\mathcal{O}(T^*V)$, and define the singular support $SS(M)$ to be the support of this module. One can show that the singular support does not depend on the choice of a good filtration on M.

This definition easily extends to the non-affine case. A D-module M on X is called *smooth* if $SS(M)$ is the zero section of T^*X. This is equivalent to being coherent over $\mathcal{O}_X$ and to be coherent and locally free over $\mathcal{O}_X$. The Bernstein inequality states that, for any non-zero M, we have $\dim SS(M) \geq \dim X$. If equality holds then M is called *holonomic*.

For a closed embedding $i : X \to Y$ of smooth affine algebraic varieties, define the functor $i^! : \mathcal{M}(D_Y) \to \mathcal{M}(D_X)$ by $i^!(M) := \{m \in M \,|\, I_X m = 0\}$, where $I_X \subset \mathcal{O}(Y)$ is the ideal of all functions that vanish on X.

If V is an affine space then the algebra $D(V)$ has an additional filtration, called the Bernstein filtration. It is defined by $\deg(\partial/\partial x_i) = \deg(x_i) = 1$, where x_i are the coordinates in V. This gives rise to the notion of Bernstein's singular support, that we will denote $SS_b(M) \subset T^*V \cong V \oplus V^*$. It is known that $\dim SS(M) = \dim SS_b(M)$.

We will also use the theory of analytic D-modules. By an *analytic D-module* on a smooth complex analytic manifold X we mean a coherent sheaf of right modules over the sheaf D_X^{An} of algebras of differential operators with analytic coefficients. All of the above notions and statements, except those concerning the Bernstein filtration, have analytic counterparts. In addition, all smooth analytic D-modules of the same rank are isomorphic.

3.1.2. *Distributions.* We will use the theory of distributions on differentiable manifolds and the theory of tempered distributions on real algebraic manifolds, see e.g. [Hör90; AG08]. For a real algebraic manifold X, we denote the space of distributions on X by $\mathcal{D}'(X) := (C_c^\infty(X))^*$ and the space of tempered distributions (a.k.a. Schwartz distributions) by $\mathcal{S}^*(X) := (\mathcal{S}(X))^*$. Similarly, for an algebraic bundle $\mathcal{E}$ over X we denote $\mathcal{D}'(X, \mathcal{E}) := (C_c^\infty(X, \mathcal{E}))^*$ and $\mathcal{S}^*(X, \mathcal{E}) := (\mathcal{S}(X, \mathcal{E}^*))^*$. The spaces $\mathcal{D}'(X)$ and $\mathcal{S}^*(X)$ form (right) D-modules over X. The space $\mathcal{D}'(X)$ is also an analytic D-module. We define the singular support of a distribution to be the singular support of the D-module it generates. It is well-known that this definition is equivalent to Definition 1.1.3. We say that a distribution is holonomic if it generates a holonomic D-module.

Lemma 3.1.1 (See Appendix A)**.** *Let $i : X \to Y$ be a closed embedding of smooth affine real algebraic varieties. Then*

$$\mathcal{D}'(X) \cong i^!(\mathcal{D}'(Y)) \text{ and } \mathcal{S}^*(X) \cong i^!(\mathcal{S}^*(Y)).$$

Lemma 3.1.2. *Let M be a smooth $D(\mathbb{C}^n)$-module of rank r. Embed the space $An(\mathbb{C}^n)$ of analytic functions on $\mathbb{C}^n$ into $\mathcal{D}'(\mathbb{R}^n)$ using the Lebesgue measure. Then* $\operatorname{Hom}(M, \mathcal{D}'(\mathbb{R}^n)) = \operatorname{Hom}(M, An(\mathbb{C}^n))$ *and* $\dim \operatorname{Hom}(M, \mathcal{D}'(\mathbb{R}^n)) = r$.

Proof. Let $M_{An} := M \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ and $D_{An}(\mathbb{C}^n) := D(\mathbb{C}^n) \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ be the analytizations of M and $D(\mathbb{C}^n)$. Then

$$\operatorname{Hom}_{D(\mathbb{C}^n)}(M, \mathcal{D}'(\mathbb{R}^n)) \cong \operatorname{Hom}_{D_{An}(\mathbb{C}^n)}(M_{An}, \mathcal{D}'(\mathbb{R}^n)).$$

Since M_{An} is also smooth, it is well-known that $M_{An} \cong An(\mathbb{C}^n)^r$. Thus it is left to prove that $\operatorname{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), \mathcal{D}'(\mathbb{R}^n)) = \operatorname{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), An(\mathbb{C}^n))$ and the latter space is one-dimensional. This follows from the fact that a distribution with vanishing partial derivatives is a multiple of the Lebesgue measure. $\square$

Corollary 3.1.3. *If a distribution generates a smooth D-module, then it is analytic.*

3.1.3. *Lie algebra actions.*

Definition 3.1.4. Let X be an algebraic manifold over a field k and $\mathfrak{g}$ be a Lie algebra over k.

