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Gelfand pairs and Invariant Distributions

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To my beloved wife

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Abstract

A Gelfand pair is a pair (G, H) consisting of a group and a subgroup such that the quazi-regular representation of G acting on the space $F(G/H)$ of functions on G/H "includes" any irreducible representation of G with multiplicity at most 1. The theory of Gelfand pairs have various applications in representation theory, harmonic analysis and the theory of automorphic forms. The main tool for proving the Gelfand property of a given pair is the Gelfand-Kazhdan method which is based on the analysis of distributions on G which are invariant with respect to the two-sided action of H.

In this thesis we present the proof of the Gelfand property for various pairs. Most of these pairs are symmetric pairs. In particular, we proved that the following pairs are Gelfand pairs:

- $(GL_{n+k}(F), GL_n(F)\times GL_k(F))$ for $F = \mathbb{R}, \mathbb{C}$. The non-archimedean case was proven in [JR].
- $(GL_n(\mathbb{C}), GL_n(\mathbb{R}))$. The non-archimedean case was proven in [Fli].
- $(O_{n+k}(\mathbb{C}), O_n(\mathbb{C}) \times O_k(\mathbb{C}))$.
- $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$.
- $(GL_{2n}(F), Sp_{2n}(F))$ for $F = \mathbb{R}, \mathbb{C}$. The non-archimedean case was proven in [HR].
- $(GL_{n+1}(F), (GL_n(F)))$ is a strong Gelfand pair (i.e. $(GL_{n+1}(F) \times (GL_n(F)), \Delta(GL_n(F)))$ is a Gelfand pair) .

The proof is based on various tools that we have developed in order to work with invariant distributions.

There are other methods for proving the Gelfand property apart from the Gelfand-Kazhdan method. Most of them are based on deducing the Gelfand property of one pair from the Gelfand property of another pair. In this thesis there are two examples of such methods.

The theory of invariant distributions has also different applications in representation theory, and we will discuss some of these applications, too.

Content of the thesis

In the introduction we discuss the results that appear in the thesis in more details. Chapter 2 consists of the papers which I wrote during my Ph.D. Studies that are discussed in the introduction. Chapter 3 contains a discussion about possible extensions of the results discussed in the thesis.

Contents

Chapter 1

Introduction

1.1 Gelfand pairs

Most of the problems in modern harmonic analysis are closely related to the following class of problems: let X be some space, and $F(X)$ a space of (complex valued) functions on X of a certain type. Suppose that X possesses some symmetries. Could one find a basis for $F(X)$ that behaves in a "good" way with respect to those symmetries?

As a first step for solving this class of problems one can look at its following concretization: suppose that there exists a group of symmetries G that acts transitively on X. How does $F(X)$ decomposes into irreducible representation of G? This question gives rise to the following notion: a pair (G, H) consisting of a group and a subgroup is said to be a Gelfand pair if any irreducible representation of G is "included" in $F(G/H)$ with multiplicity at most 1.

In any given case one should specify what kind of representations and functions we consider, and what does one mean by "included". For the case of reductive groups over local fields, there are several ways to do this (section 2 in [AGS] – see Chapter 2 below).

An overview of the theory of Gelfand pairs can be found, for example, in [vD].

Note that if a pair (G, H) is a Gelfand pair then the "decomposition" of $F(G/H)$ is unique. By Frobenius reciprocity the Gelfand property is equivalent to the fact that any irreducible representation of G, when restricted to H, "includes" the trivial representation with multiplicity at most 1.

One can consider the representation theory of the group G as the harmonic analysis of the space $F(G)$ with respect to the two-sided action of $G \times G$. From this point of view, Schur's lemma is equivalent to the Gelfand property of $(G \times G, G)$.

Gelfand pairs have various applications to classical questions of representation theory and harmonic analysis. These include the classification of representations, and constructing canonical bases for irreducible representations and spaces of functions on homogenous spaces. More recent applications of Gelfand pairs are in the theory of automorphic forms, for instance in splitting of automorphic periods and in the relative trace formula. Some of these applications are described in [Gro].

1.1.1 Strong Gelfand pairs

A stronger version of the notion of a Gelfand pair is the following one:

A pair (G, H) is said to be a strong Gelfand pair if any irreducible representation of G, when restricted to H, "includes" any irreducible representation of H with multiplicity at most 1.

The notion of Gelfand pair and strong Gelfand pair are connected in the following way: a pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair (here ΔH means the diagonal embedding of H in $G \times H$).

In this thesis we prove the Gelfand property and the strong Gelfand property for various pairs.

1.1.2 Gelfand-Kazhdan criterion

The main tool to prove that a pair (G, H) is a Gelfand pair is the following criterion by Gelfand and Kazhdan ([GK]). Suppose that there exists an involutive anti-automorphism σ of G such that any distribution on G which is invariant with respect to the $H \times H$ two-sided action is invariant with respect to σ. Then the pair (G, H) is a Gelfand pair. This criterion implies an analogous criterion for strong Gelfand pairs.

1.1.3 Symmetric pairs

Most of the results in this thesis are about symmetric pairs. Symmetric pairs are pairs (G, H) where H is the group of fixed points of some involution on G. For a symmetric pair there is a simple necessary condition to be a Gelfand pair. In the papers [AG3], [AG4], [AS] and [Aiz] (see Chapter 2 below) we proved that in many cases this condition is also sufficient. In particular, we obtain the following results

Theorem 1.1.3.1. *The following pairs are Gelfand pairs:*

- $(GL_{n+k}(F), GL_n(F) \times GL_k(F))$ *for* $F = \mathbb{R}, \mathbb{C}$. *The non-archimedean case was proven in [JR].*
- $(GL_n(\mathbb{C}), GL_n(\mathbb{R}))$. *The non-archimedean case was proven in [Fli].*
- $(O_{n+k}(\mathbb{C}), O_n(\mathbb{C}) \times O_k(\mathbb{C}))$.
- $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$.
- $(GL_{2n}(F), Sp_{2n}(F))$ *for* $F = \mathbb{R}, \mathbb{C}$. *The non-archimedean case was proven in [HR].*

These pairs include most of the symmetric pairs over the field of complex numbers. It is conjectured that all the symmetric pairs over the field of complex numbers are Gelfand pairs.

1.1.4 The strong Gelfand property of the pair $(GL_{n+1}(F), GL_n(F))$

Another important result in this thesis is the following theorem.

Theorem 1.1.4.1. *Let F be an arbitrary local field of characteristic 0. Let $GL_n(F)$ be embedded into $GL_{n+1}(F)$ in the standard way. Let $GL_n(F)$ act on $GL_{n+1}(F)$ by conjugation. Then any $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is also invariant with respect to transposition.*

This theorem was conjectured in the 1980-s by Bernstein and Rallis.

We proved this theorem for non-Archimedean F in [AGRS] and for Archimedean F in [AG5] (see Chapter 2 below). The Archimedean case was done independently in [SZ]. In [Aiz] (see Chapter 2 below), we gave a uniform proof of this theorem for all local fields of characteristic 0.

This theorem implies that the pair $(GL_{n+1}(F), GL_n(F))$ is a strong Gelfand pair.

In [AGRS] (see Chapter 2 below) we also proved analogous results for the orthogonal and the unitary groups over non-Archimedean F. These cases for Archimedean F appears in [SZ].

For some applications of those theorems, see [GP], [GGP], [Wal].

This theorem also implies Kirillov's conjecture. Kirillov's conjecture was proven before in [Ber] for non-Archimedean F, in [Sah] for the field of the complex numbers and in [Bar] for any Archimedean F.

1.1.5 General strategy for proving Gelfand property

Here is a brief description of the general strategy for verifying the conditions of Gelfand-Kazhdan criterion for reductive groups that is used in the works above.

One can easily see that a necessary condition for the Gelfand-Kazhdan criterion to work is that σ preserve any closed $H \times H$-coset. If this condition holds, then one can use an inductive argument based on geometric invariant theory (more precisely Luna's slice theorem) to reduce the problem to the study of certain equivariant distributions on a linear space. These distributions are supported in a certain small subset, called the nilpotent cone. Then we use non-geometric tools based on Fourier transform and various kinds of "uncertainty principles" in order to prove the vanishing of such distributions. The "uncertainty principles" that I used in my work were based on the Weil representation and the integrability theorem (from the theory of D-modules).

1.2 Other methods for proving Gelfand property

1.2.1 Positive characteristic versus zero characteristic

There are many methods that allow one to relate problems over fields of positive characteristic with problems over fields of zero characteristic. Most of these methods are based on approximating a field of zero characteristic with fields of positive characteristic. However, there is a different method developed in [Kaz], which is based on approximating a local field of positive characteristic with local fields of zero characteristic.

In this work the following theorem is proven: let G be a reductive group that splits over $\mathbb{Z}$. Then the Hecke algebra of compactly supported functions on $G(F)$ which are double invariant with respect to a given congruence subgroup K does not change when we replace F with a "close enough" local field F'. This theorem means that the representation theory of $G(F)$, when F is a field of positive characteristic, can be approximated by the representation theory of $G(F')$, where F' is a field of zero characteristic.

In the work [AAG] (see Chapter 2 below), we prove an analog of this theorem. It states that for certain pairs (G, H), harmonic analysis over the space $G(F)/H(F)$, when F is a field of positive characteristic, can be approximated by harmonic analysis over the space $G(F')/H(F')$ where F' is a field of zero characteristic. We apply this in order to deduce the Gelfand property of some pairs over fields of positive characteristic from the Gelfand property of these pairs over fields of zero characteristic. In particular, we prove the following theorem:

Theorem 1.2.1.1. *Let F be a local field of* ***positive characteristic.*** *Then*

- *The pair $(GL_{n+1}(F), GL_n(F))$ is a strong Gelfand pair.*
- *Suppose that $char(F) \neq 2$, then the pair $(GL_{2n}(F), GL_n(F) \times GL_n(F))$ is a Gelfand pair.*

Note that the proofs in [AGRS] (see Chapter 2 below) and [JR] of the zero characteristic case cannot be directly applied for the positive characteristic case, since they heavily rely on Jordan decomposition which is problematic in positive characteristic. The proof in [AAG] (see Chapter 2 below) relies on the theory of the Bernstein center (see [BD]) and on certain smoothness analysis of some group schemes over local rings.

1.2.2 Integration of invariant functionals

An important construction in representation theory is an averaging, with respect to the action of a subgroup $H_1 \subset G$, of an H_2-invariant functional on a representation of G, where $H_2 \subset G$ is another subgroup. Many constructions in representation theory and automorphic forms (such as intertwining operators, periods of automorphic forms, certain L-factors, etc.) can be viewed as a special case of the above construction. Note that often this integral does not converge and one has to regularize it, usually using analytic continuation. One can use this construction in order to deduce the Gelfand property of one pair from the Gelfand property of another pair. This was implemented in the work [AGJ] (see Chapter 2 below), where we proved the uniqueness of Shalika functionals in the Archimedean case. The non-Archimedean counterpart of this problem was done in [JR].

1.3 Invariant distributions

As it was mentioned earlier, the main tool to prove the Gelfand property are invariant distributions. I consider this topic to be interesting in its own right, and not only a tool to prove the Gelfand property. We studied this subject in the papers [AG1] and [AG2]. Considerable parts of the works [AG3] and [Aiz] and some parts of [AGS] and [AG6] (see Chapter 2 below) are also devoted to this subject.

1.3.1 Integrability theorem

An important tool in the theory of invariant distributions over real manifolds is the integrability theorem. The integrability theorem is a theorem from the theory of $D-$modules which implies that the characteristic variety of a distribution on a real algebraic manifold X is a co-isotropic sub-variety of T^*X. The theory of $D-$modules is not applicable to the p-adic case. However, the notion of the characteristic variety still exists in the non-Archimedean case (to be more precise, the notion of the wave front set, which is an analytic counterpart of the notion of the characteristic variety exists in the non-Archimedean case). In [Aiz] (see Chapter 2 below), we introduced a partial analog of the integrability theorem for the non-Archimedean case. Namely we introduced the notion of "weakly co-isotropic" sub-variety of T^*X and proved the following theorem

Theorem 1.3.1.1. *Let ξ be a distribution over analytic manifold X over non-archimedean filed. Then the wave front set of ξ is "weakly co-isotropic" sub-variety of T^*X.*

1.3.2 Matching problems

There are other problems in representation theory that can be solved using invariant distributions beside determination of Gelfand pairs. One such problem concerns the comparison of spaces of invariant distributions. Namely, let a group G act on a space X. The space of invariant (or equivariant) distributions on X can often be described in terms of the space of functions on the set of regular G-orbits that are obtained by taking the orbital integrals of smooth (to be precise, Schwartz) functions on X. Suppose we have another group G' that acts on a space X'. Suppose that the set of regular G-orbits on X coincides with the set of regular G'-orbits on X'. Sometimes the spaces of functions on the set of regular orbits that are obtained by taking the orbital integrals also coincide. This phenomenon is crucial in the trace formula, which is an important tool in the Langlands program. Note that usually the spaces of functions described above do not coincide, and one should introduce a matching factor in order to make them identical.

In the work [AG6] (see Chapter 2 below), we establish the following special case of this phenomenon.

Theorem 1.3.2.1. *Let N_n be the group of $n \times n$ upper unipotent matrices. Let $N(\mathbb{R}) \times N(\mathbb{R})$ act on $GL_n(\mathbb{R})$ by $(n_1, n_2)(x) = n_1^t x n_2$. Let $A \subset GL_n$ be the set of diagonal matrices. Let $\tilde{\Omega}_1 : \mathcal{S}(GL_n(\mathbb{R})) \to C^\infty(A)$ be a map defined by*

$$\tilde{\Omega}_1(f)(a) = \alpha(a) \int_{n \in N(\mathbb{R}) \times N(\mathbb{R})} f(n(a))\psi(n)dn,$$

where ψ is a "non-degenerate" character of $N(\mathbb{R}) \times N(\mathbb{R})$ and α is some normalizing factor. Let S_n be the space of non-degenerate hermitian forms and let $N(\mathbb{C})$ act on it by $n(x) = n^t x n$. Let $\tilde{\Omega}_2 : \mathcal{S}(S_n) \to C^\infty(A)$ be a map defined similarly to $\tilde{\Omega}_1$.

Then $\mathrm{Im}(\tilde{\Omega}_2) = \mathrm{Im}(\tilde{\Omega}_1)$

The non-Archimedean counterpart of this theorem was done in [Jac].

Chapter 2

Papers

In this chapter we present the papers [AGS],[AG3],[AG4], [AS], [AGRS] ,[AG5], [AAG], [AGJ],[AG6]

$(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ IS A GELFAND PAIR FOR ANY LOCAL FIELD F

AVRAHAM AIZENBUD, DMITRY GOUREVITCH, AND EITAN SAYAG

Abstract. Let F be an arbitrary local field. Consider the standard embedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$ and the two-sided action of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$.

In this paper we show that any $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$-invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.

We show that this implies that the pair $(GL_{n+1}(F), GL_n(F))$ is a Gelfand pair. Namely, for any irreducible admissible representation (π, E) of $\mathrm{GL}_{n+1}(F)$,

$$\dim Hom_{\mathrm{GL}_n(F)}(E, \mathbb{C}) \leq 1.$$

For the proof in the archimedean case we develop several tools to study invariant distributions on smooth manifolds.

Contents

Key words and phrases. Multiplicity one, invariant distribution.
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1. Introduction

Let F be an arbitrary local field. Consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the two-sided action of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. In this paper we prove the following theorem:

Theorem (A). *Any $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.*

Theorem A has the following consequence in representation theory.

Theorem (B). *Let (π, E) be an irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$. Then*

(1) $$\dim Hom_{\mathrm{GL}_n(F)}(E, \mathbb{C}) \leq 1.$$

Since any character of $\mathrm{GL}_n(F)$ can be extended to $\mathrm{GL}_{n+1}(F)$, we obtain

Corollary. *Let (π, E) be an irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$ and let χ be a character of $\mathrm{GL}_n(F)$. Then*

$$\dim Hom_{\mathrm{GL}_n(F)}(\pi, \chi) \leq 1.$$

In the non-archimedean case we use the standard notion of admissible representation (see [BZ]). In the archimedean case we consider admissible smooth Fréchet representations (see section 2).

Theorem B has some application to the theory of automorphic forms, more specifically to the factorizability of certain periods of automorphic forms on GL_n (see [Fli] and [FN]).

We deduce Theorem B from Theorem A using an argument due to Gelfand and Kazhdan adapted to the archimedean case. In our approach we use two deep results: the globalization theorem of Casselman-Wallach (see [Wal2]), and the regularity theorem of Harish-Chandra ([Wal1], chapter 8).

Clearly, Theorem B implies in particular that (1) holds for unitary irreducible representations of $\mathrm{GL}_{n+1}(F)$. That is, the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a generalized Gelfand pair in the sense of [vD] and [BvD].

The notion of Gelfand pair was studied extensively in the literature both in the setting of real groups and p-adic groups (e.g. [GK], [vD], [vDP], [BvD], [Gro], [Pra] and [JR] to mention a few). In [vD], the notion of generalized Gelfand pair is defined by requiring a condition of the form (1) for irreducible unitary representations. The definition suggested in [Gro] refers to the non-archimedean case and to a property satisfied by all irreducible admissible representations. In both cases, the verification of the said condition is achieved by means of a theorem on invariant distributions. However, the required statement on invariant distributions needed to verify condition (1) for unitary representation concerns only positive definite distributions. We elaborate on these issues in section 2.

1.1. **Related results.**

Several existing papers study related problems.

The case of non-archimedean fields of zero characteristic is covered in [AGRS] (see also [AG2]) where it is proven that the pair $(GL_{n+1}(F), GL_n(F))$ is a strong Gelfand pair i.e. $\dim_H(\pi, \sigma) \leq 1$ for any irreducible admissible representation π of G and *any* irreducible admissible representation σ of H. Here $H = \mathrm{GL}_n(F)$ and $G = \mathrm{GL}_{n+1}(F)$.

In [JR], it is proved that $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_1(F))$ is a Gelfand pair, where F is a local non-archimedean field of zero characteristic.

In [vDP] it is proved that for $n \geq 2$ the pair $(SL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$ is a generalized Gelfand pair and a similar result is obtained in [BvD] for the p-adic case, for $n \geq 3$. We emphasize that these results are proved in the realm of unitary representations. Another difference between these works and the present paper is that the embedding $GL_n(F) \subset GL_{n+1}(F)$ studied here does not factor through the embedding $GL_n(F) \hookrightarrow SL_{n+1}(F)$ of [vDP]. In particular, $(GL_2(\mathbb{R}), GL_1(\mathbb{R}))$ is a generalized Gelfand pair, and the pair $(SL_2(\mathbb{R}), GL_1(\mathbb{R}))$ is not a generalized Gelfand pair ([Mol],[vD]).

1.2. **Content of the Paper.**

We now briefly sketch the structure and content of the paper.

In section 2 we prove that Theorem A implies Theorem B. For this we clarify the relation between the theory of Gelfand pairs and the theory of invariant distributions both in the setting of [vD] and in the setting of [Gro].

In section 3 we present the proof of theorem A in the non-archimedean case. This section gives a good introduction to the rest of the paper since it contains many of the ideas but is technically simpler.

In section 4 we provide several tools to study invariant distributions on smooth manifolds. We believe that these results are of independent interest. In particular we introduce an adaption of a trick due to Bernstein which is very useful in the study of invariant distributions on vector spaces (proposition 4.3.2). These results partly relay on [AG1].

In section 5 we prove Theorem A in the archimedean case. This is the main result of the paper. The scheme of the proof is similar to the non-archimedean case. However, it is complicated by the fact that distributions on real manifolds do not behave as nicely as distributions on ℓ-spaces (see [BZ]).

We now explain briefly the main difference between the study of distributions on ℓ-spaces and distributions on real manifolds.

The space of distributions on an ℓ-space X supported on a closed subset $Z \subset X$ coincides with the space of distributions on Z. In the presence of group action on X, one can frequently use this property to reduce the study of distributions on X to distributions on orbits, that is on homogenous spaces. Although this property fails for distributions on real manifolds, one can still reduce problems to orbits. In the case of finitely many orbits this is studied in [Bru], [CHM], [AG1].

We mention that unlike the p-adic case, after the reduction to the orbits one needs to analyze generalized sections of symmetric powers of the normal bundles to the orbits, and not just distributions on those orbits. Here we employ a trick, proposition 4.3.1, which allows us to recover this information from a study of invariant distributions on a larger space.

In section A we provide the proof for the Frobenius reciprocity. The proof follows the proof in [Bar] (section 3).

In section B we prove the rest of the statements of section 4.

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2. Generalized Gelfand pairs and invariant distributions

In this section we show that Theorem A implies Theorem B. When F is non-archimedean this is a well known argument of Gelfand and Kazhdan (see [GK, Pra]). When F is archimedean and the representations in question are unitary such a reduction is due to [Tho]. We wish to consider representations which are not necessarily unitary and present here an argument which is valid in the generality of admissible smooth Fréchet representations. Our treatment is close in spirit to [Sha] (where multiplicity one result of Whittaker model is obtained for unitary representation) but at a crucial point we need to use the globalization theorem of Casselman-Wallach.

2.1. Smooth Fréchet representations.

The theory of representations in the context of Fréchet spaces is developed in [Cas2] and [Wal2]. We present here a well-known slightly modified version of that theory.

Definition 2.1.1. Let V be a complete locally convex topological vector space. A representation (π, V, G) is a continuous map $G \times V \to V$. A representation is called **Fréchet** if there exists a countable family of semi-norms ρ_i on V defining the topology of V and such that the action of G is continuous with respect to each ρ_i. We will say that V is **smooth Fréchet** representation if, for any $X \in \mathfrak{g}$ the differentiation map $v \mapsto \pi(X)v$ is a continuous linear map from V to V.

An important class of examples of smooth Fréchet representations is obtained from continuous Hilbert representations (π, H) by considering the subspace of smooth vectors H^{∞} as a Fréchet space (see [Wal1] section 1.6 and [Wal2] 11.5).

We will consider mostly smooth Fréchet representations.

Remark 2.1.2. In the language of [Wal2] and [Cas] the representations above are called smooth Fréchet representations of moderate growth.

Recall that a smooth Fréchet representation is called *admissible* if it is finitely generated and its underlying $(\mathfrak{g}, K)$-module is admissible. In what follows *admissible representation* will always refer to admissible smooth Fréchet representation.

For a smooth admissible Fréchet representation (π, E) we denote by $(\widetilde{\pi}, \widetilde{E})$ the smooth contragredient of (π, E).

We will require the following corollary of the globalization theorem of Casselman and Wallach (see [Wal2] , chapter 11).

Theorem 2.1.3. *Let E be an admissible Fréchet representation, then there exists a continuous Hilbert space representation (π, H) such that $E = H^\infty$.*

This theorem follows easily from the embedding theorem of Casselman combined with Casselman-Wallach globalization theorem.

Fréchet representations of G can be lifted to representations of $\mathcal{S}(G)$, the Schwartz space of G. This is a space consisting of functions on G which, together with all their derivatives, are rapidly decreasing (see [Cas]. For an equivalent definition see section 4.1).

For a Fréchet representation (π, E) of G, the algebra $\mathcal{S}(G)$ acts on E through

$$\pi(\phi) = \int_G \phi(g)\pi(g)dg \tag{2}$$

(see [Wal1], section 8.1.1).
The following lemma is straitforward:

Lemma 2.1.4. *Let (π, E) be an admissible Fréchet representation of G and let $\lambda \in E^*$. Then $\phi \to \pi(\phi)\lambda$ is a continuous map $\mathcal{S}(G) \to \widetilde{E}$.*

The following proposition follows from Schur's lemma for $(\mathfrak{g}, K)$ modules (see [Wal1] page 80) in light of Casselman-Wallach theorem.

Proposition 2.1.5. *Let G be a real reductive group. Let W be a Fréchet representation of G and let E be an irreducible admissible representation of G. Let $T_1, T_2 : W \hookrightarrow E$ be two embeddings of W into E. Then T_1 and T_2 are proportional.*

We need to recall the basic properties of characters of representations.

Proposition 2.1.6. *Assume that (π, E) is admissible Fréchet representation. Then $\pi(\phi)$ is of trace class, and the assignment $\phi \to trace(\pi(\phi))$ defines a continuous functional on $\mathcal{S}(G)$ i.e. a Schwartz distribution. Moreover, the distribution $\chi_\pi(\phi) = trace(\pi(\phi))$ is given by a locally integrable function on G.*

The result is well known for continuous Hilbert representations (see [Wal1] chapter 8). The case of admissible Fréchet representation follows from the case of Hilbert space representation and theorem 2.1.3.

Another useful property of the character (see loc. cit.) is the following proposition:

Proposition 2.1.7. *If two irreducible admissible representations have the same character then they are isomorphic.*

Proposition 2.1.8. *Let (π, E) be an admissible representation. Then $\widetilde{\widetilde{E}} \cong E$.*

For proof see pages 937-938 in [GP].

2.2. **Three notions of Gelfand pair.**

Let G be a real reductive group and $H \subset G$ be a subgroup. Let (π, E) be an admissible Fréchet representation of G as in the previous section. We are interested in representations (π, E) which admit a continuous H-invariant linear functional. Such representations of G are called H-distinguished.

Put differently, let $Hom_H(E, \mathbb{C})$ be the space of continuous functionals $\lambda : E \to \mathbb{C}$ satisfying

$$\forall e \in E, \forall h \in H : \lambda(he) = \lambda(e)$$

The representation (π, E) is called **H-distinguished** if $Hom_H(E, \mathbb{C})$ is non-zero. We now introduce three notions of Gelfand pair and study their inter-relations.

Definition 2.2.1. Let $H \subset G$ be a pair of reductive groups.

- We say that (G, H) satisfy **GP1** if for any irreducible admissible representation (π, E) of G we have

$$\dim Hom_H(E, \mathbb{C}) \leq 1$$

- We say that (G, H) satisfy **GP2** if for any irreducible admissible representation (π, E) of G we have

$$\dim Hom_H(E, \mathbb{C}) \cdot \dim Hom_H(\widetilde{E}, \mathbb{C}) \leq 1$$

- We say that (G, H) satisfy **GP3** if for any irreducible **unitary** representation (π, W) of G on a Hilbert space W we have

$$\dim Hom_H(W^\infty, \mathbb{C}) \leq 1$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced by [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vDP], [BvD]).

We have the following straitforward proposition:

Proposition 2.2.2. $GP1 \Rightarrow GP2 \Rightarrow GP3$.

2.3. Gelfand pairs and invariant distributions.

The theory of generalized Gelfand pairs as developed in [vDP] and [Tho] provides the following criterion to verify GP3.

Theorem 2.3.1. *Let τ be an involutive anti-automorphism of G such that $\tau(H) = H$. Suppose $\tau(T) = T$ for all bi H-invariant positive definite distributions T on G. Then (G, H) satisfies GP3.*

This is a slight reformulation of Criterion 1.2 of [vD], page 583.

We now consider an analogous criterion which allows the verification of GP2. This is inspired by the famous Gelfand-Kazhdan method in the p-adic case.

Theorem 2.3.2. *Let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(T) = T$ for all bi H-invariant distributions* [1] *on G. Then (G, H) satisfies GP2.*

Proof. Let (π, E) be an irreducible admissible Fréchet representation. If E or $\widetilde{E}$ are not distinguished by H we are done. Thus we can assume that there exists a non-zero $\lambda : E \to \mathbb{C}$ which is H-invariant. Now let ℓ_1, ℓ_2 be two non-zero H-invariant functionals on $\widetilde{E}$. We wish to show that they are proportional. For this we define two distributions D_1, D_2 as follows

$$D_i(\phi) = \ell_i(\pi(\phi)\lambda)$$

[1] In fact it is enough to check this only for Schwartz distributions.

for $i = 1, 2$. Here $\phi \in \mathcal{S}(G)$. Note that D_i are also Schwartz distributions. Both distributions are bi-H-invariant and hence, by the assumption, both distributions are τ invariant. Now consider the bilinear forms on $\mathcal{S}(G)$ defined by

$$B_i(\phi_1, \phi_2) = D_i(\phi_1 * \phi_2).$$

Since E is irreducible, the right kernel of B_1 is equal to the right kernel of B_2. We now use the fact that D_i are τ invariant. Denote by J_i the left kernels of B_i. Then $J_1 = J_2$ which we denote by J. Consider the Fréchet representation $W = \mathcal{S}(G)/J$ and define the maps $T_i : \mathcal{S}(G) \to \widetilde{\widetilde{E}} \cong E$ by $T_i(\phi) = \pi(\phi)\ell_i$. These are well defined by Lemma 2.1.4 and we use the same letters to denote the induced maps $T_i : W \to E$. By proposition 2.1.5, T_1 and T_2 are proportional and hence ℓ_1 and ℓ_2 are proportional and the proof is complete. □

2.4. **Archimedean analogue of Gelfand-Kazhdan's theorem.**

To finish the proof that Theorem A implies Theorem B we will show that in certain cases, the property $GP1$ is equivalent to $GP2$.

Proposition 2.4.1. *Let $H < \mathrm{GL}_n(F)$ be a transposition invariant subgroup. Then $GP1$ is equivalent to $GP2$ for the pair $(\mathrm{GL}_n(F), H)$.*

For the proof we need the following notation. For a representation (π, E) of $GL_n(F)$ we let $(\widehat{\pi}, E)$ be the representation of $GL_n(F)$ defined by $\widehat{\pi} = \pi \circ \theta$, where θ is the (Cartan) involution $\theta(g) = {g^{-1}}^t$. Since

$$Hom_H(\pi, \mathbb{C}) = Hom_H(\widehat{\pi}, \mathbb{C})$$

the following analogue of Gelfand-Kazhdan theorem is enough.