(i) An action of $\mathfrak{g}$ on X is a Lie algebra map from $\mathfrak{g}$ to the algebra of algebraic vector fields on X.

(ii) Assume that X is affine, fix an action of $\mathfrak{g}$ on X and let $\mathcal{E}$ be an algebraic vector bundle on X. Let M be the space of global regular sections of $\mathcal{E}$. An action of $\mathfrak{g}$ on $\mathcal{E}$ is a linear map $T : \mathfrak{g} \to \operatorname{End}_k(M)$ such that for any $\alpha \in \mathfrak{g}$, $f \in \mathcal{O}(X)$, $v \in M$ we have

$$T(\alpha)(fv) = (\alpha f)v + fT(\alpha)v.$$

(iii) The definition above extends to non-affine X in a straightforward way.

3.1.4. *Weil representation.*

Definition 3.1.5. Let V be a finite-dimensional real vector space. Let ω be the standard symplectic form on $V \oplus V^*$. Denote by $p_V : V \oplus V^* \to V$ and $p_{V^*} : V \oplus V^* \to V^*$ the natural projections. Define an action of the symplectic group $\mathrm{Sp}(V \oplus V^*, \omega)$ on the algebra $D(V)$ by

$$(\partial_v)^g := \pi(g)(\partial_v) := p_{V^*}(g(v,0)) + \partial_{p_V(g(v,0))}, \quad w^g := \pi(g)w := p_{V^*}(g(0,w)) + \partial_{p_V(g(0,w))}$$

where $v \in V$, $w \in V^*$, ∂_v denotes the derivative in the direction of v, and elements of V^* are viewed as linear polynomials and thus differential operators of order zero. For a $D(V)$-module M and an element $g \in \mathrm{Sp}(V \oplus V^*)$, we will denote by M^g the $D(V)$-module obtained by twisting the action of $D(V)$ by $\pi(g)$.

Since the above action of $\mathrm{Sp}(V \oplus V^*)$ preserves the Bernstein filtration on $D(V)$, the following lemma holds.

Lemma 3.1.6. *For a $D(V)$-module M and $g \in \mathrm{Sp}(V \oplus V^*)$ we have $SS_b(M^g) = gSS_b(M)$.*

Theorem 3.1.7 ([Wei64])**.** *There exists a two-folded cover $p : \widetilde{\mathrm{Sp}}(V \oplus V^*) \to \mathrm{Sp}(V \oplus V^*)$ and a representation Π of $\widetilde{\mathrm{Sp}}(V \oplus V^*)$ on the space $\mathcal{S}^*(V)$ of tempered distributions on V such that for any $\alpha \in D(V)$ and $g \in \widetilde{\mathrm{Sp}}(V \oplus V^*)$ we have*

$$\Pi(g)(\xi\alpha) = (\Pi(g)\xi)\alpha^{p(g)}.$$

Corollary 3.1.8. *We have an isomorphism of $D(V)$-modules $\mathcal{S}^*(V)^g \cong \mathcal{S}^*(V)$ for any $g \in \mathrm{Sp}(V \oplus V^*)$.*

In fact, this corollary can be derived directly from the Stone-von-Neumann theorem.

3.1.5. *Flat morphisms.*

Lemma 3.1.9. *Let $\phi : X \to Y$ be a proper morphism of algebraic varieties over a field k and $\mathcal{M}$ be a coherent sheaf on X. Then there exists an open dense $U \subset Y$ such that $\mathcal{M}|_{\phi^{-1}(U)}$ is flat over U.*

Proof. By [EGA IV, Théorème II.3.I], the set V of points $x \in X$ for which $\mathcal{M}$ is ϕ-flat at x is open in X. Since ϕ is proper, the set $Z := \phi(X \setminus V)$ is closed in Y. Note that $\mathcal{M}$ is flat over $U := X \setminus Z$, since $\phi^{-1}(U) \subset V$. Moreover, U contains the generic points of the irreducible components of Y. Hence, $U \subset Y$ is dense. ☐

Lemma 3.1.10 (See, e. g., [Mum74, Corollary on p. 50])**.** *Let $\phi : X \to Y$ be a proper morphism of algebraic varieties and $\mathcal{M}$ be a coherent sheaf on X that is flat over Y. For a point $y \in Y$, let $\mathcal{M}_y$ denote the pullback of $\mathcal{M}$ to $\phi^{-1}(y)$. Then the function*

$$y \mapsto \chi(\mathcal{M}_y) = \sum_{i=0}^{\infty} (-1)^i \dim_{k(y)} \mathrm{H}^i(\mathcal{M}_y)$$

is locally constant.

Corollary 3.1.11. *Let Y be an algebraic variety and $\mathcal{M}$ be a coherent sheaf on $Y \times \mathbb{P}^n$. Then there exists an open dense $U \subset Y$ such that the Hilbert polynomial[2] of $\mathcal{M}_y$ does not depend on y as long as $y \in U$.*

[2]For the definition of Hilbert polynomial see [Har77, Chapter III, Exercise 5.2].

3.2. Dimension of the space of solutions of a holonomic system.

Definition 3.2.1.

(i) Let M be a D-module over an affine space $\mathbb{A}^n$. Let F^i be a good filtration on M with respect to the Bernstein filtration on the ring $D_{\mathbb{A}^n}$. Let p be the corresponding Hilbert polynomial of M, i.e. $p(i) = \dim F^i$ for large enough i. Let k be the degree of p and a_k be the leading coefficient of p. Define the Bernstein degree of m to be $\deg_b(M) := k!a_k$. It is well-known that k and a_k do not depend on the choice of good filtration F^i.

(ii) Let M be a D-module over a smooth algebraic variety X. Let $X = \bigcup_{i=1}^{l} U_i$ be an open affine cover of X and let $\phi_i : U_i \hookrightarrow \mathbb{A}^{n_i}$ be closed embeddings. Denote

$$\deg_{\{(U_i,\phi_i)\}}(M) := \sum_{i=1}^{l} \deg_b((\phi_i)_*(M)).$$

Define the global degree of M by $\deg(M) := \min \deg_{\{(U_i,\phi_i)\}}(M)$, where the minimum is taken over the set of all possible affine covers and embeddings.

In this subsection, we prove

Theorem 3.2.2. *Let X be a real algebraic manifold. Let M be a holonomic right D-module. Then* $\dim \operatorname{Hom}(M, \mathcal{S}^*(X)) \leq \deg(M)$.

We will need the following geometric lemmas

Lemma 3.2.3. *Let V be a vector space, $L \subset V$ be a subspace and $C \subset V$ be a closed conic algebraic subvariety such that $L \cap C = \{0\}$. Then the projection $p : C \to V/L$ is a finite map.*

Proof. By induction, it is enough to prove the case $\dim L = 1$. Choose coordinates $x_1, \dots, x_n$ on V such that the coordinates $x_1, \dots, x_{n-1}$ vanish on L. Let p be a homogeneous polynomial that vanishes on C but not on L. Write $p = \sum_{i=1}^{k} g_i x_n^i$, where each g_i is a homogeneous polynomial of degree $k - i$ in $x_1, \dots, x_{n-1}$. Then $x_n|_C$ satisfies a monic polynomial equation with coefficients in $\mathcal{O}(V/L)$. □

Lemma 3.2.4. *Let W be a $2n$-dimensional symplectic vector space, and $C \subset W$ be a closed conic subvariety of dimension n. Then there exists a Lagrangian subspace $L \subset W$ such that $L \cap C = \{0\}$.*

Proof. Let $\mathcal{L}$ denote the variety of all Lagrangian subspaces of W. Note that $\dim \mathcal{L} = n(n+1)/2$. Let $P(C) \subset \mathbb{P}(W)$ be the projectivizations of C and W. Consider the configuration space

$$X := \{(x, L) \in P(C) \times \mathcal{L} \,|\, x \subset L\}.$$

We have to show that $p(X) \neq \mathcal{L}$ where $p : X \to \mathcal{L}$ is the projection. Let $q : X \to P(C)$ be the other projection. Note that $\dim q^{-1}(x) = n(n-1)/2$ for any $x \in P(C)$. Thus

$$\dim X = n(n-1)/2 + n - 1 < n(n+1)/2 = \dim \mathcal{L},$$

and thus $p : X \to \mathcal{L}$ cannot be onto. □

Corollary 3.2.5. *Let V be a vector space of dimension n and consider the standard symplectic form on $V \oplus V^*$. Let $C \subset V \oplus V^*$ be a closed conic subvariety of dimension n. Let $p : V \oplus V^* \to V$ denote the projection. Then there exists a linear symplectic automorphism $g \in \operatorname{Sp}(V \oplus V^*)$ such that $p|_{gC}$ is a finite map.*