Theorem 2.4.2. *Let (π, E) be an irreducible admissible representation of $GL_n(F)$. Then $\widehat{\pi}$ is isomorphic to $\widetilde{\pi}$.*

Remark 2.4.3. This theorem is due to Gelfand and Kazhdan in the p-adic case (they show that any distribution which is invariant to conjugation is transpose invariant, in particular this is valid for the character of an irreducible representation) and due to Shalika for unitary representations which are generic ([Sha]). We give a proof in complete generality based on Harish-Chandra regularity theorem (see chapter 8 of [Wal1]).

Proof of theorem 2.4.2. Consider the characters $\chi_{\widetilde{\pi}}$ and $\chi_{\widehat{\pi}}$. These are locally integrable functions on G that are invariant with respect to conjugation. Clearly,

$$\chi_{\widehat{\pi}}(g) = \chi_\pi({g^{-1}}^t)$$

and

$$\chi_{\widetilde{\pi}}(g) = \chi_\pi(g^{-1}).$$

But for $g \in \mathrm{GL}_n(F)$, the elements g^{-1} and ${g^{-1}}^t$ are conjugate. Thus, the characters of $\widehat{\pi}$ and $\widetilde{\pi}$ are identical. Since both are irreducible, Theorem 8.1.5 in [Wal1], implies that $\widehat{\pi}$ is isomorphic to $\widetilde{\pi}$. □

Corollary 2.4.4. *Theorem A implies Theorem B.*

Remark 2.4.5. The above argument proves also that Theorem B follows from a weaker version of Theorem A, where only Schwartz distributions are considered (these are continuous functionals on the space $\mathcal{S}(G)$ of Schwartz functions).

Remark 2.4.6. The non-archimedean analogue of theorem 2.3.2 is a special case of Lemma 4.2 of [Pra]. The rest of the argument in the non-archimedean case is identical to the above.

3. Non-archimedean case

In this section F is a non-archimedean local field of arbitrary characteristic. We will use the standard terminology of l-spaces introduced in [BZ], section 1. We denote by $\mathcal{S}(X)$ the space of Schwartz functions on an l-space X, and by $\mathcal{S}^*(X)$ the space of distributions on X equipped with the weak topology.

We fix a nontrivial additive character ψ of F.

3.1. **Preliminaries.**

Definition 3.1.1. Let V be a finite dimensional vector space over F. A subset $C \subset V$ is called a **cone** if it is homothety invariant.

Definition 3.1.2. Let V be a finite dimensional vector space over F. Note that $F^\times$ acts on V by homothety. This gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$. Let α be a character of $F^\times$.

We call a distribution $\xi \in \mathcal{S}^*(V)$ **homogeneous of type** α if for any $t \in F^\times$, we have $\rho(t)(\xi) = \alpha^{-1}(t)\xi$. That is, for any function $f \in \mathcal{S}(V)$, $\xi(\rho(t^{-1})(f)) = \alpha(t)\xi(f)$, where $\rho(t^{-1})(f)(v) = f(tv)$.

Let $LsubsetF$ be a subfield. We will call a distribution $\xi \in \mathcal{S}^*(V)$ L**-homogeneous of type** α if for any $t \in L^\times$, we have $\rho(t)(\xi) = \alpha^{-1}(t)\xi$.

Example 3.1.3. A Haar measure on V is homogeneous of type $|\cdot|^{\dim V}$. The Dirac's δ-distribution is homogeneous of type 1.

The following proposition is straightforward.

Proposition 3.1.4. *Let a l-group G act on an l-space X. Let $X = \bigcup_{i=0}^{l} X_i$ be a G-invariant stratification of X. Let χ be a character of G. Suppose that for any $i = 1 \dots l$, $\mathcal{S}^*(X_i)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.*

Proposition 3.1.5. *Let $H_i \subset G_i$ be l-groups acting on l-spaces X_i for $i = 1 \dots n$. Suppose that $\mathcal{S}^*(X_i)^{H_i} = \mathcal{S}^*(X_i)^{G_i}$ for all i. Then $\mathcal{S}^*(\prod X_i)^{\prod H_i} = \mathcal{S}^*(\prod X_i)^{\prod G_i}$.*

Proof. It is enough to prove the proposition for the case $n = 2$. Let $\xi \in \mathcal{S}^*(X_1 \times X_1)^{H_1 \times H_2}$. Fix $f_1 \in \mathcal{S}(X_1)$ and $f_2 \in \mathcal{S}(X_1)$. It is enough to prove that for any $g_1 \in G_1$ and $g_2 \in G_2$, we have $\xi(g_1(f_1) \otimes g_2(f_2)) = \xi(f_1 \otimes f_2)$. Let $\xi_1 \in \mathcal{S}^*(X_1)$ be the distribution defined by $\xi_1(f) := \xi(f \otimes f_2)$. It is H_1-invariant. Hence also G_1-invariant. Thus $\xi(f_1 \otimes f_2) = \xi(g_1(f_1) \otimes f_2)$. By the same reasons $\xi(g_1(f_1) \otimes f_2) = \xi(g_1(f_1) \otimes g_2(f_2))$. □

We will use the following important theorem proven in [Ber1], section 1.5.

Theorem 3.1.6 (Frobenius reciprocity)**.** *Let a unimodular l-group G act transitively on an l-space Z. Let $\varphi : X \to Z$ be a G-equivariant continuous map. Let $z \in Z$. Suppose that its stabilizer $\mathrm{Stab}_G(z)$ is unimodular. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{\mathrm{Stab}_G(z),\chi}$.*

The next proposition formalizes an idea from [Ber2]. The key tool used in its proof is Fourier Transform.

Proposition 3.1.7. *Let G be an l-group. Let V be a finite dimensional representation of G over F. Suppose that the action of G preserves some non-degenerate bilinear form on V. Let $V = \bigcup_{i=1}^{n} C_i$ be a stratification of V by G-invariant cones.*

Let $\mathfrak{X}$ be a set of characters of $F^\times$ such that the set $\mathfrak{X} \cdot \mathfrak{X}$ does not contain the character $|\cdot|^{\dim V}$. Let χ be a character of G. Suppose that for any i, the space $\mathcal{S}^*(C_i)^{G,\chi}$ consists of homogeneous distributions of type α for some $\alpha \in \mathfrak{X}$. Then $\mathcal{S}^*(V)^{G,\chi} = 0$.

In section B.3 we prove an archimedean analog of this proposition, and the same proof is applicable in this case.

3.2. **Proof of Theorem A for non-archimedean F.**

We need some further notations.

Notation 3.2.1. Denote $H := H_n := \mathrm{GL}_n := \mathrm{GL}_n(F)$. Denote

$$G := G_n := \{(h_1, h_2) \in \mathrm{GL}_n \times \mathrm{GL}_n | \, det(h_1) = det(h_2)\}.$$

We consider H to be diagonally embedded to G.

Consider the action of the 2-element group S_2 on G given by the involution $(h_1, h_2) \mapsto ({h_2^{-1}}^t, {h_1^{-1}}^t)$. It defines a semidirect product $\widetilde{G} := \widetilde{G}_n := G \rtimes S_2$. Denote also $\widetilde{H} := \widetilde{H}_n := H_n \rtimes S_2$.

Let $V = F^n$ and $X := X_n := \mathrm{gl}_n(F) \times V \times V^*$.

The group $\widetilde{G}$ acts on X by

$$(h_1, h_2)(A, v, \phi) := (h_1 A h_2^{-1}, h_1 v, {h_2^{-1}}^t \phi) \text{ and}$$
$$\sigma(A, v, \phi) := (A^t, \phi^t, v^t)$$

where $(h_1, h_2) \in G$ and σ is the generator of S_2. Note that $\widetilde{G}$ acts separately on gl_n and on $V \times V^*$. Define a character χ of $\widetilde{G}$ by $\chi(g, s) := sign(s)$.

We will show that the following theorem implies Theorem A.

Theorem 3.2.2. $\mathcal{S}^*(X)^{\widetilde{G},\chi} = 0$.

3.2.1. *Proof that theorem 3.2.2 implies theorem A.*

We will divide this reduction to several propositions.
Consider the action of $\widetilde{G}_n$ on GL_{n+1} and on gl_{n+1}, where G_n acts by the two-sided action and the generator of S_2 acts by transposition.

Proposition 3.2.3. *If $\mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi} = 0$ then theorem A holds.*

The proof is straightforward.

Proposition 3.2.4. *If $\mathcal{S}^*(\mathrm{gl}_{n+1})^{\widetilde{G}_n,\chi} = 0$ then $\mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi} = 0$.*

Proof.[2] Let $\xi \in \mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \mathrm{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in \mathcal{S}(F)$ be such that f vanishes in a neighborhood of zero and $f(t) \neq 0$. Consider the determinant map $\det : \mathrm{GL}_{n+1} \to F$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(\mathrm{GL}_{n+1})^{\widetilde{G}_n,\chi}$ and $p \in \mathrm{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathrm{gl}_{n+1})^{\widetilde{G}_n,\chi}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. □

[2] This proposition is an adaption of a statement in [Ber1], section 2.2.

Proposition 3.2.5. *If* $\mathcal{S}^*(X_n)^{\widetilde{G}_n,\chi} = 0$ *then* $\mathcal{S}^*(\mathrm{gl}_{n+1})^{\widetilde{G}_n,\chi} = 0$.

Proof. Note that gl_{n+1} is isomorphic as a $\widetilde{G}_n$-equivariant l-space to $X_n \times F$ where the action on F is trivial. This isomorphism is given by

$$\begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & t \end{pmatrix} \mapsto ((A, v, \phi), t).$$

The proposition now follows from proposition 3.1.5. □

This finishes the proof that theorem 3.2.2 implies Theorem A.

3.2.2. *Proof of theorem 3.2.2.*
We will now stratify $X(= gl_n \times V \times V^*)$ and deal with each strata separately.

Notation 3.2.6. Denote $W := W_n := V_n \oplus V_n^*$. Denote by $Q^i := Q_n^i \subset \mathrm{gl}_n$ the set of all matrices of rank i. Denote $Z^i := Z_n^i := Q_n^i \times W_n$.

Note that $X = \bigcup Z^i$. Hence by proposition 3.1.4, it is enough to prove the following proposition.

Proposition 3.2.7. $\mathcal{S}^*(Z^i)^{\widetilde{G},\chi} = 0$ *for any* $i = 0, 1, ..., n$.

We will use the following key lemma.

Lemma 3.2.8 (Non-archimedean Key Lemma)**.** $\mathcal{S}^*(W)^{\widetilde{H},\chi} = 0$.

For proof see section 3.3 below.

Corollary 3.2.9. *Proposition 3.2.7 holds for* $i = n$.

Proof. Clearly, one can extend the actions of $\widetilde{G}$ on Q^n and on Z^n to actions of $\widetilde{GL_n \times GL_n} := (GL_n \times GL_n) \rtimes S_2$ in the obvious way.

Step 1. $\mathcal{S}^*(Z^n)^{\widetilde{GL_n \times GL_n},\chi} = 0$.
Consider the projection on the first coordinate from Z^n to the transitive $\widetilde{GL_n \times GL_n}$-space $Q^n = GL_n$. Choose the point $Id \in Q^n$. Its stabilizer is $\widetilde{H}$ and its fiber is W. Hence by Frobenius reciprocity (theorem 3.1.6), $\mathcal{S}^*(Z^n)^{\widetilde{GL_n \times GL_n},\chi} \cong \mathcal{S}^*(W)^{\widetilde{H},\chi}$ which is zero by the key lemma.

Step 2. $\mathcal{S}^*(Z^n)^{\widetilde{G},\chi} = 0$.
Consider the space $Y := Z^n \times F^\times$ and let the group $GL_n \times GL_n$ act on it by $(h_1, h_2)(z, \lambda) := ((h_1, h_2)z, \det h_1 \det h_2^{-1} \lambda)$. Extend this action to action of $\widetilde{GL_n \times GL_n}$ by $\sigma(z, \lambda) := (\sigma(z), \lambda)$. Consider the projection $Z^n \times F^\times \to F^\times$. By Frobenius reciprocity (theorem 3.1.6),

$$\mathcal{S}^*(Y)^{\widetilde{GL_n \times GL_n},\chi} \cong \mathcal{S}^*(Z^n)^{\widetilde{G},\chi}.$$

Let Y' be equal to Y as an l-space and let $\widetilde{GL_n \times GL_n}$ act on Y' by $(h_1, h_2)(z, \lambda) := ((h_1, h_2)z, \lambda)$ and $\sigma(z, \lambda) := (\sigma(z), \lambda)$. Now Y is isomorphic to Y' as a $\widetilde{GL_n \times GL_n}$ space by $((A, v, \phi), \lambda) \mapsto ((A, v, \phi), \lambda \det A^{-1})$.

Since $\mathcal{S}^*(Z^n)^{\widetilde{GL_n \times GL_n},\chi} = 0$, proposition 3.1.5 implies that $\mathcal{S}^*(Y')^{\widetilde{GL_n \times GL_n},\chi} = 0$ and hence $\mathcal{S}^*(Y)^{\widetilde{GL_n \times GL_n},\chi} = 0$ and thus $\mathcal{S}^*(Z^n)^{\widetilde{G}_n,\chi} = 0$. □

Corollary 3.2.10. *We have*

$$\mathcal{S}^*(W_i \times W_{n-i})^{H_i \times H_{n-i}} = \mathcal{S}^*(W_i \times W_{n-i})^{\widetilde{H}_i \times \widetilde{H}_{n-i}}.$$

Proof. It follows from the key lemma and proposition 3.1.5. □

Now we are ready to prove proposition 3.2.7.

Proof of proposition 3.2.7. Fix $i < n$. Consider the projection $pr_1 : Z^i \to Q^i$. It is easy to see that the action of $\widetilde{G}$ on Q^i is transitive. We are going to use Frobenius reciprocity.

Denote

$$A_i := \begin{pmatrix} Id_{i\times i} & 0 \\ 0 & 0 \end{pmatrix} \in Q^i.$$

Denote by $G_{A_i} := \mathrm{Stab}_G(A_i)$ and $\widetilde{G}_{A_i} := \mathrm{Stab}_{\widetilde{G}}(A_i)$.

It is easy to check by explicit computation that

- G_{A_i} and $\widetilde{G}_{A_i}$ are unimodular.
- $H_i \times G_{n-i}$ can be canonically embedded into G_{A_i}.
- W is isomorphic to $W_i \times W_{n-i}$ as $H_i \times G_{n-i}$-spaces.

By Frobenius reciprocity (theorem 3.1.6),

$$\mathcal{S}^*(Z^i)^{\widetilde{G},\chi} = \mathcal{S}^*(W)^{\widetilde{G}_{A_i},\chi}.$$

Hence it is enough to show that $\mathcal{S}^*(W)^{G_{A_i}} = \mathcal{S}^*(W)^{\widetilde{G}_{A_i}}$. Let $\xi \in \mathcal{S}^*(W)^{G_{A_i}}$. By the previous corollary, ξ is $\widetilde{H}_i \times \widetilde{H}_{n-i}$-invariant. Since ξ is also G_{A_i}-invariant, it is $\widetilde{G}_{A_i}$-invariant. □

3.3. **Proof of the key lemma (lemma 3.2.8).**

Our key lemma is proved in section 10.1 of [RS]. The proof below is slightly different and more convenient to adapt to the archimedean case.

Proposition 3.3.1. *It is enough to prove the key lemma for $n = 1$.*

Proof. Consider the subgroup $T_n \subset H_n$ consisting of diagonal matrices, and $\widetilde{T}_n := T_n \rtimes S_2 \subset \widetilde{H}_n$. It is enough to prove $\mathcal{S}^*(W_n)^{\widetilde{T}_n,\chi} = 0$.

Now, by proposition 3.1.5 it is enough to prove $\mathcal{S}^*(W_1)^{\widetilde{H}_1,\chi} = 0$. □

From now on we fix $n := 1$, $H := H_1$, $\widetilde{H} := \widetilde{H}_1$ and $W := W_1$. Note that $H = F^\times$ and $W = F^2$. The action of H is given by $\rho(\lambda)(x, y) := (\lambda x, \lambda^{-1} y)$ and extended to the action of $\widetilde{H}$ by the involution $\sigma(x, y) = (y, x)$.

Let $Y := \{(x, y) \in F^2 | \, xy = 0\} \subset W$ be the **cross** and $Y' := Y \setminus \{0\}$.

By proposition 3.1.7, it is enough to prove the following proposition.

Proposition 3.3.2.
(i) $\mathcal{S}^*(\{0\})^{\widetilde{H},\chi} = 0$.
(ii) Any distribution $\xi \in \mathcal{S}^*(Y')^{\widetilde{H},\chi}$ *is homogeneous of type 1.*
(iii) $\mathcal{S}^*(W \setminus Y)^{\widetilde{H},\chi} = 0$.

Proof. (i) and (ii) are trivial.
(iii) Denote $U := W \setminus Y$. We have to show $\mathcal{S}^*(U)^{\widetilde{H},\chi} = 0$. Consider the coordinate change $U \cong F^\times \times F^\times$ given by $(x, y) \mapsto (xy, x/y)$. It is an isomorphism of $\widetilde{H}$-spaces where the action of $\widetilde{H}$ on $F^\times \times F^\times$ is only on the second coordinate, and given by $\lambda(w) = \lambda^2 w$ and $\sigma(w) = w^{-1}$. Clearly, $\mathcal{S}^*(F^\times)^{\widetilde{H},\chi} = 0$ and hence by proposition 3.1.5 $\mathcal{S}^*(F^\times \times F^\times)^{\widetilde{H},\chi} = 0$. □

4. Preliminaries on equivariant distributions in the archimedean case

From now till the end of the paper F denotes an archimedean local field, that is $\mathbb{R}$ or $\mathbb{C}$. Also, the word smooth means infinitely differentiable.

4.1. **Notations.**

4.1.1. *Distributions on smooth manifolds.*
Here we present basic notations on smooth manifolds and distributions on them.

Definition 4.1.1. Let X be a smooth manifold. Denote by $C_c^\infty(X)$ the space of complex-valued test functions on X, that is smooth compactly supported functions, with the standard topology, i.e. the topology of inductive limit of Fréchet spaces.

Denote $\mathcal{D}(X) := C_c^\infty(X)^*$ equipped with the weak topology.

For any vector bundle E over X we denote by $C_c^\infty(X, E)$ the complexification of space of smooth compactly supported sections of E and by $\mathcal{D}(X, E)$ its dual space. Also, for any finite dimensional real vector space V we denote $C_c^\infty(X, V) := C_c^\infty(X, X \times V)$ and $\mathcal{D}(X, V) := \mathcal{D}(X, X \times V)$, where $X \times V$ is a trivial bundle.

Definition 4.1.2. Let X be a smooth manifold and let $Z \subset X$ be a closed subset. We denote $\mathcal{D}_X(Z) := \{\xi \in \mathcal{D}(X) | \mathrm{Supp}(\xi) \subset Z\}$.

For locally closed subset $Y \subset X$ we denote $\mathcal{D}_X(Y) := \mathcal{D}_{X \setminus (\overline{Y} \setminus Y)}(Y)$. In the same way, for any bundle E on X we define $\mathcal{D}_X(Y, E)$.

Notation 4.1.3. Let X be a smooth manifold and Y be a smooth submanifold. We denote by $N_Y^X := (T_X|_Y)/T_Y$ the normal bundle to Y in X. We also denote by $CN_Y^X := (N_Y^X)^*$ the conormal bundle. For a point $y \in Y$ we denote by $N_{Y,y}^X$ the normal space to Y in X at the point y and by $CN_{Y,y}^X$ the conormal space.

We will also use notions of a cone in a vector space and of homogeneity type of a distribution defined in the same way as in non-archimedean case (definitions 3.1.1 and 3.1.2).

4.1.2. *Schwartz distributions on Nash manifolds.*
Our proof of Theorem A uses a trick (proposition 4.3.2) involving Fourier Transform which cannot be directly applied to distributions. For this we require a theory of Schwartz functions and distributions as developed in [AG1]. This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered (section B is a minor exception). Therefore the reader can safely replace the word *Nash* by *smooth real algebraic*.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

Notation 4.1.4. Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.

Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the space of Schwartz distributions on X.

For any Nash vector bundle E over X we denote by $\mathcal{S}(X, E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X, E)$ its dual space.

Definition 4.1.5. Let X be a smooth manifold, and let $Y \subset X$ be a locally closed (semi-)algebraic subset. Let E be a Nash bundle over X. We define $\mathcal{S}_X^*(Y)$ and $\mathcal{S}_X^*(Y, E)$ in the same way as $\mathcal{D}_X(Y)$ and $\mathcal{D}_X(Y, E)$.

Remark 4.1.6. All the classical bundles on a Nash manifold are Nash bundles. In particular the normal and conormal bundle to a Nash submanifold of a Nash manifold are Nash bundles. For proof see e.g. [AG1], section 6.1.

Remark 4.1.7. For any Nash manifold X, we have $C_c^\infty(X) \subset \mathcal{S}(X)$ and $\mathcal{S}^*(X) \subset \mathcal{D}(X)$.

Remark 4.1.8. Schwartz distributions have the following two advantages over general distributions:
(i) For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence

$$0 \to \mathcal{S}_X^*(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

(see Theorem B.2.2 in Appendix B).
(ii) Fourier transform defines an isomorphism $\mathcal{F} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.

4.2. **Basic tools.**

We present here basic tools on equivariant distributions that we will use in this paper. All the proofs are given in the appendices.

Theorem 4.2.1. *Let a real reductive group G act on a smooth affine real algebraic variety X. Let $X = \bigcup_{i=0}^{l} X_i$ be a smooth G-invariant stratification of X. Let χ be an algebraic character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and any $0 \leq i \leq l$ we have $\mathcal{D}(X_i, Sym^k(CN_{X_i}^X))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

For proof see appendix B.2.

Proposition 4.2.2. *Let $H_i \subset G_i$ be Lie groups acting on smooth manifolds X_i for $i = 1 \ldots n$. Let $E_i \to X_i$ be (finite dimensional) G_i-equivariant vector bundles. Suppose that $\mathcal{D}(X_i, E_i)^{H_i} = \mathcal{D}(X_i, E_i)^{G_i}$ for all i. Then $\mathcal{D}(\prod X_i, \boxtimes E_i)^{\prod H_i} = \mathcal{D}(\prod X_i, \boxtimes E_i)^{\prod G_i}$, where $\boxtimes$ denotes the external product of vector bundles.*

The proof of this proposition is the same as of its non-archimedean analog (proposition 3.1.5).

Theorem 4.2.3 (Frobenius reciprocity)**.** *Let a unimodular Lie group G act transitively on a smooth manifold Z. Let $\varphi : X \to Z$ be a G-equivariant smooth map. Let $z_0 \in Z$. Suppose that its stabilizer $\mathrm{Stab}_G(z_0)$ is unimodular. Let X_{z_0} be the fiber of z_0. Let χ be a character of G. Then $\mathcal{D}(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_{z_0})^{\mathrm{Stab}_G(z_0),\chi}$. Moreover, for any G-equivariant bundle E on X and a closed $\mathrm{Stab}_G(z_0)$-invariant subset $Y \subset X_{z_0}$, the space $\mathcal{D}_X(GY, E)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}_{X_{z_0}}(Y, E|_{X_{z_0}})^{\mathrm{Stab}_G(z_0),\chi}$.*

In section A we formulate and prove a more general version of this theorem.

The next theorem shows that in certain cases it is enough to show that there are no equivariant Schwartz distributions. This will allow us to use Fourier transform.

We will need the following theorem from [AG3], Theorem 4.0.2.

Theorem 4.2.4. *Let a real reductive group G act on a smooth affine real algebraic variety X. Let V be a finite-dimensional algebraic representation of G. Suppose that*

$$\mathcal{S}^*(X, V)^G = 0.$$

Then

$$\mathcal{D}(X,V)^G = 0.$$

For proof see [AG3], Theorem 4.0.2.

4.3. **Specific tools.**

We present here tools on equivariant distributions which are more specific to our problem. All the proofs are given in Appendix B.

Proposition 4.3.1. *Let a Lie group G act on a smooth manifold X. Let V be a real finite dimensional representation of G. Suppose that G preserves the Haar measure on V. Let $U \subset V$ be an open non-empty G-invariant subset. Let χ be a character of G. Suppose that $\mathcal{D}(X \times U)^{G,\chi} = 0$. Then $\mathcal{D}(X, Sym^k(V))^{G,\chi} = 0$.*

For proof see section B.4.

Proposition 4.3.2. *Let G be a Nash group. Let V be a finite dimensional representation of G over F. Suppose that the action of G preserves some non-degenerate bilinear form B on V. Let $V = \bigcup_{i=1}^{n} S_i$ be a stratification of V by G-invariant Nash cones.*

Let $\mathfrak{X}$ be a set of characters of $F^\times$ such that the set $\mathfrak{X} \cdot \mathfrak{X}$ does not contain the character $|\cdot|^{\dim_{\mathbb{R}} V}$. Let χ be a character of G. Suppose that for any i and k, the space $\mathcal{S}^*(S_i, Sym^k(CN_{S_i}^V))^{G,\chi}$ consists of homogeneous distributions of type α for some $\alpha \in \mathfrak{X}$. Then $\mathcal{S}^*(V)^{G,\chi} = 0$.

For proof see section B.3.

In order to prove homogeneity of invariant distributions we will use the following corollary of Frobenius reciprocity.

Proposition 4.3.3 (Homogeneity criterion)**.** *Let G be a Lie group. Let V be a finite dimensional representation of G over F. Let $C \subset V$ be a G-invariant G-transitive smooth cone. Consider the actions of $G \times F^\times$ on V, C and CN_C^V, where $F^\times$ acts by homotheties. Let χ be a character of G. Let α be a character of $F^\times$. Consider the character $\chi' := \chi \times \alpha^{-1}$ of $G \times F^\times$. Let $x_0 \in C$ and denote $H := Stab_G(x_0)$ and $H' := Stab_{G \times F^\times}(x_0)$. Suppose that G, H, H' are unimodular. Fix $k \in Z_{\geq 0}$.*

Then the space $\mathcal{D}(C, Sym^k(CN_C^V))^{G,\chi}$ consists of homogeneous distributions of type α if and only if

$$(Sym^k(N_{C,x_0}^V) \otimes_{\mathbb{R}} \mathbb{C})^{H,\chi} = (Sym^k(N_{C,x_0}^V) \otimes_{\mathbb{R}} \mathbb{C})^{H',\chi'}.$$

5. Proof of Theorem A for archimedean F

We will use the same notations as in the non-archimedean case (see notation 3.2.1). Again, the following theorem implies Theorem A.

Theorem 5.0.1. $\mathcal{D}(X)^{\widetilde{G},\chi} = 0$.

The implication is proven exactly in the same way as in the non-archimedean case (subsection 3.2.1).

5.1. **Proof of theorem 5.0.1.**

We will now stratify $X(= gl_n \times V \times V^*)$ and deal with each strata separately.

Notation 5.1.1. Denote $W := W_n := V_n \oplus V_n^*$. Denote by $Q^i := Q^i_n \subset \mathrm{gl}_n$ the set of all matrices of rank i. Denote $Z^i := Z^i_n := Q^i_n \times W_n$.

Note that $X = \bigcup Z^i$. Hence by theorem 4.2.1, it is enough to prove the following proposition.

Proposition 5.1.2. *$\mathcal{D}(Z^i, Sym^k(CN^X_{Z^i}))^{\widetilde{G},\chi} = 0$ for any k and i.*

We will use the following key lemma.

Lemma 5.1.3 (Key Lemma)**.** $\mathcal{D}(W)^{\widetilde{H},\chi} = 0$.

For proof see subsection 5.2 below.

Corollary 5.1.4. *Proposition 5.1.2 holds for $i = n$.*

The proof is the same as in the non-archimedean case (corollary 3.2.9).

Corollary 5.1.5. $\mathcal{D}(W_n, Sym^k(\mathrm{gl}^*_n))^{\widetilde{G},\chi} = 0$.

Proof. Consider the Killing form $K : \mathrm{gl}^*_n \to \mathrm{gl}_n$. Let $U := K^{-1}(Q^n_n)$. In the same way as in the previous corollary one can show that $\mathcal{D}(W_n \times U)^{\widetilde{G},\chi} = 0$. Hence by proposition 4.3.1, $\mathcal{D}(W_n, Sym^k(\mathrm{gl}^*_n))^{\widetilde{G},\chi} = 0$. □

Corollary 5.1.6. *We have*

$$\mathcal{D}(W_i \times W_{n-i}, Sym^k(0\times \mathrm{gl}^*_{n-i}))^{H_i\times G_{n-i}} = \mathcal{D}(W_i \times W_{n-i}, Sym^k(0\times \mathrm{gl}^*_{n-i}))^{\widetilde{H}_i\times \widetilde{G}_{n-i}}.$$

Proof. It follows from the key lemma, the last corollary and proposition 4.2.2. □

Now we are ready to prove proposition 5.1.2.

Proof of proposition 5.1.2. Fix $i < n$. Consider the projection $pr_1 : Z^i \to Q^i$. It is easy to see that the action of $\widetilde{G}$ on Q^i is transitive. Denote

$$A_i := \begin{pmatrix} Id_{i\times i} & 0 \\ 0 & 0 \end{pmatrix} \in Q^i.$$

Denote by $G_{A_i} := \mathrm{Stab}_G(A_i)$ and $\widetilde{G}_{A_i} := \mathrm{Stab}_{\widetilde{G}}(A_i)$. Note that they are unimodular. By Frobenius reciprocity (theorem 4.2.3),

$$\mathcal{D}(Z^i, Sym^k(CN^X_{Z^i}))^{\widetilde{G},\chi} = \mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{\widetilde{G}_{A_i},\chi}.$$

Hence it is enough to show that

$$\mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{G_{A_i}} = \mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{\widetilde{G}_{A_i}}.$$

It is easy to check by explicit computation that

- $H_i \times G_{n-i}$ is canonically embedded into G_{A_i},
- W is isomorphic to $W_i \times W_{n-i}$ as $H_i \times G_{n-i}$-spaces
- $CN^{\mathrm{gl}_n}_{Q^i,A_i}$ is isomorphic to $0 \times \mathrm{gl}^*_{n-i}$ as $H_i \times G_{n-i}$ representations.