Proof. By Lemma 3.2.4 there exists a Lagrangian subspace $L \subset V \oplus V^*$ such that $L \cap C = \{0\}$. Since the action of $\operatorname{Sp}(V \oplus V^*)$ on Lagrangian subspaces is transitive, there exists $g \in \operatorname{Sp}(V \oplus V^*)$ such that $V = gL$ and thus $gC \cap V = \{0\}$. From Lemma 3.2.3 we get that $p|_{gC}$ is a finite map. □

Proof of Theorem 3.2.2. Let $X = \bigcup_{i=1}^{l} U_i$ be an open affine cover of X and let $\phi_i : U_i \hookrightarrow \mathbb{A}^{n_i}$ be closed embeddings. Clearly

$$\dim \operatorname{Hom}(M, \mathcal{S}^*(X)) \leq \sum_{i=1}^{l} \dim \operatorname{Hom}(M|_{U_i}, \mathcal{S}^*(U_i)).$$

By Lemma 3.1.1

$$\operatorname{Hom}(M|_{U_i}, \mathcal{S}^*(U_i)) \cong \operatorname{Hom}(M|_{U_i}, \phi_i^!(\mathcal{S}^*(\mathbb{R}^{n_i})) \cong \operatorname{Hom}((\phi_i)_*(M|_{U_i}), \mathcal{S}^*(\mathbb{R}^{n_i})).$$

Thus it is enough to show that for any holonomic D-module N on an affine space $\mathbb{A}^n$ we have $\dim \operatorname{Hom}(N, \mathcal{S}^*(F^n)) \leq \deg_b(N)$.

Let $C \subset \mathbb{A}^{2n}$ be the singular support of N with respect to the Bernstein filtration. By Corollary 3.2.5, there exists $g \in \mathrm{Sp}_{2n}$ such that $p|_{gC}$ is a finite map, where $p : \mathbb{A}^{2n} \to \mathbb{A}^n$ is the projection on the first n coordinates. By Corollary 3.1.8 we have

$$\dim \operatorname{Hom}(N, \mathcal{S}^*(\mathbb{R}^n)) = \dim \operatorname{Hom}(N^g, \mathcal{S}^*(\mathbb{R}^n)^g) = \dim \operatorname{Hom}(N^g, \mathcal{S}^*(\mathbb{R}^n)).$$

By Lemma 3.1.6 we have $SS_b(N^g) = gC$. Let F be a good filtration on N^g (with respect to the Bernstein filtration on $D(\mathbb{A}^n)$). We see that $\operatorname{Gr} N^g$ is finitely generated over $\mathcal{O}(\mathbb{A}^n)$, and thus so is N^g. Thus N^g is a smooth D-module. Note that $\operatorname{rk}_{\mathcal{O}(\mathbb{A}^n)} N^g \leq \deg_b N^g = \deg_b N$. By Lemma 3.1.2 $\dim \operatorname{Hom}(N^g, \mathcal{S}^*(\mathbb{R}^n)) \leq \operatorname{rk}_{\mathcal{O}(\mathbb{A}^n)} N^g$. $\square$

3.3. **Families of D-modules.** In this section we discuss families of D-modules on algebraic varieties over an arbitrary field k of zero characteristic.

Notation 3.3.1. Let $\phi : X \to Y$ be a map of algebraic varieties and $\mathcal{M}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. For any $y \in Y$, denote by $\mathcal{M}_y$ the pullback of $\mathcal{M}$ to $\phi^{-1}(y)$.

Definition 3.3.2. Let X, Y be smooth algebraic varieties.

- If X and Y are affine we define the algebra $D(X, Y)$ to be $D(X) \otimes_k \mathcal{O}(Y)$.
- Extending this definition we obtain a sheaf of algebras $D_{X,Y}$ on $X \times Y$.
- By a family of D_X-modules parameterized by Y, we mean a sheaf of right modules over the sheaf of algebras $D_{X,Y}$ on $X \times Y$ which is quasicoherent as a sheaf of $\mathcal{O}_{X \times Y}$-modules.
- We call a family of D_X-modules parameterized by Y coherent if it is locally finitely generated as a $D_{X,Y}$-module.
- For a family $\mathcal{M}$ of D_X-modules parameterized by Y and a point $y \in Y$, we call $\mathcal{M}_y$ the specialization of $\mathcal{M}$ at y and consider it with the natural structure of a D_X-module.
- We say that a coherent family $\mathcal{M}$ is holonomic if every specialization is holonomic.

Theorem 3.3.3. *Let X, Y be smooth algebraic varieties and M be a family of D_X-modules parametrized by Y. Then $\deg M_y$ is bounded when y ranges over the k-points of Y.*

Proof. Without loss of generality, we can assume that $X = \mathbb{A}^n$ and Y is an affine variety, and prove that $\deg_b(\mathcal{M}_y)$ is bounded. We will prove this by induction on $\dim Y$.

The Bernstein filtration on $D(\mathbb{A}^n)$ gives rise to a filtration on $D(\mathbb{A}^n, Y)$. Choose a filtration F on $\mathcal{M}$ which is good with respect to this filtration and let $N := \operatorname{Gr} M$, considered as a graded $\mathcal{O}(A^{2n} \times Y)$-module. Associate to N a coherent sheaf $\mathcal{N}$ on $\mathbb{P}^{2n-1} \times Y$. Let $\mathcal{N}_y$ be the pullback of $\mathcal{N}$ under the embedding of $\mathbb{P}^{2n-1}$ into $\mathbb{P}^{2n-1} \times Y$ given by $x \mapsto (x, y)$. By definition, the Hilbert polynomial of $\mathcal{M}_y$ with respect to the filtration induced by F is the Hilbert polynomial of $\mathcal{N}_y$. By Corollary 3.1.11, there exists an open dense subset $U \subset Y$ such that the Hilbert polynomial of $\mathcal{N}_y$ does not depend on y as long as $y \in U$. By the induction hypothesis, $\deg_b(\mathcal{M}_y)$ is bounded on $Y \setminus U$, and thus bounded on Y. $\square$

For an application of this theorem we will need the following lemma.

Lemma 3.3.4. *Let a real Lie algebra $\mathfrak{g}$ act on a real algebraic manifold X and on an algebraic vector bundle $\mathcal{E}$ on X. Let Y be the variety of characters of $\mathfrak{g}$. Then there exists a coherent family $\mathcal{M}$ of D_X-modules parameterized by Y such that, for any $\chi \in Y$, we have*

(1) $\mathcal{S}^*(X,\mathcal{E})^{(\mathfrak{g},\chi)} = \mathrm{Hom}_{D_X}(\mathcal{M}_\chi, \mathcal{S}^*(X))$.
(2) The singular support of M_χ (with respect to the geometric filtration) is included in

$$\{(x,\phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g} \text{ we have } \langle \phi, \alpha(x)\rangle = 0\}.$$

Proof. It is enough to prove the lemma for affine X. Let N be the coherent sheaf of the regular sections of $\mathcal{E}$ (considered as a sheaf of $\mathcal{O}_X$-modules). Let $\mathcal{N}$ be the pullback of N to $X \times Y$. Let $\mathcal{N}' := \mathcal{N} \otimes_{\mathcal{O}_{X\times Y}} D_{X,Y}$, and $\mathcal{N}'' \subset \mathcal{N}'$ be the $D_{X,Y}$-submodule generated by elements of the form $\alpha n \otimes 1 + n \otimes \chi_\alpha + n \otimes \xi_\alpha$, where $\alpha \in \mathfrak{g}$, χ_α is the function on $X \times Y$ given by $\chi_\alpha(x,y) = y(\alpha)$ and ξ_α is the vector field on X corresponding to α. Then $\mathcal{M} := \mathcal{N}'/\mathcal{N}''$ satisfies the requirements. $\square$

Theorems 3.2.2 and 3.3.3 and Lemma 3.3.4 imply Theorem D.

4. Proof of Theorems A and E

In this section, we derive Theorems A and E from Theorem B and §3. We do that by embedding the multiplicity space into a certain space of spherical characters.

4.1. Preliminaries.

For a real reductive group G, we denote by $Irr(G)$ the collection of irreducible admissible smooth Fréchet representation of G of moderate growth. We refer to [Cas89; Wal88] for the background on these representations.

Theorem 4.1.1 (See [Wal88, Theorem 4.2.1])**.** *The center $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G acts finitely on every admissible smooth Fréchet representation π of G of moderate growth. This means that there exists an ideal in $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of finite codimension that annihilates π.*

Lemma 4.1.2 ([Ada, Theorem 1.2 and Corollary 1.4])**.** *For any real reductive group G, there exists an involution θ of G such that, for any $\pi \in \mathrm{Irr}(G)$, we have $\tilde{\pi} \cong \pi^\theta$.*

We also give an alternative short proof for this lemma in Appendix B, using [Sug59].