Let $\xi \in \mathcal{D}(W, Sym^k(CN^{\mathrm{gl}_n}_{Q^i,A_i}))^{G_{A_i}}$. By the previous corollary, ξ is $\widetilde{H}_i \times \widetilde{G}_{n-i}$-invariant. Since ξ is also G_{A_i}-invariant, it is $\widetilde{G}_{A_i}$-invariant. □

5.2. **Proof of the key lemma (lemma 5.1.3).**

As in the non-archimedean case, it is enough to prove the key lemma for $n = 1$ (see proposition 3.3.1).

From now on we fix $n := 1$, $H := H_1$, $\widetilde{H} := \widetilde{H}_1$ and $W := W_1$. Note that $H = F^\times$ and $W = F^2$. The action of H is given by $\rho(\lambda)(x, y) := (\lambda x, \lambda^{-1} y)$ and extended to the action of $\widetilde{H}$ by the involution $\sigma(x, y) = (y, x)$.

Let $Y := \{(x, y) \in F^2 | \, xy = 0\} \subset W$ be the **cross** and $Y' := Y \setminus \{0\}$.

Lemma 5.2.1. *Every $(\widetilde{H}, \chi)$-equivariant distribution on W is supported inside the cross Y.*

The proof of this lemma is identical to the proof of proposition 3.3.2, (iii).

To apply proposition 4.3.2 (which uses Fourier transform) we need to restrict our consideration to Schwartz distributions. By theorem 4.2.4, in order to show that $\mathcal{D}_W(Y)^{\widetilde{H},\chi} = 0$ it is enough to show that $\mathcal{S}^*(W)^{\widetilde{H},\chi} = 0$ [3]. By proposition 4.3.2, it is enough to prove the following proposition.

Proposition 5.2.2.
(i) $\mathcal{S}^*(W \setminus Y)^{\widetilde{H},\chi} = 0$.
(ii) For all $k \in \mathbb{Z}_{\geq 0}$, *any distribution* $\xi \in \mathcal{S}^*(Y', Sym^k(CN_{Y'}^W))^{\widetilde{H},\chi}$ *is* $\mathbb{R}$*-homogeneous of type* α_k *where* $\alpha_k(\lambda) := \lambda^{-2k}$.
(iii) $\mathcal{S}^*(\{0\}, Sym^k(CN_{\{0\}}^W))^{\widetilde{H},\chi} = 0$.

Proof. We have proven (i) in the proof of the previous lemma.
(ii) Fix $x_0 := (1, 0) \in Y'$. Now we want to use the homogeneity criterion (proposition 4.3.3). Note that $Stab_{\widetilde{H}}(x_0)$ is trivial and $Stab_{\widetilde{H}\times\mathbb{R}^\times}(x_0) \cong \mathbb{R}^\times$. Note that $N_{Y',x_0}^W \cong F$ and $Stab_{\widetilde{H}\times\mathbb{R}^\times}(x_0)$ acts on it by $\rho(\lambda)a = \lambda^2 a$. So we have

$$Sym^k(N_{Y',x_0}^W) = Sym^k(N_{Y',x_0}^W)^{\mathbb{R}^\times,\alpha_k^{-1}}.$$

So by the homogeneity criterion any distribution $\xi \in \mathcal{S}^*(Y', Sym^k(CN_{Y'}^W))^{\widetilde{H},\chi}$ is $\mathbb{R}$-homogeneous of type α_k.
(iii) is a simple computation. Also, it can be deduced from (i) using proposition 4.3.1. □

Appendix A. Frobenius reciprocity

In this section we obtain a slight generalization of Frobenius reciprocity proven in [Bar] (section 3). The proof will go along the same lines and is included for the benefit of the reader. To simplify the formulation and proof of Frobenius reciprocity we pass from distributions to generalized functions. Note that the space of smooth functions embeds canonically into the space of generalized functions but there is no canonical embedding of smooth functions to the space of distributions.

Notation A.0.1. Let X be a smooth manifold. We denote by D_X the bundle of densities on X. For a point $x \in X$ we denote by $D_{X,x}$ its fiber in the point x. If X is a Nash manifold then the bundle D_X has a natural structure of a Nash bundle. For its description see [AG1], section 6.1.1.

[3] Alternatively, one can show that any H-invariant distribution on W supported at Y is a Schwartz distribution since Y has finite number of orbits.

Notation A.0.2. Let X be a smooth manifold. We denote by $C^{-\infty}(X)$ the space $C^{-\infty}(X) := \mathcal{D}(X, D_X)$ of **generalized functions** on X. Let E be a vector bundle on X. We also denote by $C^{-\infty}(X, E)$ the space $C^{-\infty}(X, E) := \mathcal{D}(X, D_X \otimes E^*)$ of generalized sections of E. For a locally closed subset $Y \subset X$ we denote $C_X^{-\infty}(Y) := \mathcal{D}_X(Y, D_X)$ and $C_X^{-\infty}(Y, E) := \mathcal{D}_X(Y, D_X \otimes E^*)$.

We will prove the following version of Frobenius reciprocity.

Theorem A.0.3 (Frobenius reciprocity)**.** *Let a Lie group G act transitively on a smooth manifold Z. Let $\varphi : X \to Z$ be a G-equivariant smooth map. Let $z_0 \in Z$. Denote by G_{z_0} the stabilizer of z_0 in G and by X_{z_0} the fiber of z_0. Let E be a G-equivariant vector bundle on X. Then there exists a canonical isomorphism* $\mathrm{Fr} : C^{-\infty}(X_{z_0}, E|_{X_{z_0}})^{G_{z_0}} \to C^{-\infty}(X, E)^G$. *Moreover, for any closed G_z-invariant subset $Y \subset X_{z_0}$, Fr maps $C_{X_{z_0}}^{-\infty}(Y, E|_{X_{z_0}})^{G_{z_0}}$ to $C_X^{-\infty}(GY, E)^G$.*

First we will need the following version of Harish-Chandra's submersion principle.

Theorem A.0.4 (Harish-Chandra's submersion principle)**.** *Let X, Y be smooth manifolds. Let $E \to X$ be a vector bundle. Let $\varphi : Y \to X$ be a submersion. Then the map $\varphi^* : C^{\infty}(X, E) \to C^{\infty}(Y, \varphi^*(E))$ extends to a continuous map $\varphi^* : C^{-\infty}(X, E) \to C^{-\infty}(Y, \varphi^*(E))$.*

Proof. By partition of unity it is enough to show for the case of trivial E. In this case it can be easily deduced from [Wal1], 8.A.2.5. □

Also we will need the following fact that can be easily deduced from [Wal1], 8.A.2.9.

Proposition A.0.5. *Let $E \to Z$ be a vector bundle and G be a Lie group. Then there is a canonical isomorphism $C^{-\infty}(Z, E) \to C^{-\infty}(Z \times G, \mathrm{pr}^*(E))^G$, where $pr : Z \times G \to Z$ is the standard projection and the action of G on $Z\times$ G is the left action on the G coordinate.*

The last two statements give us the following corollary.

Corollary A.0.6. *Let a Lie group G act on a smooth manifold X. Let E be a G-equivariant bundle over X. Let $Z \subset X$ be a submanifold such that the action map $G\times Z \to X$ is submersive. Then there exists a canonical map $HC : C^{-\infty}(X, E)^G \to C^{-\infty}(Z, E|_Z)$.*

Now we can prove Frobenius reciprocity (Theorem A.0.3).

Proof of Frobenius reciprocity. We construct the map $\mathrm{Fr} : C^{-\infty}(X_{z_0}, E|_{X_{z_0}})^{G_{z_0}} \to C^{-\infty}(X, E)^G$ in the same way like in [Ber1] (1.5). Namely, fix a set-theoretic section $\nu : Z \to G$. It gives us in any point $z \in Z$ an identification between X_z and X_{z_0}. Hence we can interpret a generalized function $\xi \in C^{-\infty}(X_{z_0}, E|_{X_{z_0}})$ as a functional $\xi_z : C_c^{\infty}(X_z, E^*|_{X_z} \otimes D_{X_z}) \to \mathbb{C}$, or as a map $\xi_z : C_c^{\infty}(X_z, (E^* \otimes D_X)|_{X_z}) \to D_{Z,z}$. Now define

$$\mathrm{Fr}(\xi)(f) := \int_{z\in Z} \xi_z(f|_{X_z}).$$

It is easy to see that Fr is well-defined.

It is easy to see that the map $HC : C^{-\infty}(X, E)^G \to C^{-\infty}(X_{z_0}, E|_{X_{z_0}})$ described in the last corollary gives the inverse map.

The fact that for any closed G_z-invariant subset $Y \subset X_{z_0}$, Fr maps $C^{-\infty}_{X_{z_0}}(Y, E|_{X_{z_0}})^{G_{z_0}}$ to $C^{-\infty}_X(GY, E)^G$ follows from the fact that Fr commutes with restrictions to open sets. $\square$

Corollary A.0.7. *Theorem 4.2.3 holds.*

Proof. Without loss of generality we can assume that χ is trivial, since we can twist E by χ^{-1}. We have

$$\mathcal{D}(X, E)^G \cong C^{-\infty}(X, E^* \otimes D_X)^G \cong C^{-\infty}(X_{z_0}, (E^* \otimes D_X)|_{X_{z_0}})^{G_{z_0}} \cong (\mathcal{D}(X_{z_0}, E^*|_{X_{z_0}}) \otimes D_{Z,z_0})^{G_{z_0}}.$$

It is easy to see that in case that G and G_{z_0} are unimodular, the action of G_{z_0} on D_{Z,z_0} is trivial. $\square$

Remark A.0.8. For a Nash manifold X one can introduce the space of **generalized Schwartz functions** by $\mathcal{G}(X) := \mathcal{S}^*(X, D_X)$. Given a Nash bundle E one may consider the generalized Schwartz sections $\mathcal{G}(X, E) := \mathcal{S}^*(X, D_X \otimes E^*)$. Frobenius reciprocity in the Nash setting is obtained by restricting Fr and yields

$$Fr : \mathcal{G}(X, E)^G \cong \mathcal{G}(X_z, E|_{X_z})^{G_z}.$$

The proof goes along the same lines, but one has to prove that the corresponding integrals converge. We will not give the proof here since we will not use this fact.

Appendix B. Filtrations on spaces of distributions

B.1. Filtrations on linear spaces.

In what follows, a filtration on a vector space is always increasing and exhaustive. We make the following definition:

Definition B.1.1. Let V be a vector space. Let I be a well ordered set. Let F^i be a filtration on V indexed by $i \in I$. We denote $\mathrm{Gr}^i(V) := F^i/(\bigcup_{j<i} F^j)$.

The following lemma is obvious.

Lemma B.1.2. *Let V be a representation of an abstract group G. Let I be a well ordered set. Let F^i be a filtration of V by G invariant subspaces indexed by $i \in I$. Suppose that for any $i \in I$ we have $\mathrm{Gr}^i(V)^G = 0$. Then $V^G = 0$. An analogous statement also holds if we replace the group G by a Lie algebra $\mathfrak{g}$.*

B.2. Filtrations on spaces of distributions.

Theorem B.2.1. *Let X be a Nash manifold. Let E be a Nash bundle on X. Let $Z \subset X$ be a Nash submanifold. Then the space $\mathcal{S}^*_X(Z, E)$ has a natural filtration $F^k := F^k(\mathcal{S}^*_X(Z, E))$ such that $F^k/F^{k-1} \cong \mathcal{S}^*(Z, E|_Z \otimes Sym^k(CN_Z^X))$.*

For proof see [AG1], corollary 5.5.4.

We will also need the following important theorem

Theorem B.2.2. *Let X be a Nash manifold, $U \subset X$ be an open Nash submanifold and E be a Nash bundle over X. Then we have the following exact sequence*

$$0 \to \mathcal{S}^*_X(X \setminus U, E) \to \mathcal{S}^*(X, E) \to \mathcal{S}^*(U, E|_U) \to 0.$$

Proof. The only non-trivial part is to show that the restriction map $\mathcal{S}^*(X, E) \to \mathcal{S}^*(U, E|_U) \to 0$ is onto. It is done in [AG1], corollary 5.4.4. $\square$

Now we obtain the following corollary of theorem B.2.1 using the exact sequence from theorem B.2.2.

Corollary B.2.3. *Let X be a Nash manifold. Let E be Nash bundle over X. Let $Y \subset X$ be locally closed subset. Let $Y = \bigcup_{i=0}^{l} Y_i$ be a Nash stratification of Y.*

Then the space $\mathcal{S}^*_X(Y, E)$ has a natural filtration $F^{ik}(\mathcal{S}^*_X(Y, E))$ such that

$$\mathrm{Gr}^{ik}(\mathcal{S}^*_X(Y, E)) \cong \mathcal{S}^*(Y_i, E|_{Y_i} \otimes Sym^k(CN^X_{Y_i}))$$

for all $i \in \{1...l\}$ and $k \in \mathbb{Z}_{\geq 0}$.

Corollary B.2.4. *Let X be a Nash manifold. Let E be Nash bundle over X. Let $Y \subset X$ be locally closed subset. Let $Y = \bigcup_{i=0}^{l} Y_i$ be a Nash stratification of Y.*

Suppose that for any $0 \leq i \leq l$ and any $k \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{S}^*(Y_i, E|_{Y_i} \otimes Sym^k(CN^X_{Y_i}))^G = 0.$$

Then $\mathcal{S}^*_X(Y, E)^G = 0$.

By theorem 4.2.4, this corollary implies theorem 4.2.1.

B.3. **Fourier transform and proof of proposition 4.3.2.**

Notation B.3.1 (Fourier transform). Let V be a finite dimensional vector space over F. Let B be a non-degenerate bilinear form on V. We denote by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ the Fourier transform defined using B and the self-dual measure on V.

We will use the following well known fact.

Proposition B.3.2. *Let V be a finite dimensional vector space over F. Let B be a non-degenerate bilinear form on V. Consider the homothety action ρ of $F^\times$ on $\mathcal{S}^*(V)$. Then for any $\lambda \in F^\times$ we have*

$$\rho(\lambda) \circ \mathcal{F}_B = |\lambda|^{-\dim_{\mathbb{R}} V} \mathcal{F}_B \circ \rho(\lambda^{-1}).$$

Notation B.3.3. Let $(\rho, \mathcal{E})$ be a complex representation of $F^\times$. We denote by $JH(\rho, \mathcal{E})$ the subset of characters of $F^\times$ which are subquotients of $(\rho, \mathcal{E})$.

We will use the following straightforward lemma.

Lemma B.3.4. *Let $(\rho, \mathcal{E})$ be a complex representation of $F^\times$. Let χ be a character of $F^\times$. Suppose that there exists an invertible linear operator $A : \mathcal{E} \to \mathcal{E}$ such that for any $\lambda \in F^\times$, $\rho(\lambda) \circ A = \chi(\lambda) A \circ \rho(\lambda^{-1})$. Then $JH(\mathcal{E}) = \frac{\chi}{JH(\mathcal{E})}$.*

We will also use the following standard lemma.

Lemma B.3.5. *Let $(\rho, \mathcal{E})$ be a complex representation of $F^\times$ of countable dimension.*
(i) If $JH(\mathcal{E}) = \emptyset$ then $\mathcal{E} = 0$.
(ii) Let I be a well ordered set and F^i be a filtration on $\mathcal{E}$ indexed by $i \in I$ by subrepresentations. Then $JH(\mathcal{E}) = \bigcup_{i \in I} JH(\mathrm{Gr}^i(\mathcal{E}))$.

Now we will prove proposition 4.3.2. First we remind its formulation.

Proposition B.3.6. *Let G be a Nash group. Let V be a finite dimensional representation of G over F. Suppose that the action of G preserves some non-degenerate bilinear form B on V. Let $V = \bigcup_{i=1}^{n} S_i$ be a stratification of V by G-invariant Nash cones.*

Let $\mathfrak{X}$ be a set of characters of $F^\times$ such that the set $\mathfrak{X} \cdot \mathfrak{X}$ does not contain the character $|\cdot|^{\dim_{\mathbb{R}} V}$. Let χ be a character of G. Suppose that for any i and k, the space $\mathcal{S}^*(S_i, Sym^k(CN_{S_i}^V))^{G,\chi}$ consists of homogeneous distributions of type α for some $\alpha \in \mathfrak{X}$. Then $\mathcal{S}^*(V)^{G,\chi} = 0$.

Proof. Consider $\mathcal{S}^*(V)^{G,\chi}$ as a representation of $F^\times$. It has a canonical filtration given by corollary B.2.3. It is easy to see that $\mathrm{Gr}^{ik}(\mathcal{S}^*(V)^{G,\chi})$ is canonically imbedded into $(\mathrm{Gr}^{ik}(\mathcal{S}^*(V))^{G,\chi}$. Therefore by the previous lemma $JH(\mathcal{S}^*(V)^{G,\chi}) \subset \mathfrak{X}^{-1}$. On the other hand G preserves B and hence we have $\mathcal{F}_B : \mathcal{S}^*(V)^{G,\chi} \to \mathcal{S}^*(V)^{G,\chi}$. Therefore by lemma B.3.4 we have

$$JH(\mathcal{S}^*(V)^{G,\chi}) \subset |\cdot|^{-\dim_{\mathbb{R}} V} \mathfrak{X}.$$

Hence $JH(\mathcal{S}^*(V)^{G,\chi}) = \emptyset$. Thus $\mathcal{S}^*(V)^{G,\chi} = 0$. ☐

B.4. **Proof of proposition 4.3.1.**

The following proposition clearly implies proposition 4.3.1.

Proposition B.4.1. *Let X be a smooth manifold. Let V be a real finite dimensional vector space. Let $U \subset V$ be an open non-empty subset. Let E be a vector bundle over X. Then for any $k \geq 0$ there exists a canonical embedding $\mathcal{D}(X, E \otimes Sym^k(V)) \hookrightarrow \mathcal{D}(X \times U, E \boxtimes D_V)$.*

Proof. It is enough to construct a continuous linear epimorphism

$$\pi : C_c^\infty(X \times U, E \boxtimes D_V) \twoheadrightarrow C_c^\infty(X, E \otimes Sym^k(V)).$$

By partition of unity it is enough to do it for trivial E. Let $w \in C_c^\infty(X \times U, D_V)$ and $x \in X$ we have to define $\pi(w)(x) \in Sym^k(V)$. Consider the space $Sym^k(V)$ as the space of linear functionals on the space of homogeneous polynomials on V of degree k. Define

$$\pi(w)(x)(p) := \int_{y \in V} p(y) w(x, y).$$

It is easy to check that $\pi(w) \in C_c^\infty(X, Sym^k(V))$ and π is continuous linear epimorphism. ☐

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GENERALIZED HARISH-CHANDRA DESCENT, GELFAND PAIRS AND AN ARCHIMEDEAN ANALOG OF JACQUET-RALLIS' THEOREM

AVRAHAM AIZENBUD AND DMITRY GOUREVITCH

with Appendix D by Avraham Aizenbud, Dmitry Gourevitch and Eitan Sayag

Abstract. In the first part of the paper we generalize a descent technique due to Harish-Chandra to the case of a reductive group acting on a smooth affine variety both defined over an arbitrary local field F of characteristic zero. Our main tool is the Luna Slice Theorem.

In the second part of the paper we apply this technique to symmetric pairs. In particular we prove that the pairs $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ and $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ are Gelfand pairs for any local field F and its quadratic extension E. In the non-Archimedean case, the first result was proven earlier by Jacquet and Rallis and the second by Flicker.

We also prove that any conjugation invariant distribution on $\mathrm{GL}_n(F)$ is invariant with respect to transposition. For non-Archimedean F the latter is a classical theorem of Gelfand and Kazhdan.

Contents

Key words and phrases. Multiplicity one, Gelfand pairs, symmetric pairs, Luna Slice Theorem, invariant distributions, Harish-Chandra descent, uniqueness of linear periods.

MSC Classes: 20C99, 20G05, 22E45, 22E50, 46F10, 14L24, 14L30.

1. Introduction

Harish-Chandra developed a technique based on Jordan decomposition that allows to reduce certain statements on conjugation invariant distributions on a reductive group to the set of unipotent elements, provided that the statement is known for certain subgroups (see e.g. [HC99]).

In this paper we generalize an aspect of this technique to the setting of a reductive group acting on a smooth affine algebraic variety, using the Luna Slice Theorem. Our technique is oriented towards proving Gelfand property for pairs of reductive groups.

Our approach is uniform for all local fields of characteristic zero – both Archimedean and non-Archimedean.

1.1. Main results.

The core of this paper is Theorem 3.1.1:

Theorem. *Let a reductive group G act on a smooth affine variety X, both defined over a local field F of characteristic zero. Let χ be a character of $G(F)$.*

Suppose that for any $x \in X(F)$ with closed orbit there are no non-zero distributions on the normal space at x to the orbit $G(F)x$ which are $(G(F)_x, \chi)$-equivariant, where G_x denotes the stabilizer of x.

Then there are no non-zero $(G(F), \chi)$-equivariant distributions on $X(F)$.

In fact, a stronger version based on this theorem is given in Corollary 3.2.2. This stronger version is based on an inductive argument. It shows that it is enough to prove that there are no non-zero equivariant distributions on the normal space to the orbit $G(F)x$ at x under the assumption that all such distributions are supported in a certain closed subset which is the analog of the nilpotent cone.

We apply this stronger version to problems of the following type. Let a reductive group G act on a smooth affine variety X, and τ be an involution of X which normalizes the image of G in $\mathrm{Aut}(X)$. We want to check whether any $G(F)$-invariant distribution on $X(F)$ is also τ-invariant. Evidently, there is the following necessary condition on τ:
(*) Any closed orbit in $X(F)$ is τ-invariant.
In some cases this condition is also sufficient. In these cases we call the action of G on X *tame*.

This is a weakening of the property called *"density"* in [RR96]. However, it is sufficient for the purpose of proving Gelfand property for pairs of reductive groups.

In §6 we give criteria for tameness of actions. In particular, we introduce the notion of *"special"* action in order to show that certain actions are tame (see Theorem 6.0.5 and Proposition 7.3.5). Also, in many cases one can verify that an action is special using purely algebraic-geometric means.

In the second part of the paper we restrict our attention to the case of symmetric pairs. We transfer the terminology on actions to terminology on symmetric pairs. For example, we call a symmetric pair (G, H) *tame* if the action of $H \times H$ on G is tame.

In addition we introduce the notion of a *"regular"* symmetric pair (see Definition 7.4.2), which also helps to prove Gelfand property. Namely, we prove Theorem 7.4.5.

Theorem. *Let G be a reductive group defined over a local field F and let θ be an involution of G. Let $H := G^\theta$ and let σ be the anti-involution defined by $\sigma(g) := \theta(g^{-1})$. Consider the symmetric pair (G, H).*

Suppose that all its "descendants" *(including itself, see Definition 7.2.2) are regular. Suppose also that any closed $H(F)$-double coset in $G(F)$ is σ-invariant.*

Then every bi-$H(F)$-invariant distribution on $G(F)$ is σ-invariant. In particular, by Gelfand-Kazhdan criterion, the pair (G, H) is a Gelfand pair (see §8).

Also, we formulate an algebraic-geometric criterion for regularity of a pair (Proposition 7.3.7). We sum up the various properties of symmetric pairs and their interrelations in a diagram in Appendix E.

As an application and illustration of our methods we prove in §7.7 that the pair $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ is a Gelfand pair by proving that it is regular, along with its descendants. In the non-Archimedean case this was proven in [JR96] and our proof is along the same lines. Our technique enabled us to streamline some of the computations in the proof of [JR96] and to extend it to the Archimedean case.

We also prove (in §7.6) that the pair $(G(E), G(F))$ is tame for any reductive group G over F and a quadratic field extension E/F. This implies that the pair $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ is a Gelfand pair. In the non-Archimedean case this was proven in [Fli91]. Also we prove that the adjoint action of a reductive group on itself is tame. This is a generalization of a classical theorem by Gelfand and Kazhdan, see [GK75].

In general, we conjecture that any symmetric pair is regular. This would imply the van Dijk conjecture:

Conjecture (van Dijk)**.** *Any symmetric pair (G, H) over $\mathbb{C}$ such that G/H is connected is a Gelfand pair.*

1.2. Related work.

This paper was inspired by the paper [JR96] by Jacquet and Rallis where they prove that the pair $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ is a Gelfand pair for any non-Archimedean local field F of characteristic zero. Our aim was to see to what extent their techniques generalize.

Another generalization of Harish-Chandra descent using the Luna Slice Theorem has been carried out in the non-Archimedean case in [RR96]. In that paper Rader and Rallis investigated spherical characters of H-distinguished representations of G for symmetric pairs (G, H) and checked the validity of what they call the *"density principle"* for rank one symmetric pairs. They found out that the principle usually holds, but also found counterexamples.

In [vD86], van-Dijk investigated rank one symmetric pairs in the Archimedean case and classified the Gelfand pairs among them. In [BvD94], van-Dijk and Bosman studied the non-Archimedean case and obtained results for most rank one symmetric pairs. We hope that the second part of our paper will enhance the understanding of this question for symmetric pairs of higher rank.

1.3. Structure of the paper.

In §2 we introduce notation and terminology which allows us to speak uniformly about spaces of points of smooth algebraic varieties over Archimedean and non-Archimedean local fields, and equivariant distributions on those spaces.

In §§2.3 we formulate a version of the Luna Slice Theorem for points over local fields (Theorem 2.3.17). In §§2.5 we formulate results on equivariant distributions and equivariant Schwartz distributions. Most of those results are borrowed from [BZ76], [Ber84], [Bar03] and [AGS08], and the rest are proven in Appendix B.

In §3 we formulate and prove the Generalized Harish-Chandra Descent Theorem and its stronger version.

§4 is of interest only in the Archimedean case. In that section we prove that in the cases at hand if there are no equivariant Schwartz distributions then there are no equivariant distributions at all. Schwartz distributions are discussed in Appendix B.

In §5 we formulate a homogeneity Theorem which helps us to check the conditions of the Generalized Harish-Chandra Descent Theorem. In the non-Archimedean case this theorem had been proved earlier (see e.g. [JR96], [RS07] or [AGRS07]). We provide the proof for the Archimedean case in Appendix C.

In §6 we introduce the notion of tame actions and provide tameness criteria.

In §7 we apply our tools to symmetric pairs. In §§7.3 we provide criteria for tameness of a symmetric pair. In §§7.4 we introduce the notion of a regular symmetric pair and prove Theorem 7.4.5 alluded to above. In §§7.5 we discuss conjectures about the regularity and the Gelfand property of symmetric pairs. In §§7.6 we prove that certain symmetric pairs are tame. In §§7.7 we prove that the pair $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ is regular.

In §8 we recall basic facts on Gelfand pairs and their connections to invariant distributions. We also prove that the pairs $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ and $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ are Gelfand pairs for any local field F and its quadratic extension E.

We start Appendix A by discussing different versions of the Inverse Function Theorem for local fields. Then we prove a version of the Luna Slice Theorem for points over local fields (Theorem 2.3.17). For Archimedean F this was done by Luna himself in [Lun75].

Appendices B and C are of interest only in the Archimedean case.

In Appendix B we discuss Schwartz distributions on Nash manifolds. We prove Frobenius reciprocity for them and construct the pullback of a Schwartz distribution under a Nash submersion. Also we prove that G-invariant distributions which are (Nashly) compactly supported modulo G are Schwartz distributions.

In Appendix C we prove the Archimedean version of the Homogeneity Theorem discussed in §5.

In Appendix D we formulate and prove a version of Bernstein's Localization Principle (Theorem 4.0.1). This appendix is of interest only for Archimedean F since for l-spaces a more general version of this principle had been proven in [Ber84]. This appendix is used in §4.

In [AGS09] we formulated Localization Principle in the setting of differential geometry. Admittedly, we currently do not have a proof of this principle in such a general setting. However, in Appendix D we present a proof in the case of a reductive group G acting on a smooth affine variety X. This generality is sufficiently wide for all applications we encountered up to now, including the one considered in [AGS09].

Finally, in Appendix E we present a diagram that illustrates the interrelations of various properties of symmetric pairs.

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Part 1. **Generalized Harish-Chadra descent**

2. Preliminaries and notation

2.1. **Conventions.**

- Henceforth we fix a local field F of characteristic zero. All the algebraic varieties and algebraic groups that we will consider will be defined over F.
- For a group G acting on a set X we denote by X^G the set of fixed points of X. Also, for an element $x \in X$ we denote by G_x the stabilizer of x.

- By a reductive group we mean a (non-necessarily connected) algebraic reductive group.
- We consider an algebraic variety X defined over F as an algebraic variety over $\overline{F}$ together with action of the Galois group $Gal(\overline{F}/F)$. On X we only consider the Zariski topology. On $X(F)$ we only consider the analytic (Hausdorff) topology. We treat finite-dimensional linear spaces defined over F as algebraic varieties.
- The tangent space of a manifold (algebraic, analytic, etc.) X at x will be denoted by T_xX.
- Usually we will use the letters X, Y, Z, Δ to denote algebraic varieties and the letters G, H to denote reductive groups. We will usually use the letters $V, W, U, K, M, N, C, O, S, T$ to denote analytic spaces (such as F-points of algebraic varieties) and the letter K to denote analytic groups. Also we will use the letters L, V, W to denote vector spaces of all kinds.

2.2. **Categorical quotient.**

Definition 2.2.1. *Let an algebraic group G act on an algebraic variety X. A pair consisting of an algebraic variety Y and a G-invariant morphism $\pi : X \to Y$ is called **the quotient of** X **by the action of** G if for any pair (π', Y'), there exists a unique morphism $\phi : Y \to Y'$ such that $\pi' = \phi \circ \pi$. Clearly, if such pair exists it is unique up to a canonical isomorphism. We will denote it by $(\pi_X, X/G)$.*

Theorem 2.2.2 (cf. [Dre00])**.** *Let a reductive group G act on an affine variety X. Then the quotient X/G exists, and every fiber of the quotient map π_X contains a unique closed orbit. In fact, $X/G := \operatorname{Spec} \mathcal{O}(X)^G$.*

2.3. **Algebraic geometry over local fields.**

2.3.1. *Analytic manifolds.*
In this paper we consider distributions over l-spaces, smooth manifolds and Nash manifolds. l-spaces are locally compact totally disconnected topological spaces and Nash manifolds are semi-algebraic smooth manifolds.

For basic facts on l-spaces and distributions over them we refer the reader to [BZ76, §1].

For basic facts on Nash manifolds and Schwartz functions and distributions over them see Appendix B and [AG08a]. In this paper we consider only separated Nash manifolds.

We now introduce notation and terminology which allows a uniform treatment of the Archimedean and the non-Archimedean cases.

We will use the notion of an analytic manifold over a local field (see e.g. [Ser64, Part II, Chapter III]). When we say "**analytic manifold**" we always mean analytic manifold over some local field. Note that an analytic manifold over a non-Archimedean field is in particular an l-space and an analytic manifold over an Archimedean field is in particular a smooth manifold.

Definition 2.3.1. *A **B-analytic manifold** is either an analytic manifold over a non-Archimedean local field, or a Nash manifold.*

Remark 2.3.2. *If X is a smooth algebraic variety, then $X(F)$ is a B-analytic manifold and $(T_xX)(F) = T_x(X(F))$.*

Notation 2.3.3. *Let M be an analytic manifold and S be an analytic submanifold. We denote by $N_S^M := (T_M|_Y)/T_S$ the **normal bundle to** S **in** M. The **conormal bundle** is defined by $CN_S^M := (N_S^M)^*$. Denote by $\operatorname{Sym}^k(CN_S^M)$ the k-th symmetric power of the conormal bundle. For a point $y \in S$ we denote by $N_{S,y}^M$ the normal space to S in M at the point y and by $CN_{S,y}^M$ the conormal space.*

2.3.2. *G-orbits on X and $G(F)$-orbits on $X(F)$.*

Lemma 2.3.4 (see Appendix A.1)**.** *Let G be an algebraic group and let $H \subset G$ be a closed subgroup. Then $G(F)/H(F)$ is open and closed in $(G/H)(F)$.*

Corollary 2.3.5. *Let an algebraic group G act on an algebraic variety X. Let $x \in X(F)$. Then*

$$N_{Gx,x}^X(F) \cong N_{G(F)x,x}^{X(F)}.$$

Proposition 2.3.6. *Let an algebraic group G act on an algebraic variety X. Suppose that $S \subset X(F)$ is a non-empty closed $G(F)$-invariant subset. Then S contains a closed orbit.*

Proof. The proof is by Noetherian induction on X. Choose $x \in S$. Consider $Z := \overline{Gx} - Gx$.

If $Z(F) \cap S$ is empty then $Gx(F) \cap S$ is closed and hence $G(F)x \cap S$ is closed by Lemma 2.3.4. Therefore $G(F)x$ is closed.

If $Z(F) \cap S$ is non-empty then $Z(F) \cap S$ contains a closed orbit by the induction assumption. □

Corollary 2.3.7. *Let an algebraic group G act on an algebraic variety X. Let U be an open $G(F)$-invariant subset of $X(F)$. Suppose that U contains all closed $G(F)$-orbits. Then $U = X(F)$.*

Theorem 2.3.8 ([RR96], §2 fact A, pages 108-109)**.** *Let a reductive group G act on an affine variety X. Let $x \in X(F)$. Then the following are equivalent:*
(i) $G(F)x \subset X(F)$ is closed (in the analytic topology).
(ii) $Gx \subset X$ is closed (in the Zariski topology).

Definition 2.3.9. *Let a reductive group G act on an affine variety X. We call an element $x \in X$ G-**semisimple** if its orbit Gx is closed.*

In particular, in the case where G acts on itself by conjugation, the notion of G-semisimplicity coincides with the usual one.

Notation 2.3.10. *Let V be an F-rational finite-dimensional representation of a reductive group G. We set*
$$Q_G(V) := Q(V) := (V/V^G)(F).$$
Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V(F)$. Let $\pi : V(F) \to (V/G)(F)$ be the natural map. We set
$$\Gamma_G(V) := \Gamma(V) := \pi^{-1}(\pi(0)).$$
Note that $\Gamma(V) \subset Q(V)$. We also set
$$R_G(V) := R(V) := Q(V) - \Gamma(V).$$

Notation 2.3.11. *Let a reductive group G act on an affine variety X. For a G-semisimple element $x \in X(F)$ we set*
$$S_x := \{y \in X(F) \,|\, \overline{G(F)y} \ni x\}.$$

Lemma 2.3.12. *Let V be an F-rational finite-dimensional representation of a reductive group G. Then $\Gamma(V) = S_0$.*

This lemma follows from [RR96, fact A on page 108] for non-Archimedean F and [Brk71, Theorem 5.2 on page 459] for Archimedean F.

Example 2.3.13. *Let a reductive group G act on its Lie algebra $\mathfrak{g}$ by the adjoint action. Then $\Gamma(\mathfrak{g})$ is the set of nilpotent elements of $\mathfrak{g}$.*

Proposition 2.3.14. *Let a reductive group G act on an affine variety X. Let $x, z \in X(F)$ be G-semisimple elements which do not lie in the same orbit of $G(F)$. Then there exist disjoint $G(F)$-invariant open neighborhoods U_x of x and U_z of z.*

For the proof of this Proposition see [Lun75] for Archimedean F and [RR96, fact B on page 109] for non-Archimedean F.

Corollary 2.3.15. *Let a reductive group G act on an affine variety X. Suppose that $x \in X(F)$ is a G-semisimple element. Then the set S_x is closed.*

Proof. Let $y \in \overline{S_x}$. By Proposition 2.3.6, $\overline{G(F)y}$ contains a closed orbit $G(F)z$. If $G(F)z = G(F)x$ then $y \in S_x$. Otherwise, choose disjoint open G-invariant neighborhoods U_z of z and U_x of x. Since $z \in \overline{G(F)y}$, U_z intersects $G(F)y$ and hence contains y. Since $y \in \overline{S_x}$, this means that U_z intersects S_x. Let $t \in U_z \cap S_x$. Since U_z is $G(F)$-invariant, $G(F)t \subset U_z$. By the definition of S_x, $x \in \overline{G(F)t}$ and hence $x \in \overline{U_z}$. Hence U_z intersects U_x – contradiction! □

2.3.3. *Analytic Luna slices.*

Definition 2.3.16. *Let a reductive group G act on an affine variety X. Let $\pi : X(F) \to (X/G)(F)$ be the natural map. An open subset $U \subset X(F)$ is called* ***saturated*** *if there exists an open subset $V \subset (X/G)(F)$ such that $U = \pi^{-1}(V)$.*

We will use the following corollary of the Luna Slice Theorem:

Theorem 2.3.17 (see Appendix A.2)**.** *Let a reductive group G act on a smooth affine variety X. Let $x \in X(F)$ be G-semisimple. Consider the natural action of the stabilizer G_x on the normal space $N^X_{Gx,x}$. Then there exist*
(i) an open $G(F)$-invariant B-analytic neighborhood U of $G(F)x$ in $X(F)$ with a G-equivariant B-analytic retract $p : U \to G(F)x$ and
(ii) a G_x-equivariant B-analytic embedding $\psi : p^{-1}(x) \hookrightarrow N^X_{Gx,x}(F)$ with an open saturated image such that $\psi(x) = 0$.

Definition 2.3.18. *In the notation of the previous theorem, denote $S := p^{-1}(x)$ and $N := N^X_{Gx,x}(F)$. We call the quintuple (U, p, ψ, S, N) an* ***analytic Luna slice at*** *x.*

Corollary 2.3.19. *In the notation of the previous theorem, let $y \in p^{-1}(x)$. Denote $z := \psi(y)$. Then*
(i) $(G(F)_x)_z = G(F)_y$
(ii) $N^{X(F)}_{G(F)y,y} \cong N^N_{G(F)_x z,z}$ as $G(F)_y$-spaces
(iii) y is G-semisimple if and only if z is G_x-semisimple.

2.4. **Vector systems.** [1]

In this subsection we introduce the term *"vector system"*. This term allows to formulate statements in wider generality.

Definition 2.4.1. *For an analytic manifold M we define the notions of a* ***vector system*** *and a* ***B-vector system*** *over it.*

For a smooth manifold M, a vector system over M is a pair (E, B) where B is a smooth locally trivial fibration over M and E is a smooth (finite-dimensional) vector bundle over B.

For a Nash manifold M, a B-vector system over M is a pair (E, B) where B is a Nash fibration over M and E is a Nash (finite-dimensional) vector bundle over B.

For an l-space M, a vector system over M (or a B-vector system over M) is a sheaf of complex linear spaces.

In particular, in the case where M is a point, a vector system over M is either a $\mathbb{C}$-vector space if F is non-Archimedean, or a smooth manifold together with a vector bundle in the case where F is Archimedean. The simplest example of a vector system over a manifold M is given by the following.

Definition 2.4.2. *Let $\mathcal{V}$ be a vector system over a point pt. Let M be an analytic manifold. A* ***constant vector system with fiber*** *$\mathcal{V}$ is the pullback of $\mathcal{V}$ with respect to the map $M \to pt$. We denote it by $\mathcal{V}_M$.*

2.5. **Distributions.**

Definition 2.5.1. *Let M be an analytic manifold over F. We define $C^\infty_c(M)$ in the following way.*

If F is non-Archimedean then $C^\infty_c(M)$ is the space of locally constant compactly supported complex valued functions on M. We do not consider any topology on $C^\infty_c(M)$.

If F is Archimedean then $C^\infty_c(M)$ is the space of smooth compactly supported complex valued functions on M, endowed with the standard topology.

For any analytic manifold M, we define the space of distributions $\mathcal{D}(M)$ by $\mathcal{D}(M) := C^\infty_c(M)^*$. We consider the weak topology on it.

[1] Subsection 2.4 and in particular the notion of "vector system" along with the results at the end of §§3.1 and §§3.2 are not essential for the rest of the paper. They are merely included for future reference.

Definition 2.5.2. *Let M be a B-analytic manifold. We define $\mathcal{S}(M)$ in the following way.*

If M is an analytic manifold over non-Archimedean field, $\mathcal{S}(M) := C_c^\infty(M)$.

If M is a Nash manifold, $\mathcal{S}(M)$ is the space of Schwartz functions on M, namely smooth functions which are rapidly decreasing together with all their derivatives. See [AG08a] *for the precise definition. We consider $\mathcal{S}(M)$ as a Fréchet space.*

For any B-analytic manifold M, we define the space of ***Schwartz distributions*** *$\mathcal{S}^*(M)$ by $\mathcal{S}^*(M) := \mathcal{S}(M)^*$. Clearly, $\mathcal{S}(M)^*$ is naturally embedded into $\mathcal{D}(M)$.*

Notation 2.5.3. *Let M be an analytic manifold. For a distribution $\xi \in \mathcal{D}(M)$ we denote by* $\mathrm{Supp}(\xi)$ *the support of ξ.*

For a closed subset $N \subset M$ we denote

$$\mathcal{D}_M(N) := \{\xi \in \mathcal{D}(M) | \mathrm{Supp}(\xi) \subset N\}.$$

More generally, for a locally closed subset $N \subset M$ we denote

$$\mathcal{D}_M(N) := \mathcal{D}_{M \setminus (\overline{N} \setminus N)}(N).$$

Similarly if M is a B-analytic manifold and N is a locally closed subset we define $\mathcal{S}_M^*(N)$ in a similar vein. [2]

Definition 2.5.4. *Let M be an analytic manifold over F and $\mathcal{E}$ be a vector system over M. We define $C_c^\infty(M, \mathcal{E})$ in the following way.*

If F is non-Archimedean then $C_c^\infty(M, \mathcal{E})$ is the space of compactly supported sections of $\mathcal{E}$.

If F is Archimedean and $\mathcal{E} = (E, B)$ where B is a fibration over M and E is a vector bundle over B, then $C_c^\infty(M, \mathcal{E})$ is the complexification of the space of smooth compactly supported sections of E over B.

If $\mathcal{V}$ is a vector system over a point then we denote $C_c^\infty(M, \mathcal{V}) := C_c^\infty(M, \mathcal{V}_M)$.

We define $\mathcal{D}(M, \mathcal{E})$, $\mathcal{D}_M(N, \mathcal{E})$, $\mathcal{S}(M, \mathcal{E})$, $\mathcal{S}^*(M, \mathcal{E})$ and $\mathcal{S}_M^*(N, \mathcal{E})$ in the natural way.

Theorem 2.5.5. *Let an l-group K act on an l-space M. Let $M = \bigcup_{i=0}^l M_i$ be a K-invariant stratification of M. Let χ be a character of K. Suppose that $\mathcal{S}^*(M_i)^{K,\chi} = 0$. Then $\mathcal{S}^*(M)^{K,\chi} = 0$.*

This theorem is a direct corollary of [BZ76, Corollary 1.9].

For the proof of the next theorem see e.g. [AGS08, §B.2].

Theorem 2.5.6. *Let a Nash group K act on a Nash manifold M. Let N be a locally closed subset. Let $N = \bigcup_{i=0}^l N_i$ be a Nash K-invariant stratification of N. Let χ be a character of K. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$,*

$$\mathcal{S}^*(N_i, \mathrm{Sym}^k(CN_{N_i}^M))^{K,\chi} = 0.$$

Then $\mathcal{S}_M^*(N)^{K,\chi} = 0$.

Theorem 2.5.7 (Frobenius reciprocity)**.** *Let an analytic group K act on an analytic manifold M. Let N be an analytic manifold with a transitive action of K. Let $\phi : M \to N$ be a K-equivariant map.*

Let $z \in N$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K. Let Δ_K and Δ_{K_z} be the modular characters of K and K_z.

Let $\mathcal{E}$ be a K-equivariant vector system over M. Then

(i) there exists a canonical isomorphism

$$\mathrm{Fr} : \mathcal{D}(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z} \cong \mathcal{D}(M, \mathcal{E})^K.$$

In particular, Fr *commutes with restrictions to open sets.*

(ii) For B-analytic manifolds Fr *maps $\mathcal{S}^*(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z}$ to $\mathcal{S}^*(M, \mathcal{E})^K$.*

For the proof of (i) see [Ber84, §§1.5] and [BZ76, §§2.21 - 2.36] for the case of l-spaces and [AGS08, Theorem 4.2.3] or [Bar03] for smooth manifolds. For the proof of (ii) see Appendix B.

We will also use the following straightforward proposition.

[2] In the Archimedean case, locally closed is considered with respect to the restricted topology – cf. Appendix B.

Proposition 2.5.8. *Let K_i be analytic groups acting on analytic manifolds M_i for $i = 1 \ldots n$. Let $\Omega_i \subset K_i$ be analytic subgroups. Let $\mathcal{E}_i \to M_i$ be K_i-equivariant vector systems. Suppose that*

$$\mathcal{D}(M_i, E_i)^{\Omega_i} = \mathcal{D}(M_i, E_i)^{K_i}$$

for all i. Then

$$\mathcal{D}(\prod M_i, \boxtimes E_i)^{\prod \Omega_i} = \mathcal{D}(\prod M_i, \boxtimes E_i)^{\prod K_i},$$

where $\boxtimes$ denotes the external product.

Moreover, if Ω_i, K_i, M_i and $\mathcal{E}_i$ are B-analytic then the analogous statement holds for Schwartz distributions.

For the proof see e.g. [AGS08, proof of Proposition 3.1.5].

3. Generalized Harish-Chandra descent

3.1. **Generalized Harish-Chandra descent.**

In this subsection we will prove the following theorem.

Theorem 3.1.1. *Let a reductive group G act on a smooth affine variety X. Let χ be a character of $G(F)$. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N^X_{Gx,x}(F))^{G(F)_x,\chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{G(F),\chi} = 0.$$

Remark 3.1.2. *In fact, the converse is also true. We will not prove it since we will not use it.*

For the proof of this theorem we will need the following lemma

Lemma 3.1.3. *Let a reductive group G act on a smooth affine variety X. Let χ be a character of $G(F)$. Let $U \subset X(F)$ be an open saturated subset. Suppose that $\mathcal{D}(X(F))^{G(F),\chi} = 0$. Then $\mathcal{D}(U)^{G(F),\chi} = 0$.*

Proof. Consider the quotient X/G. It is an affine algebraic variety. Embed it in an affine space $\mathbb{A}^n$. This defines a map $\pi : X(F) \to F^n$. Since U is saturated, there exists an open subset $V \subset (X/G)(F)$ such that $U = \pi^{-1}(V)$. Clearly there exists an open subset $V' \subset F^n$ such that $V' \cap (X/G)(F) = V$.

Let $\xi \in \mathcal{D}(U)^{G(F),\chi}$. Suppose that ξ is non-zero. Let $x \in \mathrm{Supp}\xi$ and let $y := \pi(x)$. Let $g \in C_c^\infty(V')$ be such that $g(y) = 1$. Consider $\xi' \in \mathcal{D}(X(F))$ defined by $\xi'(f) := \xi(f \cdot (g \circ \pi))$. Clearly, $\mathrm{Supp}(\xi') \subset U$ and hence we can interpret ξ' as an element in $\mathcal{D}(X(F))^{G(F),\chi}$. Therefore $\xi' = 0$. On the other hand, $x \in \mathrm{Supp}(\xi')$. Contradiction. □

Proof of Theorem 3.1.1. Let x be a G-semisimple element. Let $(U_x,p_x,\psi_x, S_x,N_x)$ be an analytic Luna slice at x.

Let $\xi' = \xi|_{U_x}$. Then $\xi' \in \mathcal{D}(U_x)^{G(F),\chi}$. By Frobenius reciprocity it corresponds to $\xi'' \in \mathcal{D}(S_x)^{G_x(F),\chi}$.

The distribution ξ'' corresponds to a distribution $\xi''' \in \mathcal{D}(\psi_x(S_x))^{G_x(F),\chi}$.

However, by the previous lemma the assumption implies that $\mathcal{D}(\psi_x(S_x))^{G_x(F),\chi} = 0$. Hence $\xi' = 0$.

Let $S \subset X(F)$ be the set of all G-semisimple points. Let $U = \bigcup_{x \in S} U_x$. We saw that $\xi|_U = 0$. On the other hand, U includes all the closed orbits, and hence by Corollary 2.3.7 $U = X$. □

The following generalization of this theorem is proven in the same way.

Theorem 3.1.4. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N^X_{Gx,x}(F))^{K_x,\chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{K,\chi} = 0.$$

Now we would like to formulate a slightly more general version of this theorem concerning K-equivariant vector systems. [3]

[3]Subsection 2.4 and in particular the notion of "vector system" along with the results at the end of §§3.1 and §§3.2 are not essential for the rest of the paper. They are merely included for future reference.

Definition 3.1.5. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup. Let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$. Let $x \in X(F)$ be G-semisimple. Let $\mathcal{E}'$ be a K_x-equivariant vector system on $N^X_{Gx,x}(F)$. We say that $\mathcal{E}$ and $\mathcal{E}'$ are **compatible** if there exists an analytic Luna slice (U, p, ψ, S, N) such that $\mathcal{E}|_S = \psi^*(\mathcal{E}')$.*

Note that if $\mathcal{E}$ and $\mathcal{E}'$ are constant with the same fiber then they are compatible.

The following theorem is proven in the same way as Theorem 3.1.1.

Theorem 3.1.6. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$. Suppose that for any G-semisimple $x \in X(F)$ there exists a K-equivariant vector system $\mathcal{E}'$ on $N^X_{Gx,x}(F)$, compatible with $\mathcal{E}$ such that*

$$\mathcal{D}(N^X_{Gx,x}(F), \mathcal{E}')^{K_x} = 0.$$

Then

$$\mathcal{D}(X(F), \mathcal{E})^K = 0.$$

If $\mathcal{E}$ and $\mathcal{E}'$ are B-vector systems and K is an open B-analytic subgroup[4] of $G(F)$ then the theorem also holds for Schwartz distributions. Namely, if $\mathcal{S}^*(N^X_{Gx,x}(F), \mathcal{E}')^{K_x} = 0$ for any G-semisimple $x \in X(F)$ then $\mathcal{S}^*(X(F), \mathcal{E})^K = 0$. The proof is the same.

3.2. **A stronger version.**

In this section we provide means to validate the conditions of Theorems 3.1.1, 3.1.4 and 3.1.6 based on an inductive argument.

More precisely, the goal of this section is to prove the following theorem.

Theorem 3.2.1. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ such that*

$$\mathcal{D}(R_{G_x}(N^X_{Gx,x}))^{K_x,\chi} = 0$$

we have

$$\mathcal{D}(Q_{G_x}(N^X_{Gx,x}))^{K_x,\chi} = 0.$$

Then for any G-semisimple $x \in X(F)$ we have

$$\mathcal{D}(N^X_{Gx,x}(F))^{K_x,\chi} = 0.$$

Together with Theorem 3.1.4, this theorem gives the following corollary.

Corollary 3.2.2. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ such that*

$$\mathcal{D}(R(N^X_{Gx,x}))^{K_x,\chi} = 0$$

we have

$$\mathcal{D}(Q(N^X_{Gx,x}))^{K_x,\chi} = 0.$$

Then $\mathcal{D}(X(F))^{K,\chi} = 0$.

From now till the end of the section we fix G, X, K and χ. Let us introduce several definitions and notation.

Notation 3.2.3. *Denote*

- *$T \subset X(F)$ the set of all G-semisimple points.*
- *For $x, y \in T$ we say that $x > y$ if $G_x \supsetneq G_y$.*
- *$T_0 := \{x \in T \mid \mathcal{D}(Q(N^X_{Gx,x}))^{K_x,\chi} = 0\} = \{x \in T \mid \mathcal{D}((N^X_{Gx,x}))^{K_x,\chi} = 0\}$.*

[4]In fact, any open subgroup of a B-analytic group is B-analytic.

Proof of Theorem 3.2.1. We have to show that $T = T_0$. Assume the contrary.

Note that every chain in T with respect to our ordering has a minimum. Hence by Zorn's lemma every non-empty set in T has a minimal element. Let x be a minimal element of $T - T_0$. To get a contradiction, it is enough to show that $\mathcal{D}(R(N^X_{Gx,x}))^{K_x,\chi} = 0$.

Denote $R := R(N^X_{Gx,x})$. By Theorem 3.1.4, it is enough to show that for any $y \in R$ we have

$$\mathcal{D}(N^R_{G(F)_x y,y})^{(K_x)_y,\chi} = 0.$$

Let (U, p, ψ, S, N) be an analytic Luna slice at x.

Since $\psi(S)$ is open and contains 0, we can assume, upon replacing y by λy for some $\lambda \in F^\times$, that $y \in \psi(S)$. Let $z \in S$ be such that $\psi(z) = y$. By Corollary 2.3.19, $G(F)_z = (G(F)_x)_y \subsetneq G(F)_x$ and $N^R_{G(F)_x y,y} \cong N^X_{Gz,z}(F)$. Hence $(K_x)_y = K_z$ and therefore

$$\mathcal{D}(N^R_{G(F)_x y,y})^{(K_x)_y,\chi} \cong \mathcal{D}(N^X_{Gz,z}(F))^{K_z,\chi}.$$

However $z < x$ and hence $z \in T_0$ which means that $\mathcal{D}(N^X_{Gz,z}(F))^{K_z,\chi} = 0$. □

Remark 3.2.4. *One can rewrite this proof such that it will use Zorn's lemma for finite sets only, which does not depend on the axiom of choice.*

Remark 3.2.5. *As before, Theorem 3.2.1 and Corollary 3.2.2 also hold for Schwartz distributions, with a similar proof.*

Again, we can formulate a more general version of Corollary 3.2.2 concerning vector systems. [5]

Theorem 3.2.6. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$.*

Suppose that for any G-semisimple $x \in X(F)$ satisfying
(*) for any $K_x \times F^\times$-equivariant vector system $\mathcal{E}'$ on $R(N^X_{Gx,x})$ (where $F^\times$ acts by homothety) compatible with $\mathcal{E}$ we have $\mathcal{D}(R(N^X_{Gx,x}), \mathcal{E}')^{K_x} = 0$,

the following holds
(**) there exists a $K_x \times F^\times$-equivariant vector system $\mathcal{E}'$ on $Q(N^X_{Gx,x})$ compatible with $\mathcal{E}$ such that

$$\mathcal{D}(Q(N^X_{Gx,x}), \mathcal{E}')^{K_x} = 0.$$

Then $\mathcal{D}(X(F), \mathcal{E})^K = 0$.

The proof is the same as the proof of Theorem 3.2.1 using the following lemma which follows from the definitions.

Lemma 3.2.7. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$. Let $x \in X(F)$ be G-semisimple. Let (U, p, ψ, S, N) be an analytic Luna slice at x.*

Let $\mathcal{E}'$ be a K_x-equivariant vector system on N compatible with $\mathcal{E}$. Let $y \in S$ be G-semisimple, and let $z := \psi(y)$. Let $\mathcal{E}''$ be a $(K_x)_z$-equivariant vector system on $N^N_{G_x z,z}$ compatible with $\mathcal{E}'$. Consider the isomorphism $N^N_{G_x z,z}(F) \cong N^X_{Gy,y}(F)$ and let $\mathcal{E}'''$ be the corresponding K_y-equivariant vector system on $N^X_{Gy,y}(F)$.

Then $\mathcal{E}'''$ is compatible with $\mathcal{E}$.

Again, if $\mathcal{E}$ and $\mathcal{E}'$ are B-vector systems then the theorem holds also for Schwartz distributions.

[5]Subsection 2.4 and in particular, the notion of "vector system" along with the results at the end of §§3.1 and §§3.2 are not essential for the rest of the paper. They are merely included for future reference.

4. Distributions versus Schwartz distributions

In this section F is Archimedean. The tools developed in the previous section enable us to prove the following version of the Localization Principle.

Theorem 4.0.1 (Localization Principle). *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant map. Let χ be a character of $G(F)$. Suppose that for any $y \in Y(F)$ we have $\mathcal{D}_{X(F)}((\phi^{-1}(y))(F))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

For the proof see Appendix D.

In this section we use this theorem to show that if there are no $G(F)$-equivariant Schwartz distributions on $X(F)$ then there are no $G(F)$-equivariant distributions on $X(F)$.

Theorem 4.0.2. *Let a reductive group G act on a smooth affine variety X. Let V be a finite-dimensional algebraic representation of $G(F)$. Suppose that*

$$\mathcal{S}^*(X(F), V)^{G(F)} = 0.$$

Then

$$\mathcal{D}(X(F), V)^{G(F)} = 0.$$

For the proof we will need the following definition and theorem.

Definition 4.0.3.

(i) Let a topological group K act on a topological space M. We call a closed K-invariant subset $C \subset M$ ***compact modulo*** *K if there exists a compact subset $C' \subset M$ such that $C \subset KC'$.*

(ii) Let a Nash group K act on a Nash manifold M. We call a closed K-invariant subset $C \subset M$ ***Nashly compact modulo*** *K if there exist a compact subset $C' \subset M$ and semi-algebraic closed subset $Z \subset M$ such that $C \subset Z \subset KC'$.*

Remark 4.0.4. *Let a reductive group G act on a smooth affine variety X. Let $K := G(F)$ and $M := X(F)$. Then it is easy to see that the notions of compact modulo K and Nashly compact modulo K coincide.*

Theorem 4.0.5. *Let a Nash group K act on a Nash manifold M. Let E be a K-equivariant Nash bundle over M. Let $\xi \in \mathcal{D}(M, E)^K$ be such that* $\mathrm{Supp}(\xi)$ *is Nashly compact modulo K. Then $\xi \in \mathcal{S}^*(M, E)^K$.*

The statement and the idea of the proof of this theorem are due to J. Bernstein. For the proof see Appendix B.4.

Proof of Theorem 4.0.2. Fix any $y \in (X/G)(F)$ and denote $M := \pi_X^{-1}(y)(F)$.

By the Localization Principle (Theorem 4.0.1 and Remark D.0.4), it is enough to prove that

$$\mathcal{S}^*_{X(F)}(M, V)^{G(F)} = \mathcal{D}_{X(F)}(M, V)^{G(F)}.$$

Choose $\xi \in \mathcal{D}_{X(F)}(M, V)^{G(F)}$. M has a unique closed stable G-orbit and hence a finite number of closed $G(F)$-orbits. By Theorem 4.0.5, it is enough to show that M is Nashly compact modulo $G(F)$. Clearly M is semi-algebraic. Choose representatives x_i of the closed $G(F)$-orbits in M. Choose compact neighborhoods C_i of x_i. Let $C' := \bigcup C_i$. By Corollary 2.3.7, $G(F)C' \supset M$. $\square$

5. Applications of Fourier transform and the Weil representation

Let G be a reductive group and V be a finite-dimensional F-rational representation of G. Let χ be a character of $G(F)$. In this section we provide some tools to verify that $\mathcal{S}^*(Q(V))^{G(F),\chi} = 0$ provided that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$.