Theorem 4.1.3 (Casselman embedding theorem, see [CM82, Proposition 8.23])**.** *Let G be a real reductive group and P be a minimal parabolic subgroup of G. Let $\pi \in Irr(G)$. Then there exists a finite-dimensional representation σ of P and an epimorphism $Ind_P^G(\sigma) \twoheadrightarrow \pi$.*

4.2. Proof of Theorem A and Proposition 1.1.4.

Theorem A follows from Theorem B and Proposition 1.1.4.

Proof of Proposition 1.1.4. Let ξ be a spherical character of a smooth admissible Fréchet representation π of G of moderate growth with respect to a pair of subgroups (H_1, H_2) and their characters χ_1, χ_2. By Theorem 4.1.1 there exists an ideal $I \subset \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of finite codimension that annihilates π and thus annihilates ξ. For any element $z \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ there exists a polynomial p such that $p(z) \in I$ and thus $p(z)\xi = 0$. This implies that the symbol of any $z \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of positive degree vanishes on the singular support of ξ. It is well-known that the joint zero-set of these symbols over each point $g \in G$ is the nilpotent cone $\mathcal{N}(\mathfrak{g}^*)$. Since ξ is $(H_1 \times H_2, \chi_1 \times \chi_2)$-equivariant, this implies that the singular support of ξ lies in S. $\square$

4.3. **Proof of Theorem E.** Part (i) follows immediately from Theorem D and the Casselman embedding theorem. If G is quasi-split then so does part (ii). For the proof of part (ii) in the general case we will need the following lemma.

Lemma 4.3.1. *Let G be a real reductive group and H_1, H_2 be spherical subgroups. Let $Y = \mathrm{Spec}(\mathfrak{z}(\mathcal{U}(\mathfrak{g})))$. For any character $\lambda \in Y(\mathbb{C})$ define $U_\lambda := \mathcal{S}^*(G)^{H_1 \times H_2, (\mathfrak{z}(\mathcal{U}(\mathfrak{g})), \lambda)}$ be the space of tempered distributions on G that are left invariant with respect to H_1, right invariant with respect to H_2 and are eigendistributions with respect to the action of $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ with eigencharacter λ. Then $\dim U_\lambda$ is bounded.*

Proof. Let us construct a family of $D(G)$-modules $\mathcal{M}$ parameterized by Y. For any $\alpha \in \mathfrak{g}$ let r_α and l_α be the corresponding right and left invariant vector fields on G considered as elements in $D(X, Y)$. For any $\beta \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ let f_β be the function on Y given by $f_\beta(\lambda) = \lambda(\beta)$ and d_β be the differential operator on G corresponding to β. We consider both d_β and f_β as elements of $D(G, Y)$. Let $I \subset D(G, Y)$ be the ideal generated by $r_\alpha, l_{\theta(\alpha)}$ and $f_\beta + d_\beta$ where $\alpha \in \mathfrak{h}$ and $\beta \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$. Define $\mathcal{M} := D(G, Y)/I$.

It is easy to see that $U_\lambda \cong \mathrm{Hom}(\mathcal{M}_\lambda, \mathcal{S}^*(G))$. As in the proof of Proposition 1.1.4, the singular support of $\mathcal{M}_\lambda$ lies in S, for any λ. Thus Theorem B implies that $\mathcal{M}_\lambda$ is holonomic and thus $\mathcal{M}$ is holonomic. By Theorem 3.2.2 we have $\dim U_\lambda \leq \deg \mathcal{M}_\lambda$. By Theorem 3.3.3, $\deg \mathcal{M}_\lambda$ are bounded. $\square$

Proof of Theorem E(ii). We choose an involution θ as in Lemma 4.1.2, let $H_1 := H$, $H_2 := \theta(H)$, and define the spaces U_λ as in Lemma 4.3.1.

Now let $\pi \in \mathrm{Irr}(G)$ such that $(\pi^*)^H \neq 0$ and let λ be its infinitesimal character. By Lemma 4.1.2, $(\widetilde{\pi}^*)^{\theta(H)} \neq 0$. Fix a non-zero $\phi \in (\widetilde{\pi}^*)^{\theta(H)}$. Then ϕ defines an embedding $(\pi^*)^H \hookrightarrow U_\lambda$ by $\psi \mapsto \xi_\psi$ where ξ_ψ is the spherical character defined by $\xi_\psi(f) := \langle \psi, \pi(f)\phi \rangle$. Thus $\dim(\pi^*)^H \leq \dim U_\lambda$, which is bounded by Lemma 4.3.1. $\square$

Appendix A. Proof of Lemma 3.1.1

For the proof, we will need the following standard lemmas. Let M be a smooth manifold and $N \subset M$ be a closed smooth submanifold.