5.1. **Preliminaries.**
For this subsection let B be a non-degenerate bilinear form on a finite-dimensional vector space V over F. We also fix an additive character κ of F. If F is Archimedean we take $\kappa(x) := e^{2\pi i \operatorname{Re}(x)}$.

Notation 5.1.1. *We identify V and V^* via B and endow V with the self-dual Haar measure with respect to ψ. Denote by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ the Fourier transform. For any B-analytic manifold M over F we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the partial Fourier transform.*

Notation 5.1.2. *Consider the homothety action of $F^\times$ on V given by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$.*

Let $|\cdot|$ denote the normalized absolute value. Recall that for $F = \mathbb{R}$, $|\lambda|$ is equal to the classical absolute value but for $F = \mathbb{C}$, $|\lambda| = (\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2$.

Notation 5.1.3. *We denote by $\gamma(B)$ the Weil constant. For its definition see e.g.* [Gel76, §2.3] *for non-Archimedean F and* [RS78, §1] *for Archimedean F.*

For any $t \in F^\times$ denote $\delta_B(t) = \gamma(B)/\gamma(tB)$.

Note that $\gamma(B)$ is an 8-th root of unity and if $\dim V$ is odd and $F \neq \mathbb{C}$ then δ_B is **not** a multiplicative character.

Notation 5.1.4. *We denote*
$$Z(B) := \{x \in V \mid B(x,x) = 0\}.$$

Theorem 5.1.5 (non-Archimedean homogeneity)**.** *Suppose that F is **non-Archimedean**. Let M be a B-analytic manifold over F. Let $\xi \in \mathcal{S}^*_{V\times M}(Z(B) \times M)$ be such that $\mathcal{F}_B(\xi) \in \mathcal{S}^*_{V\times M}(Z(B) \times M)$. Then for any $t \in F^\times$, we have $\rho(t)\xi = \delta_B(t)|t|^{\dim V/2}\xi$ and $\xi = \gamma(B)^{-1}\mathcal{F}_B(\xi)$. In particular, if* $\dim V$ *is odd then $\xi = 0$.*

For the proof see e.g. [RS07, §§8.1] or [JR96, §§3.1].

For the Archimedean version of this theorem we will need the following definition.

Definition 5.1.6. *Let M be a B-analytic manifold over F. We say that a distribution $\xi \in \mathcal{S}^*(V \times M)$ is **adapted to** B if either*
(i) for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)|t|^{\dim V/2}\xi$ and ξ is proportional to $\mathcal{F}_B\xi$ or
(ii) F is Archimedean and for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)t|t|^{\dim V/2}\xi$.

Note that if $\dim V$ is odd and $F \neq \mathbb{C}$ then every B-adapted distribution is zero.

Theorem 5.1.7 (Archimedean homogeneity)**.** *Let M be a Nash manifold. Let $L \subset \mathcal{S}^*_{V\times M}(Z(B) \times M)$ be a non-zero subspace such that for all $\xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B \cdot \xi \in L$ (here B is viewed as a quadratic function).*

Then there exists a non-zero distribution $\xi \in L$ which is adapted to B.

For Archimedean F we prove this theorem in Appendix C. For non-Archimedean F it follows from Theorem 5.1.5.

We will also use the following trivial observation.

Lemma 5.1.8. *Let a B-analytic group K act linearly on V and preserving B. Let M be a B-analytic K-manifold over F. Let $\xi \in \mathcal{S}^*(V \times M)$ be a K-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also K-invariant.*

5.2. **Applications.**
The following two theorems easily follow form the results of the previous subsection.

Theorem 5.2.1. *Suppose that F is non-Archimedean. Let G be a reductive group. Let V be a finite-dimensional F-rational representation of G. Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$. Let $V = V_1 \oplus V_2$ be a G-invariant decomposition of V. Let B be a G-invariant symmetric non-degenerate bilinear form on V_1. Consider the action ρ of $F^\times$ on V by homothety on V_1.*

Then any $\xi \in \mathcal{S}^*(Q(V))^{G(F),\chi}$ satisfies $\rho(t)\xi = \delta_B(t)|t|^{\dim V_1/2}\xi$ and $\xi = \gamma(B)\mathcal{F}_B\xi$. In particular, if $\dim V_1$ *is odd then $\xi = 0$.*

Theorem 5.2.2. *Let G be a reductive group. Let V be a finite-dimensional F-rational representation of G. Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$. Let $Q(V) = W \oplus (\bigoplus_{i=1}^k V_i)$ be a G-invariant decomposition of $Q(V)$. Let B_i be G-invariant symmetric non-degenerate bilinear forms on V_i. Suppose that any $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F),\chi}$ which is adapted to each B_i is zero.*

Then $\mathcal{S}^*(Q(V))^{G(F),\chi} = 0$.

Remark 5.2.3. *One can easily generalize Theorems 5.2.2 and 5.2.1 to the case of constant vector systems.*

6. Tame actions

In this section we consider problems of the following type. A reductive group G acts on a smooth affine variety X, and τ is an automorphism of X which normalizes the image of G in $\mathrm{Aut}(X)$. We want to check whether any $G(F)$-invariant Schwartz distribution on $X(F)$ is also τ-invariant.

Definition 6.0.1. *Let π be an action of a reductive group G on a smooth affine variety X. We say that an algebraic automorphism τ of X is G-**admissible** if*
(i) τ normalizes $\pi(G(F))$ and $\tau^2 \in \pi(G(F))$.
(ii) For any closed $G(F)$-orbit $O \subset X(F)$, we have $\tau(O) = O$.

Proposition 6.0.2. *Let π be an action of a reductive group G on a smooth affine variety X. Let τ be a G-admissible automorphism of X. Let $K := \pi(G(F))$ and let $\widetilde{K}$ be the group generated by $\pi(G(F))$ and τ. Let $x \in X(F)$ be a point with closed $G(F)$-orbit. Let $\tau' \in \widetilde{K}_x - K_x$. Then $d\tau'|_{N^X_{Gx,x}}$ is G_x-admissible.*

Proof. Let $\widetilde{G}$ denote the group generated by $\pi(G)$ and τ. We check that the two properties of G_x-admissibility hold for $d\tau'|_{N^X_{Gx,x}}$. The first one is obvious. For the second, let $y \in N^X_{Gx,x}(F)$ be an element with closed G_x-orbit. Let $y' = d\tau'(y)$. We have to show that there exists $g \in G_x(F)$ such that $gy = y'$. Let (U, p, ψ, S, N) be an analytic Luna slice at x with respect to the action of $\widetilde{G}$. We can assume that there exists $z \in S$ such that $y = \psi(z)$. Let $z' = \tau'(z)$. By Corollary 2.3.19, z is G-semisimple. Since τ is admissible, this implies that there exists $g \in G(F)$ such that $gz = z'$. Clearly, $g \in G_x(F)$ and $gy = y'$. ◻

Definition 6.0.3. *We call an action of a reductive group G on a smooth affine variety X **tame** if for any G-admissible $\tau : X \to X$, we have $\mathcal{S}^*(X(F))^{G(F)} \subset \mathcal{S}^*(X(F))^{\tau}$.*

Definition 6.0.4. *We call an F-rational representation V of a reductive group G **linearly tame** if for any G-admissible linear map $\tau : V \to V$, we have $\mathcal{S}^*(V(F))^{G(F)} \subset \mathcal{S}^*(V(F))^{\tau}$.*

*We call a representation **weakly linearly tame** if for any G-admissible linear map $\tau : V \to V$, such that $\mathcal{S}^*(R(V))^{G(F)} \subset \mathcal{S}^*(R(V))^{\tau}$ we have $\mathcal{S}^*(Q(V))^{G(F)} \subset \mathcal{S}^*(Q(V))^{\tau}$.*

Theorem 6.0.5. *Let a reductive group G act on a smooth affine variety X. Suppose that for any G-semisimple $x \in X(F)$, the action of G_x on $N^X_{Gx,x}$ is weakly linearly tame. Then the action of G on X is tame.*

The proof is rather straightforward except for one minor complication: the group of automorphisms of $X(F)$ generated by the action of $G(F)$ is not necessarily a group of F-points of any algebraic group.

Proof. Let $\tau : X \to X$ be an admissible automorphism.

Let $\widetilde{G} \subset \mathrm{Aut}(X)$ be the algebraic group generated by the actions of G and τ. Let $K \subset \mathrm{Aut}(X(F))$ be the B-analytic group generated by the action of $G(F)$. Let $\widetilde{K} \subset \mathrm{Aut}(X(F))$ be the B-analytic group generated by the actions of G and τ. Note that $\widetilde{K} \subset \widetilde{G}(F)$ is an open subgroup of finite index. Note that for any $x \in X(F)$, x is $\widetilde{G}$-semisimple if and only if it is G-semisimple. If $K = \widetilde{K}$ we are done, so we will assume $K \neq \widetilde{K}$. Let χ be the character of $\widetilde{K}$ defined by $\chi(K) = \{1\}$, $\chi(\widetilde{K} - K) = \{-1\}$.

It is enough to prove that $\mathcal{S}^*(X)^{\widetilde{K},\chi} = 0$. By Generalized Harish-Chandra Descent (Corollary 3.2.2) it is enough to prove that for any G-semisimple $x \in X$ such that

$$\mathcal{S}^*(R(N^X_{Gx,x}))^{\widetilde{K}_x,\chi} = 0$$

we have

$$\mathcal{S}^*(Q(N^X_{Gx,x}))^{\widetilde{K}_x,\chi} = 0.$$

Choose any automorphism $\tau' \in \widetilde{K}_x - K_x$. Note that τ' and K_x generate $\widetilde{K}_x$. Denote

$$\eta := d\tau'|_{N^X_{Gx,x}(F)}.$$

By Proposition 6.0.2, η is G_x-admissible. Note that

$$\mathcal{S}^*(R(N^X_{Gx,x}))^{K_x} = \mathcal{S}^*(R(N^X_{Gx,x}))^{G(F)_x} \text{ and } \mathcal{S}^*(Q(N^X_{Gx,x}))^{K_x} = \mathcal{S}^*(Q(N^X_{Gx,x}))^{G(F)_x}.$$

Hence we have

$$\mathcal{S}^*(R(N^X_{Gx,x}))^{G(F)_x} \subset \mathcal{S}^*(R(N^X_{Gx,x}))^{\eta}.$$

Since the action of G_x is weakly linearly tame, this implies that

$$\mathcal{S}^*(Q(N^X_{Gx,x}))^{G(F)_x} \subset \mathcal{S}^*(Q(N^X_{Gx,x}))^{\eta}$$

and therefore $\mathcal{S}^*(Q(N^X_{Gx,x}))^{\widetilde{K}_x,\chi} = 0$. □

Definition 6.0.6. *We call an F-rational representation V of a reductive group G* ***special*** *if there is no non-zero $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F)}$ such that for any G-invariant decomposition $Q(V) = W_1 \oplus W_2$ and any two G-invariant symmetric non-degenerate bilinear forms B_i on W_i the Fourier transforms $\mathcal{F}_{B_i}(\xi)$ are also supported in $\Gamma(V)$.*

Proposition 6.0.7. *Every special representation V of a reductive group G is weakly linearly tame.*

The proposition follows immediately from the following lemma.

Lemma 6.0.8. *Let V be an F-rational representation of a reductive group G. Let τ be an admissible linear automorphism of V. Let $V = W_1 \oplus W_2$ be a G-invariant decomposition of V and B_i be G-invariant symmetric non-degenerate bilinear forms on W_i. Then W_i and B_i are also τ-invariant.*

This lemma follows in turn from the following one.

Lemma 6.0.9. *Let V be an F-rational representation of a reductive group G. Let τ be an admissible automorphism of V. Then $\mathcal{O}(V)^G \subset \mathcal{O}(V)^\tau$.*

Proof. Consider the projection $\pi : V \to V/G$. We have to show that τ acts trivially on V/G and let $x \in \pi(V(F))$. Let $X := \pi^{-1}(x)$. By Proposition 2.3.6 $G(F)$ has a closed orbit in $X(F)$. The automorphism τ preserves this orbit and hence preserves x. Thus τ acts trivially on $\pi(V(F))$, which is Zariski dense in V/G. Hence τ acts trivially on V/G. □

Now we introduce a criterion that allows to prove that a representation is special. It follows immediately from Theorem 5.1.7.

Lemma 6.0.10. *Let V be an F-rational representation of a reductive group G. Let $Q(V) = \bigoplus W_i$ be a G-invariant decomposition. Let B_i be symmetric non-degenerate G-invariant bilinear forms on W_i. Suppose that any $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F)}$ which is adapted to all B_i is zero. Then V is special.*

Part 2. Symmetric and Gelfand pairs

7. Symmetric pairs

In this section we apply our tools to symmetric pairs. We introduce several properties of symmetric pairs and discuss their interrelations. In Appendix E we present a diagram that illustrates the most important ones.

7.1. **Preliminaries and notation.**

Definition 7.1.1. *A* ***symmetric pair*** *is a triple (G,H,θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$. We call a symmetric pair* ***connected*** *if G/H is connected.*

For a symmetric pair (G,H,θ) we define an antiinvolution $\sigma : G \to G$ by

$$\sigma(g) := \theta(g^{-1}),$$

denote $\mathfrak{g} := \operatorname{Lie} G$, $\mathfrak{h} := \operatorname{Lie} H$. Let θ and σ act on $\mathfrak{g}$ by their differentials and denote

$$\mathfrak{g}^\sigma := \{a \in \mathfrak{g} \mid \sigma(a) = a\} = \{a \in \mathfrak{g} \mid \theta(a) = -a\}.$$

Note that H acts on $\mathfrak{g}^\sigma$ by the adjoint action. Denote also

$$G^\sigma := \{g \in G \mid \sigma(g) = g\}$$

and define a ***symmetrization map*** *$s : G \to G^\sigma$ by*

$$s(g) := g\sigma(g).$$

We will consider the action of $H \times H$ on G by left and right translation and the conjugation action of H on G^σ.

Definition 7.1.2. *Let (G_1,H_1,θ_1) and (G_2,H_2,θ_2) be symmetric pairs. We define their* ***product*** *to be the symmetric pair $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$.*

Theorem 7.1.3. *For any connected symmetric pair (G,H,θ) we have $\mathcal{O}(G)^{H\times H} \subset \mathcal{O}(G)^\sigma$.*

Proof. Consider the multiplication map $H \times G^\sigma \to G$. It is étale at 1×1 and hence its image HG^σ contains an open neighborhood of 1 in G. Hence the image of HG^σ in G/H is dense. Thus $HG^\sigma H$ is dense in G. Clearly $\mathcal{O}(HG^\sigma H)^{H\times H} \subset \mathcal{O}(HG^\sigma H)^\sigma$ and hence $\mathcal{O}(G)^{H\times H} \subset \mathcal{O}(G)^\sigma$. ☐

Corollary 7.1.4. *For any connected symmetric pair (G,H,θ) and any closed $H \times H$ orbit $\Delta \subset G$, we have $\sigma(\Delta) = \Delta$.*

Proof. Denote $\Upsilon := H \times H$. Consider the action of the 2-element group $(1,\tau)$ on Υ given by $\tau(h_1,h_2) := (\theta(h_2),\theta(h_1))$. This defines the semi-direct product $\widetilde{\Upsilon} := (1,\tau) \ltimes \Upsilon$. Extend the two-sided action of Υ to $\widetilde{\Upsilon}$ by the antiinvolution σ. Note that the previous theorem implies that $G/\Upsilon = G/\widetilde{\Upsilon}$. Let Δ be a closed Υ-orbit. Let $\widetilde{\Delta} := \Delta \cup \sigma(\Delta)$. Let $a := \pi_G(\widetilde{\Delta}) \subset G/\widetilde{\Upsilon}$. Clearly, a consists of one point. On the other hand, $G/\widetilde{\Upsilon} = G/\Upsilon$ and hence $\pi_G^{-1}(a)$ contains a unique closed G-orbit. Therefore $\Delta = \widetilde{\Delta} = \sigma(\Delta)$. ☐

Corollary 7.1.5. *Let (G,H,θ) be a connected symmetric pair. Let $g \in G(F)$ be $H \times H$-semisimple. Suppose that the Galois cohomology $H^1(F,(H\times H)_g)$ is trivial. Then $\sigma(g) \in H(F)gH(F)$.*

For example, if $(H \times H)_g$ is a product of general linear groups over some field extensions then $H^1(F,(H\times H)_g)$ is trivial.

Definition 7.1.6. *A symmetric pair (G,H,θ) is called* ***good*** *if for any closed $H(F) \times H(F)$ orbit $O \subset G(F)$, we have $\sigma(O) = O$.*

Corollary 7.1.7. *Any connected symmetric pair over $\mathbb{C}$ is good.*

Definition 7.1.8. *A symmetric pair (G,H,θ) is called a* ***GK-pair*** *if*

$$\mathcal{S}^*(G(F))^{H(F)\times H(F)} \subset \mathcal{S}^*(G(F))^\sigma.$$

We will see later in §8 that GK-pairs satisfy a Gelfand pair property that we call GP2 (see Definition 8.1.2 and Theorem 8.1.5). Clearly every GK-pair is good and we conjecture that the converse is also true. We will discuss it in more detail in §§7.5.

Lemma 7.1.9. *Let (G,H,θ) be a symmetric pair. Then there exists a G-invariant θ-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}$. In particular, $\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{h}$ is an orthogonal direct sum with respect to B.*

Proof.

Step 1. Proof for semisimple $\mathfrak{g}$.

Let B be the Killing form on $\mathfrak{g}$. Since it is non-degenerate, it is enough to show that $\mathfrak{h}$ is orthogonal to $\mathfrak{g}^\sigma$. Let $A \in \mathfrak{h}$ and $C \in \mathfrak{g}^\sigma$. We have to show $\mathrm{Tr}(\mathrm{ad}(A)\,\mathrm{ad}(C)) = 0$. This follows from the fact that $\mathrm{ad}(A)\,\mathrm{ad}(C)(\mathfrak{h}) \subset \mathfrak{g}^\sigma$ and $\mathrm{ad}(A)\,\mathrm{ad}(C)(\mathfrak{g}^\sigma) \subset \mathfrak{h}$.

Step 2. Proof in the general case.

Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ such that $\mathfrak{g}'$ is semisimple and $\mathfrak{z}$ is the center. It is easy to see that this decomposition is invariant under $Aut(\mathfrak{g})$ and hence θ-invariant. Now the proposition easily follows from the previous case. □

Remark 7.1.10. *Let (G, H, θ) be a symmetric pair. Let $\mathcal{U}(G)$ be the set of unipotent elements in $G(F)$ and $\mathcal{N}(\mathfrak{g})$ the set of nilpotent elements in $\mathfrak{g}(F)$. Then the exponent map $exp : \mathcal{N}(\mathfrak{g}) \to \mathcal{U}(G)$ is σ-equivariant and intertwines the adjoint action with conjugation.*

Lemma 7.1.11. *Let (G, H, θ) be a symmetric pair. Let $x \in \mathfrak{g}^\sigma$ be a nilpotent element. Then there exists a group homomorphism $\phi : \mathrm{SL}_2 \to G$ such that*

$$d\phi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = x, \quad d\phi(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \in \mathfrak{g}^\sigma \text{ and } \quad \phi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) \in H.$$

In particular $0 \in \overline{\mathrm{Ad}(H)(x)}$.

This lemma was essentially proven for $F = \mathbb{C}$ in [KR73]. The same proof works for any F and we repeat it here for the convenience of the reader.

Proof. By the Jacobson-Morozov Theorem (see [Jac62, Chapter III, Theorems 17 and 10]) we can complete x to an $\mathfrak{sl}_2$-triple (x_-, s, x). Let $s' := \frac{s+\theta(s)}{2}$. It satisfies $[s', x] = 2x$ and lies in the ideal $[x, \mathfrak{g}]$ and hence by the Morozov Lemma (see [Jac62, Chapter III, Lemma 7]), x and s' can be completed to an $\mathfrak{sl}_2$ triple (x_-, s', x). Let $x'_- := \frac{x_- - \theta(x_-)}{2}$. Note that (x'_-, s', x) is also an $\mathfrak{sl}_2$-triple. Exponentiating this $\mathfrak{sl}_2$-triple to a map $\mathrm{SL}_2 \to G$ we get the required homomorphism. □

Notation 7.1.12. *In the notation of the previous lemma we denote*

$$D_t(x) := \phi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) \in H \text{ and } d(x) := d\phi(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \in \mathfrak{h}.$$

These elements depend on the choice of ϕ. However, whenever we use this notation, nothing will depend on their choice.

7.2. **Descendants of symmetric pairs.** Recall that for a symmetric pair (G, H, θ) we consider the $H \times H$ action on G by left and right translation and the conjugation action of H on G^σ.

Proposition 7.2.1. *Let (G, H, θ) be a symmetric pair. Let $g \in G(F)$ be $H \times H$-semisimple. Let $x = s(g)$. Then*
(i) x is semisimple (both as an element of G and with respect to the H-action).
(ii) $H_x \cong (H \times H)_g$ and $(\mathfrak{g}_x)^\sigma \cong N^G_{HgH,g}$ as H_x-spaces.

Proof.

(i) Since the symmetrization map is closed, it is clear that the H-orbit of x is closed. This means that x is semisimple with respect to the H-action. Now we have to show that x is semisimple as an element of G . Let $x = x_s x_u$ be the Jordan decomposition of x. The uniqueness of the Jordan decomposition implies that both x_u and x_s belong to G^σ. To show that $x_u = 1$ it is enough to show that $\overline{\mathrm{Ad}(H)(x)} \ni x_s$. We will do that in several steps.

Step 1. Proof for the case when $x_s = 1$.

It follows immediately from Remark 7.1.10 and Lemma 7.1.11.

Step 2. Proof for the case when $x_s \in Z(G)$.

This case follows from Step 1 since conjugation acts trivially on $Z(G)$.

Step 3. Proof in the general case.

Note that $x \in G_{x_s}$ and G_{x_s} is θ-invariant. The statement follows from Step 2 for the group G_{x_s}.

(ii) The symmetrization map gives rise to an isomorphism $(H \times H)_g \cong H_x$. Let us now show that $(\mathfrak{g}_x)^\sigma \cong N^G_{HgH,g}$. First of all, $N^G_{HgH,g} \cong \mathfrak{g}/(\mathfrak{h} + \mathrm{Ad}(g)\mathfrak{h})$. Let θ' be the involution of G defined by $\theta'(y) = x\theta(y)x^{-1}$. Note that $\mathrm{Ad}(g)\mathfrak{h} = \mathfrak{g}^{\theta'}$. Fix a non-degenerate G-invariant symmetric bilinear form B on $\mathfrak{g}$ as in Lemma 7.1.9. Note that B is also θ'-invariant and hence

$$(\mathrm{Ad}(g)\mathfrak{h})^\perp = \{a \in \mathfrak{g} | \theta'(a) = -a\}.$$

Now

$$N^G_{HgH,g} \cong (\mathfrak{h} + \mathrm{Ad}(g)\mathfrak{h})^\perp = \mathfrak{h}^\perp \cap \mathrm{Ad}(g)\mathfrak{h}^\perp = \{a \in \mathfrak{g} | \theta(a) = \theta'(a) = -a\} = (\mathfrak{g}_x)^\sigma.$$

□

It is easy to see that the isomorphism $N^G_{HgH,g} \cong (\mathfrak{g}_x)^\sigma$ is independent of the choice of B.

Definition 7.2.2. *In the notation of the previous proposition we will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a* ***descendant*** *of (G, H, θ).*

7.3. **Tame symmetric pairs.**

Definition 7.3.1. *We call a symmetric pair (G, H, θ)*
(i) ***tame*** *if the action of $H \times H$ on G is tame.*
(ii) ***linearly tame*** *if the action of H on $\mathfrak{g}^\sigma$ is linearly tame.*
(iii) ***weakly linearly tame*** *if the action of H on $\mathfrak{g}^\sigma$ is weakly linearly tame.*

Remark 7.3.2. *Evidently, any good tame symmetric pair is a GK-pair.*

The following theorem is a direct corollary of Theorem 6.0.5.

Theorem 7.3.3. *Let (G, H, θ) be a symmetric pair. Suppose that all its descendants (including itself) are weakly linearly tame. Then (G, H, θ) is tame and linearly tame.*

Definition 7.3.4. *We call a symmetric pair (G, H, θ)* ***special*** *if $\mathfrak{g}^\sigma$ is a special representation of H (see Definition 6.0.6).*

The following proposition follows immediately from Proposition 6.0.7.

Proposition 7.3.5. *Any special symmetric pair is weakly linearly tame.*

Using Lemma 7.1.9 it is easy to prove the following proposition.

Proposition 7.3.6. *A product of special symmetric pairs is special.*

Now we would like to give a criterion of speciality for symmetric pairs. Recall the notation $d(x)$ of 7.1.12.

Proposition 7.3.7 (Speciality criterion)**.** *Let (G, H, θ) be a symmetric pair. Suppose that for any nilpotent $x \in \mathfrak{g}^\sigma$ either*
(i) $\mathrm{Tr}(\mathrm{ad}(d(x))|_{\mathfrak{h}_x}) < \dim \mathfrak{g}^\sigma$ *or*
(ii) F is non-Archimedean and $\mathrm{Tr}(\mathrm{ad}(d(x))|_{\mathfrak{h}_x}) \neq \dim \mathfrak{g}^\sigma$.
Then the pair (G, H, θ) is special.

For the proof we will need the following auxiliary results.

Lemma 7.3.8. *Let (G, H, θ) be a symmetric pair. Then $\Gamma(\mathfrak{g}^\sigma)$ is the set of all nilpotent elements in $Q(\mathfrak{g}^\sigma)$.*

This lemma is a direct corollary from Lemma 7.1.11.

Lemma 7.3.9. *Let (G, H, θ) be a symmetric pair. Let $x \in \mathfrak{g}^\sigma$ be a nilpotent element. Then all the eigenvalues of* $\mathrm{ad}(d(x))|_{\mathfrak{g}^\sigma/[x,\mathfrak{h}]}$ *are non-positive integers.*

This lemma follows from the existence of a natural surjection $\mathfrak{g}/[x, \mathfrak{g}] \twoheadrightarrow \mathfrak{g}^\sigma/[x, \mathfrak{h}]$ (given by the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^\sigma$)
using the following straightforward lemma.

Lemma 7.3.10. *Let V be a representation of an $\mathfrak{sl}_2$ triple (e,h,f). Then all the eigenvalues of $h|_{V/e(V)}$ are non-positive integers.*

Now we are ready to prove the speciality criterion.

Proof of Proposition 7.3.7. We will give a proof in the case where F is Archimedean. The case of non-Archimedean F is done in the same way but with less complications.

By Lemma 6.0.10 and the definition of adapted it is enough to prove

$$\mathcal{S}^*_{Q(\mathfrak{g}^\sigma)}(\Gamma(\mathfrak{g}^\sigma))^{H(F)\times F^\times,(1,\chi)}=0$$

for any character χ of $F^\times$ of the form $\chi(\lambda)=u(\lambda)|\lambda|^{\dim\mathfrak{g}^\sigma/2}$ or $\chi(\lambda)=u(\lambda)|\lambda|^{\dim\mathfrak{g}^\sigma/2+1}$, where u is some unitary character.

The set $\Gamma(\mathfrak{g}^\sigma)$ has a finite number of $H(F)$-orbits (it follows from Lemma 7.3.8 and the introduction of [KR73]). Hence it is enough to show that for any $x\in\Gamma(\mathfrak{g}^\sigma)$ we have

$$\mathcal{S}^*(\mathrm{Ad}(H(F))x,\mathrm{Sym}^k(CN^{\mathfrak{g}^\sigma}_{\mathrm{Ad}(H(F))x}))^{H(F)\times F^\times,(1,\chi)}=0 \text{ for any } k.$$

Let $K:=\{(D_t(x),t^2)|t\in F^\times\}\subset(H(F)\times F^\times)_x$.

Note that

$$\Delta_{(H(F)\times F^\times)_x}((D_t(x),t^2))=|\det(\mathrm{Ad}(D_t(x))|_{\mathfrak{g}^\sigma_x})|=|t|^{\mathrm{Tr}(\mathrm{ad}(d(x))|_{\mathfrak{h}_x})}.$$

By Lemma 7.3.9 the eigenvalues of the action of $(D_t(x),t^2)$ on $(\mathrm{Sym}^k(\mathfrak{g}^\sigma/[x,\mathfrak{h}]))$ are of the form t^l where l is a non-positive integer.

Now by Frobenius reciprocity (Theorem 2.5.7) we have

$$\begin{aligned}
&\mathcal{S}^*\left((H(F))x,\mathrm{Sym}^k(CN^{\mathfrak{g}^\sigma}_{\mathrm{Ad}(H(F))x})\right)^{H(F)\times F^\times,(1,\chi)}=\\
&=\mathcal{S}^*\left(\{x\},\mathrm{Sym}^k(CN^{\mathfrak{g}^\sigma}_{\mathrm{Ad}(H(F))x,x})\otimes\Delta_{H(F)\times F^\times}|_{(H(F)\times F^\times)_x}\cdot\Delta^{-1}_{(H(F)\times F^\times)_x}\otimes(1,\chi)\right)^{(H(F)\times F^\times)_x}=\\
&=\left(\mathrm{Sym}^k(\mathfrak{g}^\sigma/[x,\mathfrak{h}])\otimes\Delta_{(H(F)\times F^\times)_x}\otimes(1,\chi)^{-1}\otimes_{\mathbb{R}}\mathbb{C}\right)^{(H(F)\times F^\times)_x}\subset\\
&\subset\left(\mathrm{Sym}^k(\mathfrak{g}^\sigma/[x,\mathfrak{h}])\otimes\Delta_{(H(F)\times F^\times)_x}\otimes(1,\chi)^{-1}\otimes_{\mathbb{R}}\mathbb{C}\right)^{K}
\end{aligned}$$

which is zero since all the absolute values of the eigenvalues of the action of any $(D_t(x),t^2)\in K$ on

$$\mathrm{Sym}^k(\mathfrak{g}^\sigma/[x,\mathfrak{h}])\otimes\Delta_{(H(F)\times F^\times)_x}\otimes(1,\chi)^{-1}$$

are of the form $|t|^l$ where $l<0$. ☐

7.4. **Regular symmetric pairs.**

In this subsection we will formulate a property which is weaker than weakly linearly tame but still enables us to prove the GK property for good pairs.

Definition 7.4.1. *Let (G,H,θ) be a symmetric pair. We call an element $g\in G(F)$* ***admissible*** *if*
(i) $\mathrm{Ad}(g)$ *commutes with θ (or, equivalently, $s(g)\in Z(G)$) and*
(ii) $\mathrm{Ad}(g)|_{\mathfrak{g}^\sigma}$ *is H-admissible.*

Definition 7.4.2. *We call a symmetric pair (G,H,θ)* ***regular*** *if for any admissible $g\in G(F)$ such that $\mathcal{S}^*(R(\mathfrak{g}^\sigma))^{H(F)}\subset\mathcal{S}^*(R(\mathfrak{g}^\sigma))^{\mathrm{Ad}(g)}$ we have*

$$\mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{H(F)}\subset\mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{\mathrm{Ad}(g)}.$$

Remark 7.4.3. *Clearly, every weakly linearly tame pair is regular.*

Proposition 7.4.4. *Let (G_1,H_1,θ_1) and (G_2,H_2,θ_2) be regular symmetric pairs. Then their product $(G_1\times G_2,H_1\times H_2,\theta_1\times\theta_2)$ is also a regular pair.*

Proof. This follows from Proposition 2.5.8, since a product of admissible elements is admissible, and $R(\mathfrak{g}_1^{\sigma_2})\times R(\mathfrak{g}_2^{\sigma_2})$ is an open saturated subset of $R((\mathfrak{g}_1\times\mathfrak{g}_2)^{\sigma_1\times\sigma_2})$. ☐

The goal of this subsection is to prove the following theorem.

Theorem 7.4.5. *Let (G, H, θ) be a good symmetric pair such that all its descendants are regular. Then it is a GK-pair.*

We will need several definitions and lemmas.

Definition 7.4.6. *Let (G, H, θ) be a symmetric pair. An element $g \in G$ is called* ***normal*** *if g commutes with $\sigma(g)$.*

Note that if g is normal then

$$g\sigma(g)^{-1} = \sigma(g)^{-1}g \in H.$$

The following lemma is straightforward.

Lemma 7.4.7. *Let (G, H, θ) be a symmetric pair. Then any σ-invariant $H(F) \times H(F)$-orbit in $G(F)$ contains a normal element.*

Proof.
Let $g' \in O$. We know that $\sigma(g') = h_1 g' h_2$ where $h_1, h_2 \in H(F)$. Let $g := g'h_1$. Then

$$\sigma(g)g = h_1^{-1}\sigma(g')g'h_1 = h_1^{-1}\sigma(g')\sigma(\sigma(g'))h_1 = \\ = h_1^{-1}h_1g'h_2\sigma(h_1g'h_2))h_1 = g'\sigma(g') = g'h_1h_1^{-1}\sigma(g') = g\sigma(g).$$

Thus g in O is normal. □

Notation 7.4.8. *Let (G, H, θ) be a symmetric pair. We denote*

$$\widetilde{H \times H} := H \times H \rtimes \{1, \sigma\}$$

where

$$\sigma \cdot (h_1, h_2) = (\theta(h_2), \theta(h_1)) \cdot \sigma.$$

The two-sided action of $H \times H$ on G is extended to an action of $\widetilde{H \times H}$ in the natural way. We denote by χ the character of $\widetilde{H \times H}$ defined by

$$\chi(\widetilde{H \times H} - H \times H) = \{-1\}, \quad \chi(H \times H) = \{1\}.$$

Proposition 7.4.9. *Let (G, H, θ) be a good symmetric pair. Let $O \subset G(F)$ be a closed $H(F) \times H(F)$-orbit.*

Then for any $g \in O$ there exist $\tau \in (\widetilde{H \times H})_g(F) - (H \times H)_g(F)$ and $g' \in G_{s(g)}(F)$ such that $\mathrm{Ad}(g')$ commutes with θ on $G_{s(g)}$ and the action of τ on $N^G_{O,g}$ corresponds via the isomorphism given by Proposition 7.2.1 to the adjoint action of g' on $\mathfrak{g}^\sigma_{s(g)}$.

Proof. Clearly, if the statement holds for some $g \in O$ then it holds for all $g \in O$.

Let $g \in O$ be a normal element. Let $h := g\sigma(g)^{-1}$. Recall that $h \in H(F)$ and $gh = hg = \sigma(g)$. Let $\tau := (h^{-1}, 1) \cdot \sigma$. Evidently, $\tau \in (\widetilde{H \times H})_g(F) - (H \times H)_g(F)$. Consider $d\tau_g : T_gG \to T_gG$. It corresponds via the identification $dg : \mathfrak{g} \cong T_gG$ to some $A : \mathfrak{g} \to \mathfrak{g}$. Clearly, $A = da$ where $a : G \to G$ is defined by $a(\alpha) = g^{-1}h^{-1}\sigma(g\alpha)$. However, $g^{-1}h^{-1}\sigma(g\alpha) = \theta(g)\sigma(\alpha)\theta(g)^{-1}$. Hence $A = \mathrm{Ad}(\theta(g)) \circ \sigma$. By Lemma 7.1.9, there exists a non-degenerate G-invariant σ-invariant symmetric bilinear form B on $\mathfrak{g}$. By Theorem 7.1.3, A preserves B. Therefore τ corresponds to $A|_{\mathfrak{g}^\sigma_{s(g)}}$ via the isomorphism given by Proposition 7.2.1. However, σ is trivial on $\mathfrak{g}^\sigma_{s(g)}$ and hence $A|_{\mathfrak{g}^\sigma_{s(g)}} = \mathrm{Ad}(\theta(g))|_{\mathfrak{g}^\sigma_{s(g)}}$. Since g is normal, $\theta(g) \in G_{s(g)}$. It is easy to see that $\mathrm{Ad}(\theta(g))$ commutes with θ on $G_{s(g)}$. Hence we take $g' := \theta(g)$. □

Now we are ready to prove Theorem 7.4.5.

Proof of Theorem 7.4.5. We have to show that $\mathcal{S}^*(G(F))^{\widetilde{H \times H}, \chi} = 0$. By Theorem 3.2.2 it is enough to show that for any $H \times H$-semisimple $x \in G(F)$ such that

$$\mathcal{D}(R(N^G_{HxH,x}))^{(\widetilde{H(F) \times H(F)})_x, \chi} = 0$$

we have

$$\mathcal{D}(Q(N^G_{HxH,x}))^{(\widetilde{H(F)\times H(F)})_x,\chi} = 0.$$

This follows immediately from the regularity of the pair (G_x, H_x) using the last proposition. □

7.5. **Conjectures.**

Conjecture 1 (van Dijk). *If $F = \mathbb{C}$ then any connected symmetric pair is a Gelfand pair (GP3, see Definition 8.1.2 below).*

By Theorem 8.1.5 this would follow from the following stronger conjecture.

Conjecture 2. *If $F = \mathbb{C}$ then any connected symmetric pair is a GK-pair.*

By Corollary 7.1.7 this in turn would follow from the following more general conjecture.

Conjecture 3. *Every good symmetric pair is a GK-pair.*

which in turn follows (by Theorem 7.4.5) from the following one.

Conjecture 4. *Any symmetric pair is regular.*

Remark 7.5.1. *In the next two subsections we prove this conjecture for certain symmetric pairs. In subsequent works* [AG08c, Say08a, AS08, Say08b, Aiz08] *this conjecture was verified for most classical symmetric pairs and several exceptional ones.*

Remark 7.5.2. *An indirect evidence for this conjecture is that every GK-pair is regular. One can easily show this by analyzing a Luna slice for an orbit of an admissible element.*

Remark 7.5.3. *It is well known that if F is Archimedean, G is connected and H is compact then the pair (G, H, θ) is good, Gelfand (GP1, see Definition 8.1.2 below) and in fact also GK. See e.g.* [Yak04].

Remark 7.5.4. *In general, not every symmetric pair is good. For example, $(\mathrm{SL}_2(\mathbb{R}), T)$ where T is the split torus. Also, it is not a Gelfand pair (not even GP3, see Definition 8.1.2 below).*

Remark 7.5.5. *It seems unlikely that every symmetric pair is special. However, in the next two subsections we will prove that certain symmetric pairs are special.*

7.6. **The pairs $(G \times G, \Delta G)$ and $(G_{E/F}, G)$ are tame.**

Notation 7.6.1. *Let E be a quadratic extension of F. Let G be an algebraic group defined over F. We denote by $G_{E/F}$ the restriction of scalars from E to F of G viewed as a group over E. Thus, $G_{E/F}$ is an algebraic group defined over F and $G_{E/F}(F) = G(E)$.*

In this section we will prove the following theorem.

Theorem 7.6.2. *Let G be a reductive group.*
(i) Consider the involution θ of $G \times G$ given by $\theta((g, h)) := (h, g)$. Its fixed points form the diagonal subgroup ΔG. Then the symmetric pair $(G \times G, \Delta G, \theta)$ is tame.
(ii) Let E be a quadratic extension of F. Consider the involution γ of $G_{E/F}$ given by the nontrivial element of $Gal(E/F)$. Its fixed points form G. Then the symmetric pair $(G_{E/F}, G, \gamma)$ is tame.

Corollary 7.6.3. *Let G be a reductive group. Then the adjoint action of G on itself is tame. In particular, every conjugation invariant distribution on $\mathrm{GL}_n(F)$ is transposition invariant* [6].

For the proof of the theorem we will need the following straightforward lemma.

Lemma 7.6.4.
(i) Every descendant of $(G \times G, \Delta G, \theta)$ is of the form $(H \times H, \Delta H, \theta)$ for some reductive group H.
(ii) Every descendant of $(G_{E/F}, G, \gamma)$ is of the form $(H_{E/F}, H, \gamma)$ for some reductive group H.

Now in view of Theorem 7.4.5, Theorem 7.6.2 follows from the following theorem.

[6] In the non-Archimedean case, the latter is a classical result of Gelfand and Kazhdan, see [GK75].

Theorem 7.6.5. *The pairs* $(G \times G, \Delta G, \theta)$ *and* $(G_{E/F}, G, \gamma)$ *are special for any reductive group* G.

By the speciality criterion (Proposition 7.3.7) this theorem follows from the following lemma.

Lemma 7.6.6. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Let* $\{e, h, f\} \subset \mathfrak{g}$ *be an* $\mathfrak{sl}_2$ *triple. Then* $\mathrm{Tr}(\mathrm{ad}(h)|_{\mathfrak{g}_e})$ *is an integer smaller than* $\dim \mathfrak{g}$.

Proof. Consider $\mathfrak{g}$ as a representation of $\mathfrak{sl}_2$ via the triple (e, h, f). Decompose it into irreducible representations $\mathfrak{g} = \bigoplus V_i$. Let λ_i be the highest weights of V_i. Clearly

$$\mathrm{Tr}(\mathrm{ad}(h)|_{\mathfrak{g}_e}) = \sum \lambda_i \text{ while } \dim \mathfrak{g} = \sum (\lambda_i + 1).$$

□

7.7. **The pair** $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ **is a GK pair.**

Notation 7.7.1. *We define an involution* $\theta_{n,k} : \mathrm{GL}_{n+k} \to \mathrm{GL}_{n+k}$ *by* $\theta_{n,k}(x) = \varepsilon x \varepsilon$ *where* $\varepsilon = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix}$. *Note that* $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k, \theta_{n,k})$ *is a symmetric pair. If there is no ambiguity we will denote* $\theta_{n,k}$ *simply by* θ.

Theorem 7.7.2. *The pair* $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k, \theta_{n,k})$ *is a GK-pair.*

By Theorem 7.4.5 it is enough to prove that our pair is good and all its descendants are regular.
In §§§7.7.1 we compute the descendants of our pair and show that the pair is good.
In §§§7.7.2 we prove that all the descendants are regular.

7.7.1. *The descendants of the pair* $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$.

Theorem 7.7.3. *All the descendants of the pair* $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k, \theta_{n,k})$ *are products of pairs of the types*
(i) $((\mathrm{GL}_m)_{E/F} \times (\mathrm{GL}_m)_{E/F}, \Delta(\mathrm{GL}_m)_{E/F}, \theta)$ *for some field extension* E/F
(ii) $((\mathrm{GL}_m)_{E/F}, (\mathrm{GL}_m)_{L/F}, \gamma)$ *for some field extension* L/F *and its quadratic extension* E/L
(iii) $(\mathrm{GL}_{m+l}, \mathrm{GL}_m \times \mathrm{GL}_l, \theta_{m,l})$.

Proof. Let $x \in \mathrm{GL}_{n+k}^{\sigma}(F)$ be a semisimple element. We have to compute G_x and H_x. Since $x \in G^{\sigma}$, we have $\varepsilon x \varepsilon = x^{-1}$. Let $V = F^{n+k}$. Decompose $V := \bigoplus_{i=1}^{s} V_i$ such that the minimal polynomial of $x|_{V_i}$ is irreducible. Now $G_x(F)$ decomposes as a product of $\mathrm{GL}_{E_i}(V_i)$, where E_i is the extension of F defined by the minimal polynomial of $x|_{V_i}$ and the E_i-vector space structure on V_i is given by x.

Clearly, ε permutes the V_i's. Now we see that V is a direct sum of spaces of the following two types
A. $W_1 \oplus W_2$ such that the minimal polynomials of $x|_{W_i}$ are irreducible and $\varepsilon(W_1) = W_2$.
B. W such that the minimal polynomial of $x|_W$ is irreducible and $\varepsilon(W) = W$.

It is easy to see that in case A we get the symmetric pair (i).

In case B there are two possibilities: either $x = x^{-1}$ or $x \neq x^{-1}$. It is easy to see that these cases correspond to types (iii) and (ii) respectively. □

Corollary 7.7.4. *The pair* $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ *is good.*

Proof. Theorem 7.7.3 implies that for any $(\mathrm{GL}_n \times \mathrm{GL}_k) \times (\mathrm{GL}_n \times \mathrm{GL}_k)$-semisimple element $x \in \mathrm{GL}_{n+k}(F)$, the stabilizer $((\mathrm{GL}_n \times \mathrm{GL}_k) \times (\mathrm{GL}_n \times \mathrm{GL}_k))_x$ is a product of groups of types $(\mathrm{GL}_m)_{E/F}$ for some extensions E/F. Hence $H^1(F, ((\mathrm{GL}_n \times \mathrm{GL}_k) \times (\mathrm{GL}_n \times \mathrm{GL}_k))_x) = 0$ and hence by Corollary 7.1.5 the pair $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ is good. □

7.7.2. *All the descendants of the pair* $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ *are regular.*
Clearly, for any field extension E/F, if a pair (G, H, θ) is regular as a symmetric pair over E then the pair $(G_{E/F}, H_{E/F}, \theta)$ is regular. Therefore by Theorem 7.7.3 and Theorem 7.6.2 it is enough to prove that the pair $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k, \theta_{n,k})$ is regular as a symmetric pair over F.

In the case $n \neq k$ this follows from the definition since in this case the normalizer of $\mathrm{GL}_n \times \mathrm{GL}_k$ in GL_{k+n} is $\mathrm{GL}_n \times \mathrm{GL}_k$ and hence, any admissible $g \in \mathrm{GL}_{n+k}$ lies in $\mathrm{GL}_n \times \mathrm{GL}_k$.

So we can assume $n = k > 0$. Hence by Proposition 7.3.7 it suffices to prove the following Key Lemma.

Lemma 7.7.5 (Key Lemma)**.** [7] *Let $x \in \mathfrak{gl}_{2n}^{\sigma}(F)$ be a nilpotent element and $d := d(x)$. Then*

$$\mathrm{Tr}(\mathrm{ad}(d)|_{(\mathfrak{gl}_n(F)\times\mathfrak{gl}_n(F))_x}) < 2n^2.$$

We will need the following definition and lemmas.

Definition 7.7.6. *We fix a grading on $\mathfrak{sl}_2(F)$ given by $h \in \mathfrak{sl}_2(F)_0$ and $e, f \in \mathfrak{sl}_2(F)_1$ where (e, h, f) is the standard $\mathfrak{sl}_2$-triple. A* ***graded representation of*** $\mathfrak{sl}_2$ *is a representation of $\mathfrak{sl}_2$ on a graded vector space $V = V_0 \oplus V_1$ such that $\mathfrak{sl}_2(F)_i(V_j) \subset V_{i+j}$ where $i, j \in \mathbb{Z}/2\mathbb{Z}$.*

The following lemma is standard.

Lemma 7.7.7.
(i) Every irreducible graded representation of $\mathfrak{sl}_2$ is irreducible (as a usual representation of $\mathfrak{sl}_2$).
(ii) Every irreducible representation V of $\mathfrak{sl}_2$ admits exactly two gradings. In one grading the highest weight vector lies in V_0 and in the other grading it lies in V_1.

Notation 7.7.8. *Denote by V_λ^w be the irreducible graded representation of $\mathfrak{sl}_2$ with highest weight λ and highest weight vector of parity $w \in \mathbb{Z}/2\mathbb{Z}$.*

Lemma 7.7.9. [8] *Consider* $\mathrm{Hom}((V_{\lambda_1}^{w_1}, V_{\lambda_2}^{w_2})^e)_0$ *- the even part of the space of e-equivariant linear maps $V_{\lambda_1}^{w_1} \to V_{\lambda_2}^{w_2}$. Let $r_i := \dim V_{\lambda_i}^{w_i} = \lambda_i + 1$ and let*

$$m := \mathrm{Tr}(h|_{(\mathrm{Hom}((V_{\lambda_1}^{w_1}, V_{\lambda_2}^{w_2})^e)_0}) + \mathrm{Tr}(h|_{\mathrm{Hom}((V_{\lambda_2}^{w_2}, V_{\lambda_1}^{w_1})^e)_0}) - r_1 r_2.$$

Then

$$m = \begin{cases} -\min(r_1, r_2), & \text{if } r_1 \neq r_2 \pmod 2; \\ -2\min(r_1, r_2), & \text{if } r_1 \equiv r_2 \equiv 0 \pmod 2 \text{ and } w_1 = w_2; \\ 0, & \text{if } r_1 \equiv r_2 \equiv 0 \pmod 2 \text{ and } w_1 \neq w_2; \\ |r_1 - r_2| - 1, & \text{if } r_1 \equiv r_2 \equiv 1 \pmod 2 \text{ and } w_1 = w_2; \\ -(r_1 + r_2 - 1), & \text{if } r_1 \equiv r_2 \equiv 1 \pmod 2 \text{ and } w_1 \neq w_2; \end{cases}$$

This lemma follows by a direct computation from the following straightforward lemma.

Lemma 7.7.10. *One has*

$$\mathrm{Tr}(h|_{((V_\lambda^w)^e)_0}) = \begin{cases} \lambda, & \text{if } w = 0 \\ 0, & \text{if } w = 1 \end{cases} \tag{1}$$

$$(V_\lambda^w)^* = V_\lambda^{w+\lambda} \tag{2}$$

$$V_{\lambda_1}^{w_1} \otimes V_{\lambda_2}^{w_2} = \bigoplus_{i=0}^{\min(\lambda_1,\lambda_2)} V_{\lambda_1+\lambda_2-2i}^{w_1+w_2+i}. \tag{3}$$

Proof of the Key Lemma. Let $V_0 := V_1 := F^n$. Let $V := V_0 \oplus V_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. We consider $\mathfrak{gl}_{2n}(F)$ as the $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra $\mathrm{End}(V)$. Note that $\mathfrak{gl}_n(F) \times \mathfrak{gl}_n(F)$ is the even part of $\mathrm{End}(V)$ with respect to this grading. Consider V as a graded representation of the $\mathfrak{sl}_2$ triple (x, d, x_-). Decompose V into graded irreducible representations W_i. Let $r_i := \dim W_i$ and w_i be the parity of the highest weight vector of W_i. Note that if r_i is even then $\dim(W_i \cap V_0) = \dim(W_i \cap V_1)$. If r_i is odd then $\dim(W_i \cap V_0) = \dim(W_i \cap V_1) + (-1)^{w_i}$. Since $\dim V_0 = \dim V_1$, we get that the number of indices i such that r_i is odd and $w_i = 0$ is equal to the number of indices i such that r_i is odd and $w_i = 1$. We denote this number by l. Now

$$\mathrm{Tr}(\mathrm{ad}(d)|_{(\mathfrak{gl}_n(F)\times\mathfrak{gl}_n(F))_x}) - 2n^2 = \mathrm{Tr}(d|_{(\mathrm{Hom}(V,V)^x)_0}) - 2n^2 = \frac{1}{2}\sum_{i,j} m_{ij},$$

where

$$m_{ij} := \mathrm{Tr}(d|_{(\mathrm{Hom}(W_i,W_j)^x)_0}) + \mathrm{Tr}(d|_{(\mathrm{Hom}(W_j,W_i)^x)_0}) - r_i r_j.$$

The m_{ij} can be computed using Lemma 7.7.9.

[7]This Lemma is similar to [JR96, §§3.2, Lemma 3.1]. The proofs are also similar.
[8]This Lemma is similar to [JR96, Lemma 3.2] but computes a different quantity.

As we see from the lemma, if either r_i or r_j is even then m_{ij} is non-positive and m_{ii} is negative. Therefore, if all r_i are even then we are done. Otherwise $l > 0$ and we can assume that all r_i are odd. Reorder the spaces W_i so that $w_i = 0$ for $i \leq l$ and $w_i = 1$ for $i > l$. Now

$$\sum_{1\leq i,j\leq 2l} m_{ij} = \sum_{i\leq l, j\leq l} (|r_i - r_j| - 1) + \sum_{i>l, j>l} (|r_i - r_j| - 1) - \sum_{i\leq l, j>l} (r_i + r_j - 1) - \sum_{i>l, j\leq l} (r_i + r_j - 1) =$$
$$= \sum_{i\leq l, j\leq l} |r_i - r_j| + \sum_{i>l, j>l} |r_i - r_j| - \sum_{i\leq l, j>l} (r_i + r_j) - \sum_{i>l, j\leq l} (r_i + r_j) <$$
$$< \sum_{i\leq l, j\leq l} (r_i + r_j) + \sum_{i>l, j>l} (r_i + r_j) - \sum_{i\leq l, j>l} (r_i + r_j) - \sum_{i>l, j\leq l} (r_i + r_j) = 0.$$

The Lemma follows. □

8. Applications to Gelfand pairs

8.1. **Preliminaries on Gelfand pairs and distributional criteria.**

In this section we recall a technique due to Gelfand-Kazhdan which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS08, §2].

Definition 8.1.1. *Let G be a reductive group. By an* ***admissible representation of*** *G we mean an admissible representation of $G(F)$ if F is non-Archimedean (see* [BZ76]*) and admissible smooth Fréchet representation of $G(F)$ if F is Archimedean.*

We now introduce three a-priori distinct notions of Gelfand pair.

Definition 8.1.2. *Let $H \subset G$ be a pair of reductive groups.*

- *We say that (G, H) satisfy* **GP1** *if for any irreducible admissible representation (π, E) of G we have*
$$\dim \operatorname{Hom}_{H(F)}(E, \mathbb{C}) \leq 1.$$
- *We say that (G, H) satisfy* **GP2** *if for any irreducible admissible representation (π, E) of G we have*
$$\dim \operatorname{Hom}_{H(F)}(E, \mathbb{C}) \cdot \dim \operatorname{Hom}_{H}(\widetilde{E}, \mathbb{C}) \leq 1.$$
- *We say that (G, H) satisfy* **GP3** *if for any irreducible* **unitary** *representation $(\pi, \mathcal{H})$ of $G(F)$ on a Hilbert space $\mathcal{H}$ we have*
$$\dim \operatorname{Hom}_{H(F)}(\mathcal{H}^{\infty}, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK75]). Property GP2 was introduced in [Gro91] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vDP90], [vD86], [BvD94]).

We have the following straightforward proposition.

Proposition 8.1.3. *$GP1 \Rightarrow GP2 \Rightarrow GP3$.*

Remark 8.1.4. *It is not known whether some of these notions are equivalent.*

We will use the following theorem from [AGS08] which is a version of a classical theorem of Gelfand and Kazhdan (see [GK75]).

Theorem 8.1.5. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(F)$-invariant Schwartz distributions ξ on $G(F)$. Then (G, H) satisfies GP2.*

Corollary 8.1.6. *Any symmetric GK-pair satisfies GP2.*

In some cases, GP2 is known to be equivalent to GP1. For example, see Corollary 8.2.3 below.

8.2. Applications to Gelfand pairs.

Theorem 8.2.1. *Let G be a reductive group and let σ be an $\mathrm{Ad}(G)$-admissible anti-automorphism of G. Let θ be the automorphism of G defined by $\theta(g) := \sigma(g^{-1})$. Let (π, E) be an irreducible admissible representation of G.*

Then $\widetilde{E} \cong E^{\theta}$, where $\widetilde{E}$ denotes the smooth contragredient representation and E^{θ} is E twisted by θ.

Proof. By Corollary 7.6.3, the characters of $\widetilde{E}$ and E^{θ} are identical. Since these representations are irreducible, this implies that they are isomorphic (see e.g. [Wal88, Theorem 8.1.5]). ☐

Remark 8.2.2. *This theorem has an alternative proof using Harish-Chandra's Regularity Theorem, which says that the character of an admissible representation is a locally integrable function.*

Corollary 8.2.3. *Let $H \subset G$ be reductive groups and let τ be an $\mathrm{Ad}(G)$-admissible anti-automorphism of G such that $\tau(H) = H$. Then $GP1$ is equivalent to $GP2$ for the pair (G, H).*

This corollary, together with Corollary 8.1.6 and Theorem 7.7.2 implies the following result.

Theorem 8.2.4. *The pair $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ satisfies GP1.*

For non-Archimedean F this theorem is proven in [JR96].

Theorem 8.2.5. *Let E be a quadratic extension of F. Then the pair $((\mathrm{GL}_n)_{E/F}, \mathrm{GL}_n)$ satisfies GP1.*

For non-Archimedean F this theorem is proven in [Fli91].

Proof. By Theorem 7.6.2 this pair is tame. Hence it is enough to show that this symmetric pair is good. Consider the adjoint action of GL_n on itself. Let $x \in \mathrm{GL}_n(E)^{\sigma}$ be semisimple. The stabilizer $(\mathrm{GL}_n)_x$ is a product of groups of the form $(\mathrm{GL}_n)_{F'/F}$ for some extensions F'/F. Hence $H^1(F, (\mathrm{GL}_n)_x) = 0$. Therefore, by Corollary 7.1.5, the symmetric pair in question is good. ☐

Part 3. Appendices

Appendix A. Algebraic geometry over local fields

A.1. Implicit Function Theorems.

Definition A.1.1. *An analytic map $\phi : M \to N$ is called **étale** if $d_x\phi : T_xM \to T_{\phi(x)}N$ is an isomorphism for any $x \in M$. An analytic map $\phi : M \to N$ is called a **submersion** if $d_x\phi : T_xM \to T_{\phi(x)}N$ is onto for any $x \in M$.*

We will use the following version of the Inverse Function Theorem.

Theorem A.1.2 (cf. [Ser64], Theorem 2 in §9 of Chapter III in part II)**.** *Let $\phi : M \to N$ be an étale map of analytic manifolds. Then it is locally an isomorphism.*

Corollary A.1.3. *Let $\phi : X \to Y$ be a morphism of (not necessarily smooth) algebraic varieties. Suppose that ϕ is étale at $x \in X(F)$. Then there exists an open neighborhood $U \subset X(F)$ of x such that $\phi|_U$ is a homeomorphism to its open image in $Y(F)$.*

For the proof see e.g. [Mum99, Chapter III, §5, proof of Corolary 2]. There, the proof is given for the case $F = \mathbb{C}$ but it works in general.

Remark A.1.4. *If F is Archimedean then one can choose U to be semi-algebraic.*

The following proposition is well known (see e.g. §10 of Chapter III in part II of [Ser64]).

Proposition A.1.5. *Any submersion $\phi : M \to N$ is open.*

Corollary A.1.6. *Lemma 2.3.4 holds. Namely, for any algebraic group G and a closed algebraic subgroup $H \subset G$ the subset $G(F)/H(F)$ is open and closed in $(G/H)(F)$.*

Proof. Consider the map $\phi : G(F) \to (G/H)(F)$ defined by $\phi(g) = gH$. Clearly, it is a submersion and its image is exactly $G(F)/H(F)$. Hence, $G(F)/H(F)$ is open. Since each $G(F)$-orbit in $(G/H)(F)$ is open for the same reason, $G(F)/H(F)$ is also closed. ☐

A.2. **The Luna Slice Theorem.**
In this subsection we formulate the Luna Slice Theorem and show how it implies Theorem 2.3.17. For a survey on the Luna Slice Theorem we refer the reader to [Dre00] and the original paper [Lun73].

Definition A.2.1 (cf. [Dre00])**.** *Let a reductive group G act on affine varieties X and Y. A G-equivariant algebraic map $\phi : X \to Y$ is called* ***strongly étale*** *if*
(i) $\phi/G : X/G \to Y/G$ is étale
(ii) ϕ and the quotient morphism $\pi_X : X \to X/G$ induce a G-isomorphism $X \cong Y \times_{Y/G} X/G$.