Lemma A.0.1. *Denote $I_N := \{f \in C_c^\infty(N) \,|\, f|_M = 0\}$. Let $J \subset I_N$ be an ideal in $C_c^\infty(M)$ such that*

(1) For any $x \in N$ the space $\{d_x f | f \in J\}$ is the conormal space to N in M at the point x.
(2) For any $x \in M \setminus N$ there exists $f \in J$ s.t. $f(x) \neq 0$.

Then $J = I_N$.

Proof. Using partition of unity it is enough to show that for any $f \in I_N$ and $x \in M$ there exists $f' \in J$ such that f coincides with f' in a neighborhood of x. For $x \notin N$ this is obvious, so we assume that $x \in N$. We prove the statement by induction on the codimension d of N in M. The base case $d = 1$ follows, using the implicit function theorem, from the case $N = \mathbb{R}^{n-1} \subset \mathbb{R}^n = M$, which is obvious.

For the induction step take $g \in J$ such that $d_x g \neq 0$. Let

$$Z := \{y \in M \,|\, g(y) = 0\} \text{ and } U := \{y \in M \,|\, d_y g \neq 0\}.$$

By the implicit function theorem $U \cap Z$ is a closed submanifold of U. Choose $\rho \in C_c^\infty(M)$ such that $\rho = 1$ in a neighborhood of x and $\mathrm{Supp}(\rho) \subset U$. Let $\bar{f} := (\rho f)|_{U \cap Z}$. Let

$$\bar{J} := \{\alpha|_{U \cap S} | \alpha \in J \text{ and } \mathrm{Supp}\,\alpha \subset U\}.$$

By the induction hypothesis $\bar{f} \in \bar{J}$. Thus there exists $f'' \in J$ such that $f - f''$ vanishes in a neighborhood of x in Z. Now, the case $d = 1$ implies that there exists $\alpha \in C_c^\infty(M)$ such that $f - f'$ coincides with αg in a neighborhood of x. $\square$

Lemma A.0.2. *The restriction* $C_c^\infty(M) \to C_c^\infty(N)$ *is an open map.*

Proof. Let $K \subset M$ be a compact subset. It is easy to see that there exists a compact $K' \supset K$ such that the restriction map $C_{K'}^\infty(M) \to C_{K' \cap N}^\infty(N)$ is onto, using the partition of unity. By the Banach open map theorem this map is open. Thus the restriction $C_c^\infty(M) \to C_c^\infty(N)$ is an open map. $\square$

Let Y be a real algebraic manifold and X be a closed algebraic submanifold. Let $i : X \to Y$ denote the embedding.

Lemma A.0.3. *Let ξ be a distribution on X such that $i_*\xi$ is a tempered distribution. Then ξ is a tempered distribution.*

Proof. The map i_* is dual to the pullback map $C_c^\infty(Y) \to C_c^\infty(X)$. This can be extended to a continuous map $i^* : \mathcal{S}(Y) \to \mathcal{S}(X)$ which is onto by [AG08, Theorem 4.6.1]. The Banach open map theorem implies that i^* is an open map. It is easy to see that $i_*\xi : \mathcal{S}(Y) \to \mathbb{C}$ vanishes on $Ker(i^*)$, and thus it gives rise to a continuous map $\mathcal{S}(X) \to \mathbb{C}$, which extends ξ. $\square$

Lemma A.0.4. *Let ξ be a distribution on Y such that $p\xi = 0$ for any polynomial p on Y that vanishes on X. Then ξ is a pushforward of a distribution on X.*

Proof. Let $J(X)$ be the ideal of all polynomials on Y that vanish on X. Let $J := J(X)C_c^\infty(Y)$. By Lemma A.0.1 we have $J = I_X$. Thus ξ vanishes on I_X and thus, by Lemma A.0.2, ξ is a pushforward of a distribution on X. $\square$

Lemma 3.1.1 follows now from Lemmas A.0.3 and A.0.4 and the definition of $i^!$ for closed embedding of smooth affine varieties.

Appendix B. Proof of Lemma 4.1.2, by Andrey Minchenko

In this appendix, unlike the rest of the paper, we will not ignore the difference between real algebraic varieties and their real points.

Lemma 4.1.2 follows from [AG09b, Theorem 8.2.1][3] and the next lemma.

Lemma B.0.1. *Let G be a connected real reductive algebraic group and $G_0 := G(\mathbb{R})$. Then there exists an automorphism θ of G_0 such that, for all semisimple $g \in G_0$, $\theta(g)$ and g^{-1} are conjugate in G_0.*

Proof. Let $\mathfrak{g}$ stand for the Lie algebra of G. Let us show that there exists an automorphism θ of $\mathfrak{g}$ and a subset $X \subset \mathfrak{g}$ such that $\theta(x) = -x$ for all $x \in X$ and every Cartan subalgebra of $\mathfrak{g}$ is conjugate to a subset of X. Let τ be a Cartan involution of $\mathfrak{g}$ and let $\mathfrak{a} \subset \mathfrak{g}$ be a maximal split torus. Then $\mathfrak{t} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is a Cartan subalgebra of $\mathfrak{g}$. Let Σ be a root system of $\mathfrak{g}(\mathbb{C})$. By a theorem of Chevalley, one can choose a vector $e_\alpha \neq 0$, $\alpha \in \Sigma$, in each root subspace of $\mathfrak{g}(\mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{t}(\mathbb{C})$ such that $[e_\alpha, e_\beta] = r_{\alpha,\beta}e_{\alpha+\beta}$, where $r_{\alpha,\beta} = r_{-\alpha,-\beta} \in \mathbb{Z}$. Note that elements $ih_\alpha, i(e_\alpha + e_{-\alpha}), d_\alpha := e_\alpha - e_{-\alpha}$ span a compact form $\mathfrak{u}$ of $\mathfrak{g}$, which is stable under τ. We may assume without loss of generality that $\mathfrak{g} = \mathfrak{u}^\tau + i\mathfrak{u}^-$, where $\mathfrak{u}^- := \{x \in \mathfrak{u} \ : \ \tau(x) = -x\}$.

We claim, that we can take $X := \mathrm{Span}_{\mathbb{R}}\{\mathfrak{t}, d_\alpha - d_{\tau\alpha}\}$ and θ defined by $\theta(h) = -h$, $h \in \mathfrak{t}$, $\theta(e_\alpha) = -e_{-\alpha}$. Using $\tau(\mathfrak{u}) = \mathfrak{u}$, one verifies $\theta\tau = \tau\theta$, hence, $\theta(\mathfrak{g}) = \mathfrak{g}$. By [Sug59, Theorem 5], every Cartan subalgebra of $\mathfrak{g}$ is conjugate to one of the form $\mathfrak{h} = \mathfrak{h}^\tau + \mathfrak{h}^-$, where $\mathfrak{h}^- \subset \mathfrak{a}$ and the orthogonal complement to $\mathfrak{h}^-$ in $\mathfrak{a}$ is spanned by h_β, where β runs through some $Y \subset \Sigma$ with $\beta_1 \pm \beta_2 \notin \Sigma$ for all $\beta_1 \neq \beta_2 \in Y$. Moreover, by [Sug59, Proposition

[3] One can also use Harish-Chandra's regularity theorem instead [AG09b, Theorem 8.2.1].

5], $\mathfrak{h}$ is determined by $\mathfrak{h}^-$ uniquely, up to conjugacy. On the other hand, for every such Y, $\mathfrak{h}' := \mathfrak{h}^- + \mathfrak{t}^\tau + \mathrm{Span}_{\mathbb{R}}\{d_\alpha\}_{\alpha \in Y} \subset X$ is a Cartan subalgebra of $\mathfrak{g}$.

Let us take $U \subset G_0$ to be the set of semisimple elements. Clearly, the lift of θ to an automorphism of the universal cover $\widehat{G}$ acts on the center by inversion. Hence, it preserves any central subgroup and, therefore, induces an automorphism of G. By [Kna02, Theorem 7.108], U lies in the union of conjugates of Cartan subgroups of G, which are centralizers of Cartan subalgebras of $\mathfrak{g}$. Since every Cartan subgroup of $G(\mathbb{R})$ belongs to a Cartan subgroup of $G(\mathbb{C})$, it follows that θ acts by inversion on every Cartan subgroup of G whose Lie algebra belongs to X. This completes the proof. $\square$

Remark B.0.2. A thorough check of the proof of Lemma B.0.1 and references there reveals that one may as well take G_0 to be any Lie group satisfying $G(\mathbb{R})^\circ \subset G_0 \subset G(\mathbb{R})$.

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