Definition A.2.2. *Let G be a reductive group and H be a closed reductive subgroup. Suppose that H acts on an affine variety X. Then $G \times_H X$ denotes $(G \times X)/H$ with respect to the action $h(g, x) = (gh^{-1}, hx)$.*

Theorem A.2.3 (Luna Slice Theorem)**.** *Let a reductive group G act on a smooth affine variety X. Let $x \in X$ be G-semisimple.*

Then there exists a locally closed smooth affine G_x-invariant subvariety $Z \ni x$ of X and a strongly étale algebraic map of G_x spaces $\nu : Z \to N^X_{Gx,x}$ such that the G-morphism $\phi : G \times_{G_x} Z \to X$ induced by the action of G on X is strongly étale .

Proof. It follows from [Dre00, Proposition 4.18, Lemma 5.1 and Theorems 5.2 and 5.3], noting that one can choose Z and ν (in our notation) to be defined over F. □

Corollary A.2.4. *Theorem 2.3.17 holds. Namely:*
Let a reductive group G act on a smooth affine variety X. Let $x \in X(F)$ be G-semisimple.

Then there exist
(i) an open $G(F)$-invariant B-analytic neighborhood U of $G(F)x$ in $X(F)$ with a G-equivariant B-analytic retract $p : U \to G(F)x$ and
(ii) a G_x-equivariant B-analytic embedding $\psi : p^{-1}(x) \hookrightarrow N^X_{Gx,x}(F)$ with an open saturated image such that $\psi(x) = 0$.

Proof. Let Z, ϕ and ν be as in the last theorem.

Let $Z' := Z/G_x \cong (G \times_{G_x} Z)/G$ and $X' := X/G$. Consider the natural map $\phi' : Z'(F) \to X'(F)$. By Corollary A.1.3 there exists a neighborhood $S' \subset Z'(F)$ of $\pi_Z(x)$ such that $\phi'|_{S'}$ is a homeomorphism to its open image.

Consider the natural map $\nu' : Z'(F) \to N^X_{Gx,x}/G_x(F)$. Let $S'' \subset Z(F)$ be a neighborhood of $\pi_Z(x)$ such that $\nu'|_{S''}$ is an isomorphism to its open image. In case that F is Archimedean we choose S' and S'' to be semi-algebraic.

Let $S := \pi_Z^{-1}(S'' \cap S') \cap Z(F)$. Clearly, S is B-analytic.

Let $\rho : (G \times_{G_x} Z)(F) \to Z'(F)$ be the natural projection. Let $O = \rho^{-1}(S'' \cap S')$. Let $q : O \to (G/G_x)(F)$ be the natural map. Let $O' := q^{-1}(G(F)/G_x(F))$ and $q' := q|_{O'}$.

Now put $U := \phi(O')$ and put $p : U \to G(F)x$ be the morphism that corresponds to q'. Note that $p^{-1}(x) \cong S$ and put $\psi : p^{-1}(x) \to N^X_{Gx,x}(F)$ to be the imbedding that corresponds to $\nu|_S$. □

Appendix B. Schwartz distributions on Nash manifolds

B.1. **Preliminaries and notation.**
In this appendix we will prove some properties of K-equivariant Schwartz distributions on Nash manifolds. We work in the notation of [AG08a], where one can read about Nash manifolds and Schwartz distributions over them. More detailed references on Nash manifolds are [BCR98] and [Shi87].

Nash manifolds are equipped with the **restricted topology**, in which open sets are open semi-algebraic sets. This is not a topology in the usual sense of the word as infinite unions of open sets are not necessarily open sets in the restricted topology. However, finite unions of open sets are open and therefore in the restricted topology we consider only finite covers. In particular, if $E \to M$ is a Nash vector bundle it means that there exists a finite open cover U_i of M such that $E|_{U_i}$ is trivial.

Notation B.1.1. *Let M be a Nash manifold. We denote by D_M the Nash bundle of densities on M. It is the natural bundle whose smooth sections are smooth measures. For the precise definition see e.g.* [AG08a].

An important property of Nash manifolds is

Theorem B.1.2 (Local triviality of Nash manifolds; [Shi87], Theorem I.5.12)**.** *Any Nash manifold can be covered by a finite number of open submanifolds Nash diffeomorphic to* $\mathbb{R}^n$*.*

Definition B.1.3. *Let M be a Nash manifold. We denote by $\mathcal{G}(M) := \mathcal{S}^*(M, D_M)$ the* ***space of Schwartz generalized functions*** *on M. Similarly, for a Nash bundle $E \to M$ we denote by $\mathcal{G}(M, E) := \mathcal{S}^*(M, E^* \otimes D_M)$ the* ***space of Schwartz generalized sections*** *of E.*

In the same way, for any smooth manifold M we denote by $C^{-\infty}(M) := \mathcal{D}(M, D_M)$ the ***space of generalized functions*** *on M and for a smooth bundle $E \to M$ we denote by $C^{-\infty}(M, E) := \mathcal{D}(M, E^* \otimes D_M)$ the* ***space of generalized sections*** *of E.*

Usual L^1-functions can be interpreted as Schwartz generalized functions but not as Schwartz distributions. We will need several properties of Schwartz functions from [AG08a].

Property B.1.4 ([AG08a], Theorem 4.1.3)**.** $\mathcal{S}(\mathbb{R}^n)$ *= Classical Schwartz functions on* $\mathbb{R}^n$*.*

Property B.1.5 ([AG08a], Theorem 5.4.3)**.** *Let $U \subset M$ be a (semi-algebraic) open subset, then*

$$\mathcal{S}(U, E) \cong \{\phi \in \mathcal{S}(M, E) | \quad \phi \text{ is 0 on } M \setminus U \text{ with all derivatives}\}.$$

Property B.1.6 (see [AG08a], §5)**.** *Let M be a Nash manifold. Let $M = \bigcup U_i$ be a finite open cover of M. Then a function f on M is a Schwartz function if and only if it can be written as $f = \sum_{i=1}^{n} f_i$ where $f_i \in \mathcal{S}(U_i)$ (extended by zero to M).*

Moreover, there exists a smooth partition of unity $1 = \sum_{i=1}^{n} \lambda_i$ such that for any Schwartz function $f \in \mathcal{S}(M)$ the function $\lambda_i f$ is a Schwartz function on U_i (extended by zero to M).

Property B.1.7 (see [AG08a], §5)**.** *Let M be a Nash manifold and E be a Nash bundle over it. Let $M = \bigcup U_i$ be a finite open cover of M. Let $\xi_i \in \mathcal{G}(U_i, E)$ such that $\xi_i|_{U_j} = \xi_j|_{U_i}$. Then there exists a unique $\xi \in \mathcal{G}(M, E)$ such that $\xi|_{U_i} = \xi_i$.*

We will also use the following notation.

Notation B.1.8. *Let M be a metric space and $x \in M$. We denote by $B(x, r)$ the open ball with center x and radius r.*

B.2. **Submersion principle.**

Theorem B.2.1 ([AG08b], Theorem 2.4.16)**.** *Let M and N be Nash manifolds and $s : M \to N$ be a surjective submersive Nash map. Then locally it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that s has a Nash section on each U_i.*

Corollary B.2.2. *An étale map $\phi : M \to N$ of Nash manifolds is locally an isomorphism. That means that there exists a finite cover $M = \bigcup U_i$ such that $\phi|_{U_i}$ is an isomorphism onto its open image.*

Theorem B.2.3. *Let $p : M \to N$ be a Nash submersion of Nash manifolds. Then there exist a finite open (semi-algebraic) cover $M = \bigcup U_i$ and isomorphisms $\phi_i : U_i \cong W_i$ and $\psi_i : p(U_i) \cong V_i$ where $W_i \subset \mathbb{R}^{d_i}$ and $V_i \subset \mathbb{R}^{k_i}$ are open (semi-algebraic) subsets, $k_i \leq d_i$ and $p|_{U_i}$ correspond to the standard projections.*

Proof. The problem is local, hence without loss of generality we can assume that $N = \mathbb{R}^k$, M is an equidimensional closed submanifold of $\mathbb{R}^n$ of dimension d, $d \geq k$, and p is given by the standard projection $\mathbb{R}^n \to \mathbb{R}^k$.

Let Ω be the set of all coordinate subspaces of $\mathbb{R}^n$ of dimension d which contain N. For any $V \in \Omega$ consider the projection $pr : M \to V$. Define $U_V = \{x \in M | d_x pr \text{ is an isomorphism } \}$. It is easy to see that $pr|_{U_V}$ is étale and $\{U_V\}_{V \in \Omega}$ gives a finite cover of M. The theorem now follows from the previous corollary (Corollary B.2.2). □

Theorem B.2.4. *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a Nash bundle over N. Then*
(i) there exists a unique continuous linear map $\phi_* : \mathcal{S}(M, \phi^*(E) \otimes D_M) \to \mathcal{S}(N, E \otimes D_N)$ such that for any $f \in \mathcal{S}(N, E^*)$ and $\mu \in \mathcal{S}(M, \phi^*(E) \otimes D_M)$ we have

$$\int_{x \in N} \langle f(x), \phi_* \mu(x) \rangle = \int_{x \in M} \langle \phi^* f(x), \mu(x) \rangle.$$

In particular, we mean that both integrals converge.
(ii) If ϕ is surjective then ϕ_* is surjective.

Proof.
(i)
Step 1. Proof for the case when $M = \mathbb{R}^n$, $N = \mathbb{R}^k$, $k \leq n$, ϕ is the standard projection and E is trivial.
Fix Haar measure on $\mathbb{R}$ and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l. Define

$$\phi_*(f)(x) := \int_{y \in \mathbb{R}^{n-k}} f(x, y) dy.$$

Convergence of the integral and the fact that $\phi_*(f)$ is a Schwartz function follows from standard calculus.
Step 2. Proof for the case when $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ are open (semi-algebraic) subsets, ϕ is the standard projection and E is trivial.
Follows from the previous step and Property B.1.5.
Step 3. Proof for the case when E is trivial.
Follows from the previous step, Theorem B.2.3 and partition of unity (Property B.1.6).
Step 4. Proof in the general case.
Follows from the previous step and partition of unity (Property B.1.6).
(ii) The proof is the same as in (i) except of Step 2. Let us prove (ii) in the case of Step 2. Again, fix Haar measure on $\mathbb{R}$ and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l. By Theorem B.2.1 and partition of unity (Property B.1.6) we can assume that there exists a Nash section $\nu : N \to M$. We can write ν in the form $\nu(x) = (x, s(x))$.

For any $x \in N$ define $R(x) := \sup\{r \in \mathbb{R}_{\geq 0} | B(\nu(x), r) \subset M\}$. Clearly, R is continuous and positive. By Tarski - Seidenberg principle (see e.g. [AG08a, Theorem 2.2.3]) it is semi-algebraic. Hence (by [AG08a, Lemma A.2.1]) there exists a positive Nash function $r(x)$ such that $r(x) < R(x)$. Let $\rho \in \mathcal{S}(\mathbb{R}^{n-k})$ such that ρ is supported in the unit ball and its integral is 1. Now let $f \in \mathcal{S}(N)$. Let $g \in C^\infty(M)$ defined by $g(x, y) := f(x)\rho((y - s(x))/r(x))/r(x)$ where $x \in N$ and $y \in \mathbb{R}^{n-k}$. It is easy to see that $g \in \mathcal{S}(M)$ and $\phi_* g = f$. ☐

Notation B.2.5. *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a bundle on N. We denote by $\phi^* : \mathcal{G}(N, E) \to \mathcal{G}(M, \phi^*(E))$ the dual map to ϕ_*.*

Remark B.2.6. *Clearly, the map $\phi^* : \mathcal{G}(N, E) \to \mathcal{G}(M, \phi^*(E))$ extends to the map $\phi^* : C^{-\infty}(N, E) \to C^{-\infty}(M, \phi^*(E))$ described in* [AGS08, Theorem A.0.4].

Proposition B.2.7. *Let $\phi : M \to N$ be a surjective Nash submersion of Nash manifolds. Let E be a bundle on N. Let $\xi \in C^{-\infty}(N)$. Suppose that $\phi^*(\xi) \in \mathcal{G}(M)$. Then $\xi \in \mathcal{G}(N)$.*

Proof. It follows from Theorem B.2.4 and Banach Open Map Theorem (see [Rud73, Theorem 2.11]). ☐

B.3. **Frobenius reciprocity.**
In this subsection we prove Frobenius reciprocity for Schwartz functions on Nash manifolds.

Proposition B.3.1. *Let M be a Nash manifold. Let K be a Nash group. Let $E \to M$ be a Nash bundle. Consider the standard projection $p : K \times M \to M$. Then the map $p^* : \mathcal{G}(M, E) \to \mathcal{G}(M \times K, p^*E)^K$ is an isomorphism.*

This proposition follows from in [AG08b, Proposition 4.0.11].

Corollary B.3.2. *Let a Nash group K act on a Nash manifold M. Let E be a K-equivariant Nash bundle over M. Let $N \subset M$ be a Nash submanifold such that the action map $K \times N \to M$ is submersive. Then there exists a canonical map*

$$\mathrm{HC} : \mathcal{G}(M, E)^K \to \mathcal{G}(N, E|_N).$$

Theorem B.3.3. *Let a Nash group K act on a Nash manifold M. Let N be a K-transitive Nash manifold. Let $\phi : M \to N$ be a Nash K-equivariant map.*

Let $z \in N$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K. Let E be a K-equivariant Nash vector bundle over M.

Then there exists a canonical isomorphism

$$\mathrm{Fr} : \mathcal{G}(M_z, E|_{M_z})^{K_z} \cong \mathcal{G}(M, \mathcal{E})^K.$$

Proof. Consider the map $a_z : K \to N$ given by $a_z(g) = gz$. It is a submersion. Hence by Theorem B.2.1 there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that a_z has a Nash section s_i on each U_i. This gives an isomorphism $\phi^{-1}(U_i) \cong U_i \times M_z$ which defines a projection $p : \phi^{-1}(U_i) \to M_z$. Let $\xi \in \mathcal{G}(M_z, E|_{M_z})^{K_z}$. Denote $\xi_i := p^*\xi$. Clearly it does not depend on the section s_i. Hence $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ and hence by Property B.1.7 there exists $\eta \in \mathcal{G}(M, \mathcal{E})$ such that $\eta|_{U_i} = \xi_i$. Clearly η does not depend on the choices. Hence we can define $\mathrm{Fr}(\xi) = \eta$.

It is easy to see that the map $\mathrm{HC} : \mathcal{G}(M, E)^K \to \mathcal{G}(M_z, E|_{M_z})$ described in the last corollary gives the inverse map. ☐

Since our construction coincides with the construction of Frobenius reciprocity for smooth manifolds (see e.g. [AGS08, Theorem A.0.3]) we obtain the following corollary.

Corollary B.3.4. *Part (ii) of Theorem 2.5.7 holds.*

B.4. K-invariant distributions compactly supported modulo K.

In this subsection we prove Theorem 4.0.5. Let us first remind its formulation.

Theorem B.4.1. *Let a Nash group K act on a Nash manifold M. Let E be a K-equivariant Nash bundle over M. Let $\xi \in \mathcal{D}(M, E)^K$ such that $\mathrm{Supp}(\xi)$ is Nashly compact modulo K. Then $\xi \in \mathcal{S}^*(M, E)^K$.*

For the proof we will need the following lemmas.

Lemma B.4.2. *Let M be a Nash manifold. Let $C \subset M$ be a compact subset. Then there exists a relatively compact open (semi-algebraic) subset $U \subset M$ that includes C.*

Proof. For any point $x \in C$ choose an affine chart, and let U_x be an open ball with center at x inside this chart. Those U_x give an open cover of C. Choose a finite subcover $\{U_i\}_{i=1}^n$ and let $U := \bigcup_{i=1}^n U_i$. ☐

Lemma B.4.3. *Let M be a Nash manifold. Let E be a Nash bundle over M. Let $U \subset M$ be a relatively compact open (semi-algebraic) subset. Let $\xi \in \mathcal{D}(M, E)$. Then $\xi|_U \in \mathcal{S}^*(U, E|_U)$.*

Proof. It follows from the fact that extension by zero $ext : \mathcal{S}(U, E|_U) \to C_c^\infty(M, E)$ is a continuous map. ☐

Proof of Theorem B.4.1. Let $Z \subset M$ be a semi-algebraic closed subset and $C \subset M$ be a compact subset such that $Supp(\xi) \subset Z \subset KC$.

Let $U \supset C$ be as in Lemma B.4.2. Let $\xi' := \xi|_{KU}$. Since $\xi|_{M-Z} = 0$, it is enough to show that ξ' is Schwartz.

Consider the surjective submersion $m_U : K \times U \to KU$. Let

$$\xi'' := m_U^*(\xi') \in \mathcal{D}(K \times U, m_U^*(E))^K.$$

By Proposition B.2.7, it is enough to show that

$$\xi'' \in \mathcal{S}^*(K \times U, m_U^*(E)).$$

By Frobenius reciprocity, ξ'' corresponds to $\eta \in \mathcal{D}(U, E)$. It is enough to prove that $\eta \in \mathcal{S}^*(U, E)$. Consider the submersion $m : K \times M \to M$ and let

$$\xi''' := m^*(\xi) \in \mathcal{D}(K \times M, m^*(E)).$$

By Frobenius reciprocity, ξ''' corresponds to $\eta' \in \mathcal{D}(M, E)$. Clearly $\eta = \eta'|_U$. Hence by Lemma B.4.3, $\eta \in \mathcal{S}^*(U, E)$. $\square$

Appendix C. Proof of the Archimedean Homogeneity Theorem

The goal of this appendix is to prove Theorem 5.1.7 for Archimedean F. First we remind its formulation.

Theorem C.0.1 (Archimedean Homogeneity)**.** *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a Nash manifold. Let $L \subset \mathcal{S}^*_{V\times M}(Z(B) \times M)$ be a non-zero subspace such that for all $\xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).*

Then there exists a non-zero distribution $\xi \in L$ which is adapted to B.

Till the end of the section we assume that F is Archimedean and we fix V and B.

First we will need some facts about the Weil representation. For a survey on the Weil representation in the Archimedean case we refer the reader to [RS78, §1].

(1) There exists a unique (infinitesimal) action π of $\mathfrak{sl}_2(F)$ on $\mathcal{S}^*(V)$ such that
(i) $\pi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})\xi = -\mathrm{i}\pi Re(B)\xi$ and $\pi(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix})\xi = -\mathcal{F}_B^{-1}(\mathrm{i}\pi Re(B)\mathcal{F}_B(\xi))$.
(ii) If $F = \mathbb{C}$ then $\pi(\begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix}) = \pi(\begin{pmatrix} 0 & 0 \\ -\mathrm{i} & 0 \end{pmatrix}) = 0$

(2) It can be lifted to an action of the metaplectic group $\mathrm{Mp}(2, F)$. We will denote this action by Π.

(3) In case $F = \mathbb{C}$ we have $\mathrm{Mp}(2, F) = \mathrm{SL}_2(F)$ and in case $F = \mathbb{R}$ the group $\mathrm{Mp}(2, F)$ is a connected 2-fold covering of $\mathrm{SL}_2(F)$. We will denote by $\varepsilon \in \mathrm{Mp}(2, F)$ the central element of order 2 satisfying $\mathrm{SL}_2(F) = \mathrm{Mp}(2, F)/\{1, \varepsilon\}$.

(4) In case $F = \mathbb{R}$ we have $\Pi(\varepsilon) = (-1)^{\dim V}$ and therefore if $\dim V$ is even then Π factors through $\mathrm{SL}_2(F)$ and if $\dim V$ is odd then no nontrivial subrepresentation of Π factors through $\mathrm{SL}_2(F)$. In particular if $\dim V$ is odd then Π has no nontrivial finite-dimensional representations, since every finite-dimensional representation of $\mathrm{Mp}(2, F)$ factors through $\mathrm{SL}_2(F)$.

(5) In case $F = \mathbb{C}$ or in case $\dim V$ is even we have $\Pi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = \delta^{-1}(t)|t|^{-\dim V/2}\rho(t)\xi$ and $\Pi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\xi = \gamma(B)^{-1}\mathcal{F}_B\xi$.

We also need the following straightforward lemma.

Lemma C.0.2. *Let (Λ, L) be a continuous finite-dimensional representation of $\mathrm{SL}_2(\mathbb{R})$. Then there exists a non-zero $\xi \in L$ such that either*

$$\Lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = \xi \text{ and } \Lambda(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\xi \text{ is proportional to } \xi$$

or

$$\Lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = t\xi,$$

for all t.

Now we are ready to prove the theorem.

Proof of Theorem 5.1.7. Without loss of generality assume $M = pt$.

Let $\xi \in L$ be a non-zero distribution. Let $L' := U_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{R}))\xi \subset L$. Here, $U_{\mathbb{C}}$ means the complexified universal enveloping algebra.

We are given that $\xi, \mathcal{F}_B(\xi) \in \mathcal{S}^*_V(Z(B))$. By Lemma C.0.3 below this implies that $L' \subset \mathcal{S}^*(V)$ is finite-dimensional. Clearly, L' is also a subrepresentation of Π. Therefore by Fact (4), $F = \mathbb{C}$ or $\dim V$ is even. Hence Π factors through $\mathrm{SL}_2(F)$.

Now by Lemma C.0.2 there exists $\xi' \in L'$ which is B-adapted. □

Lemma C.0.3. *Let V be a representation of $\mathfrak{sl}_2$. Let $v \in V$ be a vector such that $e^k v = f^n v = 0$ for some n, k. Then the representation generated by v is finite-dimensional.*[9]

This lemma is probably well-known. Since we have not found any reference we include the proof.

Proof. The proof is by induction on k.

Base k=1:
It is easy to see that

$$e^l f^l v = l!(\prod_{i=0}^{l-1}(h-i))v$$

for all l. This can be checked by direct computation, and also follows from the fact that $e^l f^l$ is of weight 0, hence it acts on the singular vector v by its Harish-Chandra projection which is

$$\mathrm{HC}(e^l f^l) = l! \prod_{i=0}^{l-1}(h-i).$$

Therefore $(\prod_{i=0}^{n-1}(h-i))v = 0$.

Hence $W := U_{\mathbb{C}}(h)v$ is finite-dimensional and h acts on it semi-simply. Here, $U_{\mathbb{C}}(h)$ denotes the universal enveloping algebra of h. Let $\{v_i\}_{i=1}^m$ be an eigenbasis of h in W. It is enough to show that $U_{\mathbb{C}}(\mathfrak{sl}_2)v_i$ is finite-dimensional for any i. Note that $e|_W = f^n|_W = 0$. Now, $U_{\mathbb{C}}(\mathfrak{sl}_2)v_i$ is finite-dimensional by the Poincare-Birkhoff-Witt Theorem.

Induction step:
Let $w := e^{k-1}v$. Let us show that $f^{n+k-1}w = 0$. Consider the element $f^{n+k-1}e^{k-1} \in U_{\mathbb{C}}(\mathfrak{sl}_2)$. It is of weight $-2n$, hence by the Poincare-Birkhoff-Witt Theorem it can be rewritten as a combination of elements of the form $e^a h^b f^c$ such that $c - a = n$ and hence $c \geq n$. Therefore $f^{n+k-1}e^{k-1}v = 0$.

Now let $V_1 := U_{\mathbb{C}}(\mathfrak{sl}_2)v$ and $V_2 := U_{\mathbb{C}}(\mathfrak{sl}_2)w$. By the base of the induction V_2 is finite-dimensional, by the induction hypotheses V_1/V_2 is finite-dimensional, hence V_1 is finite-dimensional. □

Appendix D. Localization Principle

by Avraham Aizenbud, Dmitry Gourevitch and Eitan Sayag

In this appendix we formulate and prove the Localization Principle in the case of a reductive group G acting on a smooth affine variety X. This is of interest only for Archimedean F since for l-spaces, a more general version of this principle has been proven in [Ber84]. In [AGS09], we formulated without proof a Localization Principle in the setting of differential geometry. Admittedly, we currently do not have a proof of this principle in such a general setting. However, the current generality is sufficiently wide for all applications we encountered up to now, including the one in [AGS09].

Theorem D.0.1 (Localization Principle)**.** *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant map. Let χ be a character of $G(F)$. Suppose that for any $y \in Y(F)$ we have $\mathcal{D}_{X(F)}((\phi^{-1}(y))(F))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

[9]For our purposes it is enough to prove this lemma for k=1.

Proof. Clearly, it is enough to prove the theorem for the case when X is affine, $Y = X/G$ and $\phi = \pi_X(F)$. By the Generalized Harish-Chandra Descent (Corollary 3.2.2), it is enough to prove that for any G-semisimple $x \in X(F)$, we have
$$\mathcal{D}_{N^X_{Gx,x}(F)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} = 0.$$
Let (U, p, ψ, S, N) be an analytic Luna slice at x. Clearly,
$$\mathcal{D}_{N^X_{Gx,x}(F)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} \cong \mathcal{D}_{\psi(S)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} \cong \mathcal{D}_S(\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G_x(F),\chi}.$$
By Frobenius reciprocity (Theorem 2.5.7),
$$\mathcal{D}_S(\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G_x(F),\chi} = \mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi}.$$
By Lemma 2.3.12,
$$G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})) = \{y \in X(F) | x \in \overline{G(F)y}\}.$$
Hence by Corollary 2.3.15, $G(F)\psi^{-1}(\Gamma(N^X_{Gx,x}))$ is closed in $X(F)$. Hence
$$\mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi} = \mathcal{D}_{X(F)}(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi}.$$
Now,
$$G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})) \subset \pi_X(F)^{-1}(\pi_X(F)(x))$$
and we are given
$$\mathcal{D}_{X(F)}(\pi_X(F)^{-1}(\pi_X(F)(x)))^{G(F),\chi} = 0$$
for any G-semisimple x. $\square$

Remark D.0.2. *An analogous statement holds for Schwartz distributions and the proof is the same.*

Corollary D.0.3. *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant submersion. Suppose that for any $y \in Y(F)$ we have $\mathcal{S}^*(\phi^{-1}(y))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

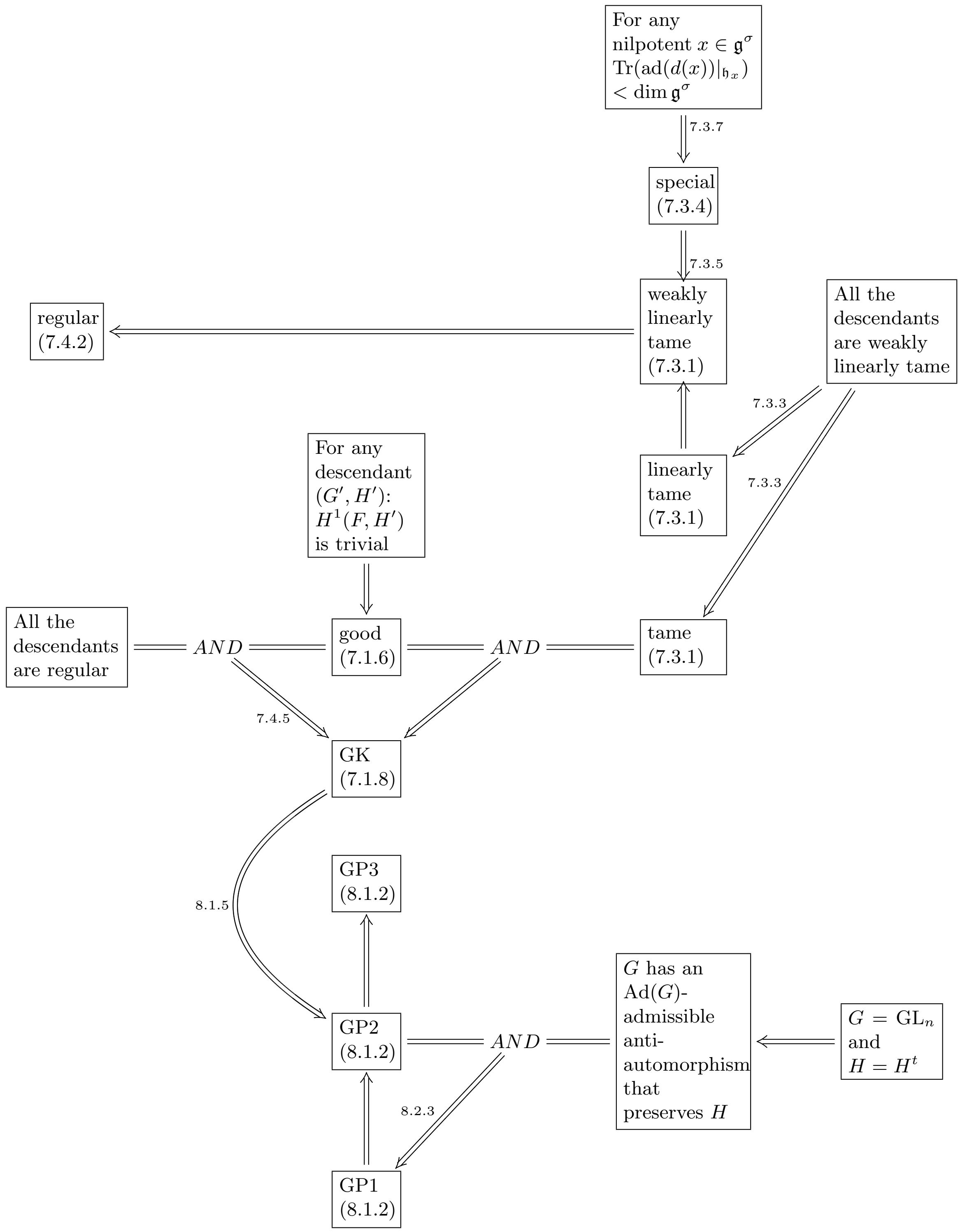
Proof. For any $y \in Y(F)$, denote $X(F)_y := (\phi^{-1}(y))(F)$. Since ϕ is a submersion, for any $y \in Y(F)$ the set $X(F)_y$ is a smooth manifold. Moreover, $d\phi$ defines an isomorphism between $N^{X(F)}_{X(F)_y,z}$ and $T_{Y(F),y}$ for any $z \in X(F)_y$. Hence the bundle $CN^{X(F)}_{X(F)_y}$ is a trivial $G(F)$-equivariant bundle.

We know that
$$\mathcal{S}^*(X(F)_y)^{G(F),\chi} = 0.$$
Therefore for any k, we have
$$\mathcal{S}^*(X(F)_y, \mathrm{Sym}^k(CN^{X(F)}_{X(F)_y}))^{G(F),\chi} = 0.$$
Thus by Theorem 2.5.6, $\mathcal{S}^*_{X(F)}(X(F)_y)^{G(F),\chi} = 0$. Now, by Theorem D.0.1 (and Remark D.0.2) this implies that $\mathcal{S}^*(X(F))^{G(F),\chi} = 0$. Finally, by Theorem 4.0.2 this implies $\mathcal{D}(X(F))^{G(F),\chi} = 0$. $\square$

Remark D.0.4. *Theorem 4.0.1 and Corollary D.0.3 admit obvious generalizations to constant vector systems. The same proofs hold.*

Appendix E. Diagram

The following diagram illustrates the interrelations of the various properties of a symmetric pair (G, H). On the non-trivial implications we put the numbers of the statements that prove them. Near the important notions we put the numbers of the definitions which define those notions.



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SOME REGULAR SYMMETRIC PAIRS

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ABSTRACT. In [AG2] we explored the question what symmetric pairs are Gelfand pairs. We introduced the notion of regular symmetric pair and conjectured that all symmetric pairs are regular. This conjecture would imply that many symmetric pairs are Gelfand pairs, including all connected symmetric pair over $\mathbb{C}$.

In this paper we show that the pairs

$(GL(V), O(V))$, $(GL(V), U(V))$, $(U(V), O(V))$, $(O(V \oplus W), O(V) \times O(W))$, $(U(V \oplus W), U(V) \times U(W))$

are regular where V and W are quadratic or hermitian spaces over arbitrary local field of characteristic zero. We deduce from this that the pairs $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$ and $(O_{n+m}(\mathbb{C}), O_n(\mathbb{C}) \times O_m(\mathbb{C}))$ are Gelfand pairs.

CONTENTS

1. INTRODUCTION

In [AG2] we explored the question what symmetric pairs are Gelfand pairs. We introduced the notion of regular symmetric pair and conjectured that all symmetric pairs are regular. This conjecture would imply that many symmetric pairs are Gelfand pairs, including all connected symmetric pair over $\mathbb{C}$.

Key words and phrases. Uniqueness, multiplicity one, Gelfand pair, symmetric pair, unitary group, orthogonal group.
MSC Classes: 20G05, 20G25, 22E45.

In this paper we show that the pairs

$(GL(V),O(V))$, $(GL(V),U(V))$, $(U(V),O(V))$, $(O(V\oplus W),O(V)\times O(W))$, $(U(V\oplus W),U(V)\times U(W))$

are regular where V and W are quadratic or hermitian spaces over arbitrary local field of characteristic zero. We deduce from this that the pairs $(GL_n(\mathbb{C}),O_n(\mathbb{C}))$ and $(O_{n+m}(\mathbb{C}),O_n(\mathbb{C})\times O_m(\mathbb{C}))$ are Gelfand pairs.

In general, if we would know that all symmetric pairs are regular, then in order to show that a given symmetric pair (G,H) is a Gelfand pair it would be enough to check the following condition that we called "goodness":
(*) Every closed H-double coset in G is invariant with respect to σ. Here, σ is the anti-involution defined by $\sigma(g):=\theta(g^{-1})$ and θ is an involution (i.e. automorphism of order 2) of G such that $H=G^{\theta}$.

This condition always holds for connected symmetric pairs over $\mathbb{C}$.

Meanwhile, before the conjecture is proven, in order to show that a given symmetric pair is a Gelfand pair one has to verify that the pair is good, to prove that it is regular and also to compute its "descendants" and show that they are also regular. The "descendants" are certain symmetric pairs related to centralizers of semisimple elements.

In this paper we develop further the tools from [AG2] for proving regularity of symmetric pairs. We also introduce a systematic way to compute descendants of classical symmetric pairs.

Based on that we show that all the descendants of the above symmetric pairs are regular.

1.1. **Structure of the paper.**

In section 2 we introduce the notions that we discuss in this paper. In subsection 2.1 we discuss the notion of Gelfand pair and review a classical technique for proving Gelfand property due to Gelfand and Kazhdan. In subsection 2.2 we review the results of [AG2], introduce the notions of symmetric pair, descendants of a symmetric pair, good symmetric pair and regular symmetric pair mentioned above and discuss their relations to Gelfand property.

In section 3 we formulate the main results of the paper. We also explain how they follow from the rest of the paper.

In section 4 we introduce terminology that enables us to prove regularity for symmetric pairs in question.

In section 5 we prove regularity for symmetric pairs in question.

In section 6 we compute the descendants of those symmetric pairs.

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2. Preliminaries and notations

- Throughout the paper we fix an arbitrary local field F of characteristic zero.
- All the algebraic varieties and algebraic groups that we will consider will be defined over F.
- For a group G acting on a set X and an element $x\in X$ we denote by G_x the stabilizer of x.
- By a reductive group we mean an algebraic reductive group.

In this paper we will refer to distributions on algebraic varieties over archimedean and non-archimedean fields. In the non-archimedean case we mean the notion of distributions on l-spaces from [BZ], that is linear functionals on the space of locally constant compactly supported functions. In the archimedean case one can consider the usual notion of distributions, that is continuous functionals on the space of smooth compactly supported functions, or the notion of Schwartz distributions (see e.g. [AG1]). It does not matter here which notion to choose since in the cases of consideration of this paper, if there are no nonzero equivariant Schwartz distributions then there are no nonzero equivariant distributions at all (see Theorem 4.0.2 in [AG2]).

Notation 2.0.1. *Let E be an extension of F. Let G be an algebraic group defined over F. We denote by $G_{E/F}$ the canonical algebraic group defined over F such that $G_{E/F}(F) = G(E)$.*

2.1. Gelfand pairs.

In this section we recall a technique due to Gelfand and Kazhdan ([GK]) which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS], section 2.

Definition 2.1.1. *Let G be a reductive group. By an* ***admissible representation of*** *G we mean an admissible representation of $G(F)$ if F is non-archimedean (see* [BZ]*) and admissible smooth Fréchet representation of $G(F)$ if F is archimedean.*

We now introduce three notions of Gelfand pair.

Definition 2.1.2. *Let $H \subset G$ be a pair of reductive groups.*

- *We say that (G, H) satisfy* **GP1** *if for any irreducible admissible representation (π, E) of G we have*
$$\dim Hom_{H(F)}(E, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy* **GP2** *if for any irreducible admissible representation (π, E) of G we have*
$$\dim Hom_{H(F)}(E, \mathbb{C}) \cdot \dim Hom_H(\widetilde{E}, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy* **GP3** *if for any irreducible* **unitary** *representation $(\pi, \mathcal{H})$ of $G(F)$ on a Hilbert space $\mathcal{H}$ we have*
$$\dim Hom_{H(F)}(\mathcal{H}^\infty, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vD, BvD]).

We have the following straightforward proposition.

Proposition 2.1.3. *$GP1 \Rightarrow GP2 \Rightarrow GP3$.*

We will use the following theorem from [AGS] which is a version of a classical theorem of Gelfand and Kazhdan.

Theorem 2.1.4. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(F)$-invariant distributions ξ on $G(F)$. Then (G, H) satisfies GP2.*

In the cases we consider in this paper GP2 is equivalent to GP1 by the following proposition.

Proposition 2.1.5.
(i) Let V be a quadratic space (i.e. a linear space with a non-degenerate quadratic form) and let $H \subset GL(V)$ be any transpose invariant subgroup. Then $GP1$ is equivalent to $GP2$ for the pair $(\mathrm{GL}(V), H)$.
(ii) Let V be a quadratic space and let $H \subset O(V)$ be any subgroup. Then $GP1$ is equivalent to $GP2$ for the pair $(O(V), H)$.

It follows from the following 2 propositions.

Proposition 2.1.6. *Let $H \subset G$ be reductive groups and let τ be an anti-automorphism of G such that*
(i) $\tau^2 \in Ad(G(F))$
(ii) τ preserves any closed conjugacy class in $G(F)$
(iii) $\tau(H) = H$.
Then $GP1$ is equivalent to $GP2$ for the pair (G, H).

For proof see [AG2], Corollary 8.2.3.

Proposition 2.1.7.
(i) Let V be a quadratic space and let $g \in GL(V)$. Then g is conjugate to g^t.
(ii) Let V be a quadratic space and let $g \in O(V)$. Then g is conjugate to g^{-1} inside $O(V)$.

Part (i) is well known. For the proof of (ii) see [MVW], Proposition I.2 in chapter 4.

2.2. **Symmetric pairs.**
In this subsection we review some tools developed in [AG2] that enable to prove that a symmetric pair is a Gelfand pair. The main results discussed in this subsection are Theorem 2.2.16, Theorem 2.2.24 and Proposition 2.2.19.

Definition 2.2.1. *A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$. We call a symmetric pair **connected** if G/H is connected.*

*For a symmetric pair (G, H, θ) we define an antiinvolution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$, denote $\mathfrak{g} := LieG$, $\mathfrak{h} := LieH$, $\mathfrak{g}^\sigma := \{a \in \mathfrak{g} | \theta(a) = -a\}$. Note that H acts on $\mathfrak{g}^\sigma$ by the adjoint action. Denote also $G^\sigma := \{g \in G | \sigma(g) = g\}$ and define a **symmetrization map** $s : G \to G^\sigma$ by $s(g) := g\sigma(g)$.*

In case when the involution is obvious we will omit it.

Remark 2.2.2. *Let (G, H, θ) be a symmetric pair. Then $\mathfrak{g}$ has a $\mathbb{Z}/2\mathbb{Z}$ grading given by θ.*

Definition 2.2.3. *Let (G_1, H_1, θ_1) and (G_2, H_2, θ_2) be symmetric pairs. We define their **product** to be the symmetric pair $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$.*

Definition 2.2.4. *We call a symmetric pair (G, H, θ) **good** if for any closed $H(F) \times H(F)$ orbit $O \subset G(F)$, we have $\sigma(O) = O$.*

Proposition 2.2.5. *Every connected symmetric pair over $\mathbb{C}$ is good.*

For proof see e.g. [AG2], Corollary 7.1.7.

Definition 2.2.6. *We say that a symmetric pair (G, H, θ) is a **GK pair** if any $H(F) \times H(F)$ - invariant distribution on $G(F)$ is σ - invariant.*

Remark 2.2.7. *Theorem 2.1.4 implies that any GK pair satisfies GP2.*

2.2.1. *Descendants of symmetric pairs.*

Proposition 2.2.8. *Let (G, H, θ) be a symmetric pair. Let $g \in G(F)$ such that HgH is closed. Let $x = s(g)$. Then x is a semisimple element of G.*

For proof see e.g. [AG2], Proposition 7.2.1.

Definition 2.2.9. *In the notations of the previous proposition we will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a **descendant** of (G, H, θ).*

2.2.2. *Tame symmetric pairs.*

Definition 2.2.10. *Let π be an action of a reductive group G on a smooth affine variety X. We say that an algebraic automorphism τ of X is G-**admissible** if*
(i) $\pi(G(F))$ is of index at most 2 in the group of automorphisms of X generated by $\pi(G(F))$ and τ.
(ii) For any closed $G(F)$ orbit $O \subset X(F)$, we have $\tau(O) = O$.

Definition 2.2.11. *We call an action of a reductive group G on a smooth affine variety X **tame** if for any G-admissible $\tau : X \to X$, every $G(F)$-invariant distribution on $X(F)$ is τ-invariant.*

*We call a symmetric pair (G, H, θ) **tame** if the action of $H \times H$ on G is tame.*

Remark 2.2.12. *Evidently, any good tame symmetric pair is a GK pair.*

Notation 2.2.13. *Let V be an algebraic finite dimensional representation over F of a reductive group G. Denote $Q(V) := V/V^G$. Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V$.*

Notation 2.2.14. *Let (G, H, θ) be a symmetric pair. We denote by $\mathcal{N}_{G,H}$ the subset of all the nilpotent elements in $Q(\mathfrak{g}^\sigma)$. Denote $R_{G,H} := Q(\mathfrak{g}^\sigma) - \mathcal{N}_{G,H}$.*

Note that our notion of $R_{G,H}$ coincides with the notion $R(\mathfrak{g}^\sigma)$ used in [AG2], Notation 2.3.10. This follows from Lemma 7.1.11 in [AG2].

Definition 2.2.15. *We call a symmetric pair (G, H, θ)* ***weakly linearly tame*** *if for any H-admissible transformation τ of $\mathfrak{g}^\sigma$ such that every $H(F)$-invariant distribution on $R_{G,H}$ is also τ-invariant, we have (*) every $H(F)$-invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also τ-invariant.*

Theorem 2.2.16. *Let (G, H, θ) be a symmetric pair. Suppose that all its descendants (including itself) are weakly linearly tame. Then (G, H, θ) is tame.*

For proof see Theorem 7.3.3 in [AG2].

Now we would like to formulate a criterion for being weakly linearly tame. For it we will need the following lemma and notation.

Lemma 2.2.17. *Let (G, H, θ) be a symmetric pair. Then any nilpotent element $x \in \mathfrak{g}^\sigma$ can be extended to an sl_2 triple $(x, d(x), x_-)$ such that $d(x) \in \mathfrak{h}$ and $x_- \in \mathfrak{g}^\sigma$.*

For proof see e.g. [AG2], Lemma 7.1.11.

Notation 2.2.18. *We will use the notation $d(x)$ from the last lemma in the future. It is not uniquely defined but whenever we will use this notation nothing will depend on its choice.*

Proposition 2.2.19. *Let (G, H, θ) be a symmetric pair. Suppose that for any nilpotent $x \in \mathfrak{g}^\sigma$ we have*

$$\mathrm{Tr}(ad(d(x))|_{\mathfrak{h}_x}) < \dim Q(\mathfrak{g}^\sigma).$$

Then the pair (G, H, θ) is weakly linearly tame.

This proposition follows from [AG2] (Propositions 7.3.7 and 7.3.5).

2.2.3. *Regular symmetric pairs.*

Definition 2.2.20. *Let (G, H, θ) be a symmetric pair. We call an element $g \in G(F)$* ***admissible*** *if*
(i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
(ii) $Ad(g)|_{\mathfrak{g}^\sigma}$ is H-admissible.

Definition 2.2.21. *We call a symmetric pair (G, H, θ)* ***regular*** *if for any admissible $g \in G(F)$ such that every $H(F)$-invariant distribution on $R_{G,H}$ is also $Ad(g)$-invariant, we have*
(*) every $H(F)$-invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also $Ad(g)$-invariant.

The following two propositions are evident.

Proposition 2.2.22. *Let (G, H, θ) be symmetric pair. Suppose that any $g \in G(F)$ satisfying $\sigma(g)g \in Z(G(F))$ lies in $Z(G(F))H(F)$. Then (G, H, θ) is regular. In particular if the normalizer of $H(F)$ lies inside $Z(G(F))H(F)$ then (G, H, θ) is regular.*

Proposition 2.2.23.
(i) Any weakly linearly tame pair is regular.
(ii) A product of regular pairs is regular (see [AG2]*, Proposition 7.4.4).*

In section 4 we will introduce terminology that will help to verify the condition of Proposition 2.2.19.

The importance of the notion of regular pair is demonstrated by the following theorem.

Theorem 2.2.24. *Let (G, H, θ) be a good symmetric pair such that all its descendants (including itself) are regular. Then it is a GK pair.*

For proof see [AG2], Theorem 7.4.5.

3. Main Results

Here we formulate the main results of the paper and explain how they follow from the rest of the paper.

Definition 3.0.1. *A quadratic space is a linear space with a fixed non-degenerate quadratic form.*

Let F' be an extension of F and V be a quadratic space over it. We denote by $O(V)$ the canonical algebraic group such that its F-points form the group of orthogonal transformations of V.

Definition 3.0.2. *Let D be a field with an involution τ. A hermitian space over (D, τ) is a linear space over D with a fixed non-degenerate hermitian form.*

Suppose that D is an extension of F and $F \subset D^\tau$. Let V be a hermitian space over (D, τ). We denote by $U(V)$ the canonical algebraic group such that its F-points form the group of unitary transformations of V.

Definition 3.0.3. *Let G be a reductive group and $\varepsilon \in G$ be an element of order 2. We denote by (G, G_ε) the symmetric pair defined by the involution $x \mapsto \varepsilon x \varepsilon$.*

The following lemma is straightforward.

Lemma 3.0.4. *Let V be a quadratic space.*
(i) Let $\varepsilon \in GL(V)$ be an element of order 2. Then $GL(V)_\varepsilon \cong GL(V_1) \times GL(V_2)$ for some decomposition $V = V_1 \oplus V_2$.
(ii) Let $\varepsilon \in O(V)$ be an element of order 2. Then $O(V)_\varepsilon \cong O(V_1) \times O(V_2)$ for some orthogonal decomposition $V = V_1 \oplus V_2$.
(iii) Let V be a hermitian space.
Let $\varepsilon \in U(V)$ be an element of order 2. Then $U(V)_\varepsilon \cong U(V_1) \times U(V_2)$ for some orthogonal decomposition $V = V_1 \oplus V_2$.

Theorem 3.0.5. *Let V be a quadratic space over F. Then all the descendants of the pair $(O(V), O(V)_\varepsilon)$ are regular.*

Proof. By Theorem 6.5.1 below, the descendants of the pair $(O(V), O(V)_\varepsilon)$ are products of pairs of the types
(i) $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F
(ii) $(U(W_E), O(W))$ for some quadratic space W over some field F' that extends F, and some quadratic extension E of F'. Here, $W_E := W \otimes_{F'} E$ is the extension of scalars with the corresponding hermitian structure.
(iii) $(O(W), O(W)_\varepsilon)$ for some quadratic space W over F.

The pair (i) is regular by Theorem 5.1.1 below. The pair (ii) is regular by subsection 5.3 below. The pair (iii) is regular by subsection 5.2 below. ☐

Corollary 3.0.6. *Suppose that $F = \mathbb{C}$ and Let V be a quadratic space over it. Then the pair $(O(V), O(V)_\varepsilon)$ satisfies GP1.*

Proof. This pair is good by Proposition 2.2.5 and all its descendants are regular. Hence by Theorem 2.2.24 it is a GK pair. Therefore by Theorem 2.1.4 it satisfies GP2. Now, by Proposition 2.1.5, it satisfies GP1. ☐

Theorem 3.0.7. *Let D/F be a quadratic extension and $\tau \in Gal(D/F)$ be the non-trivial element. Let V be a hermitian space over (D, τ). Then all the descendants of the pair $(U(V), U(V)_\varepsilon)$ are regular.*

Proof. By theorem 6.6.1 below, the descendants of the pair $(U(V), U(V)_\varepsilon)$ are products of pairs of the types
(a) $(G \times G, \Delta G)$ for some reductive group G.
(b) $(GL(W), U(W))$ for some hermitian space W over some extension (D', τ') of (D, τ)
(c) $(G_{E/F}, G)$ for some reductive group G and some quadratic extension E/F.
(d) $(GL(W), GL(W)_\varepsilon)$ where W is a linear space over D and $\varepsilon \in GL(W)$ is an element of order ≤ 2.
(e) $(U(W), U(W)_\varepsilon)$ where W is a hermitian space over (D, τ).

The pairs (a) and (c) are regular by Theorem 4.2.12 below. The pairs (b) and (e) are regular by subsection 5.3 below. The pair (d) is regular by Theorem 4.2.13 below. □

Theorem 3.0.8. *Let V be a quadratic space over F. Then all the descendants of the pair $(GL(V), O(V))$ are weakly linearly tame. In particular, this pair is tame.*

Proof. By Theorem 6.2.1 below, the descendants of the pair $(GL(V), O(V))$ are products of pairs of the type $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F. By Theorem 5.1.1 below, these pairs are weakly linearly tame. Now, the pair $(GL(V), O(V))$ is tame by Theorem 2.2.16. □

Corollary 3.0.9. *Suppose that $F = \mathbb{C}$ and Let V be a quadratic space over it. Then the pair $(GL(V), O(V))$ is GP1.*

Theorem 3.0.10. *Let D/F be a quadratic extension and $\tau \in Gal(D/F)$ be the non-trivial element. Let V be a hermitian space over (D, τ). Then all the descendants of the pair $(GL(V), U(V))$ are weakly linearly tame. In particular, this pair is tame.*

Proof. By Theorem 6.3.1 below, all the descendants of the pair $(GL(V), U(V))$ are products of pairs of the types
(i) $(GL(W) \times GL(W), \Delta GL(W))$ for some linear space W over some field D' that extends D
(ii) $(GL(W), U(W))$ for some hermitian space W over some (D', τ') that extends (D, τ).

The pair (i) is weakly linearly tame by Theorem 4.2.12 below and the pair (ii) is weakly linearly tame by subsection 5.3 below. Now, the pair $(GL(V), U(V))$ is tame by Theorem 2.2.16. □

Theorem 3.0.11. *Let V be a quadratic space over F. Let D/F be a quadratic extension and $\tau \in Gal(D/F)$ be the non-trivial element. Let $V_D := V \otimes_F D$ be its extension of scalars with the corresponding hermitian structure. Then all the descendants of the pair $(U(V_D), O(V))$ are weakly linearly tame. In particular, this pair is tame.*

Proof. By Theorem 6.4.1 below, all the descendants of the pair $(U(V_D), O(V))$ are products of pairs of the types
(i) $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F.
(ii) $(U(W_{D'}), O(W))$ for some extension (D', τ') of (D, τ) and some quadratic space W over $D'^{\tau'}$.

The pair (i) is weakly linearly tame by Theorem 5.1.1 below and the pair (ii) is weakly linearly tame by subsection 5.3 below. Now, the pair $(GL(V), U(V))$ is tame by Theorem 2.2.16. □

4. $\mathbb{Z}/2\mathbb{Z}$ graded representations of sl_2 and their defects

In this section we will introduce terminology that will help to verify the condition of Proposition 2.2.19.

4.1. Graded representations of sl_2.

Definition 4.1.1. *We fix standard basis e, h, f of $sl_2(F)$. We fix a grading on $sl_2(F)$ given by $h \in sl_2(F)_0$ and $e, f \in sl_2(F)_1$. A* ***graded representation of*** sl_2 *is a representation of sl_2 on a graded vector space $V = V_0 \oplus V_1$ such that $sl_2(F)_i(V_j) \subset V_{i+j}$ where $i, j \in \mathbb{Z}/2\mathbb{Z}$.*

The following lemma is standard.

Lemma 4.1.2.
(i) Every graded representation of sl_2 which is irreducible as a graded representation is irreducible just as a representation.
(ii) Every irreducible representation V of sl_2 admits exactly two gradings. In one highest weight vector lies in V_0 and in the other in V_1.

Definition 4.1.3. *We denote by V_λ^w the irreducible graded representation of sl_2 with highest weight λ and highest weight vector of parity p where $w = (-1)^p$.*

The following lemma is straightforward.

Lemma 4.1.4.

$$(V_\lambda^w)^* = V_\lambda^{w(-1)^\lambda} \tag{1}$$

$$V_{\lambda_1}^{w_1} \otimes V_{\lambda_2}^{w_2} = \bigoplus_{i=0}^{\min(\lambda_1,\lambda_2)} V_{\lambda_1+\lambda_2-2i}^{w_1 w_2 (-1)^i} \tag{2}$$

$$\Lambda^2(V_\lambda^w) = \bigoplus_{i=0}^{\lfloor \frac{\lambda-1}{2} \rfloor} V_{2\lambda-4i-2}^{-1}. \tag{3}$$

4.2. **Defects.**

Definition 4.2.1. *Let π be a graded representation of sl_2. We define the* ***defect*** *of π to be*

$$def(\pi) = \mathrm{Tr}(h|_{(\pi^e)_0}) - \dim(\pi_1)$$

The following lemma is straightforward

Lemma 4.2.2.

$$def(\pi \oplus \tau) = def(\pi) + def(\tau) \tag{4}$$

$$def(V_\lambda^w) = \frac{1}{2}(\lambda w + w(\frac{1+(-1)^\lambda}{2}) - 1) = \frac{1}{2}\begin{cases} \lambda w + w - 1 & \lambda \text{ is even} \\ \lambda w - 1 & \lambda \text{ is odd} \end{cases} \tag{5}$$

Definition 4.2.3. *Let $\mathfrak{g}$ be a $(\mathbb{Z}/2\mathbb{Z})$ graded Lie algebra. We say that $\mathfrak{g}$* ***is of negative defect*** *if for any graded homomorphism $\pi : sl_2 \to \mathfrak{g}$, the defect of $\mathfrak{g}$ with respect to the adjoint action of sl_2 is negative.*

We say that $\mathfrak{g}$ ***is of negative normalized defect*** *if the semi-simple part of $\mathfrak{g}$ (i.e. the quotient of $\mathfrak{g}$ by its center) is of negative defect.*

Remark 4.2.4. *Clearly, $\mathfrak{g}$ is of negative normalized defect if and only if for any graded homomorphism $\pi : sl_2 \to \mathfrak{g}$, the defect of $\mathfrak{g}$ with respect to the adjoint action of sl_2 is less than minus the dimension of the odd part of the center of $\mathfrak{g}$.*

Definition 4.2.5. *We say that a symmetric pair (G, H, θ)* ***is of negative normalized defect*** *if the Lie algebra $\mathfrak{g}$ with the grading defined by θ is of negative normalized defect.*

Lemma 4.2.6. *Let (G, H, θ) be a symmetric pair. Assume that $\mathfrak{g}$ is semi-simple. Then $Q(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$.*

Proof.
Assume the contrary: there exists $0 \neq x \in \mathfrak{g}^\sigma$ such that $Hx = x$. Then $\dim(CN_{Hx,x}^{\mathfrak{g}^\sigma}) = \dim \mathfrak{g}^\sigma$, hence $CN_{Hx,x}^{\mathfrak{g}^\sigma} = \mathfrak{g}^\sigma$. On the other hand, $CN_{Hx,x}^{\mathfrak{g}^\sigma} \cong [\mathfrak{h}, x]^\perp = (\mathfrak{g}^\sigma)^x$ (here $(\cdot)^\perp$ means the orthogonal compliment w.r.t. the Killing form). Therefore $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma)^x$ and hence x lies in the center of $\mathfrak{g}$, which is impossible. □

Proposition 2.2.19 can be rewritten now in the following form

Theorem 4.2.7. *A symmetric pair of negative normalized defect is weakly linearly tame.*

Evidently, a product of pairs of negative normalized defect is again of negative normalized defect.
The following lemma is straightforward.

Lemma 4.2.8. *Let (G, H, θ) be a symmetric pair. Let F' be any field extending F. Let $(G_{F'}, H_{F'}, \theta)$ be the extension of (G, H, θ) to F'. Suppose that it is of negative normalized defect (as a pair over F') . Then (G, H, θ) and $(G_{F'/F}, H_{F'/F}, \theta)$ are of negative normalized defect (as pairs over F).*

In [AG2] we proved the following (easy) proposition (see [AG2], Lemma 7.6.6).

Proposition 4.2.9. *Let π be a representation of sl_2. Then $\mathrm{Tr}(h|_{(\pi^e)}) < \dim(\pi)$.*

We would like to reformulate it in terms of defect. For this we will need the following notation.

Notation 4.2.10.
(i) Let π be a representation of sl_2. We denote by $\overline{\pi}$ the representation of sl_2 on the same space defined by $\overline{\pi}(e) := -\pi(e)$, $\overline{\pi}(f) := -\pi(f)$ and $\overline{\pi}(h) := \pi(h)$.
(ii) We define grading on $\pi \oplus \overline{\pi}$ by the involution $s(v \oplus w) := w \oplus v$.

Proposition 4.2.9 can be reformulated in the following way.

Proposition 4.2.11. *Let π be a representation of sl_2. Then $def(\pi \oplus \overline{\pi}) < 0$.*

In [AG2] we also deduced from this proposition the following theorem (see [AG2], 7.6.2).

Theorem 4.2.12. *For any reductive group G, the pairs $(G \times G, \Delta G)$ and $(G_{E/F}, G)$ are of negative normalized defect and hence weakly linearly tame. Here ΔG is the diagonal in $G \times G$.*

In [AG2, §§7.7] we proved the following theorem.

Theorem 4.2.13. *The pair $(GL(V \oplus V), GL(V) \times GL(V))$ is of negative normalized defect and hence regular.*

Note that in the case $dimV \neq dimW$ the pair $(GL(V \oplus W), GL(V) \times GL(W))$ is obviously regular by Proposition 2.2.22.

5. Proof of regularity and tameness

5.1. The pair $(GL(V), O(V))$.

In this subsection we prove that the pair $(GL(V), O(V))$ is weakly linearly tame. For $\dim V \leq 1$ it is obvious. Hence it is enough to prove the following theorem.

Theorem 5.1.1. *Let V be a quadratic space of dimension at least 2. Then the pair $(GL(V), O(V))$ has negative normalized defect.*

We will need the following notation.

Notation 5.1.2. *Let π be a representation of sl_2. We define grading on $\pi \otimes \overline{\pi}$ by the involution $s(v \otimes w) := -w \otimes v$.*

Theorem 5.1.1 immediately follows from the following one.

Theorem 5.1.3. *Let π be a representation of sl_2 of dimension at least 2. Then $def(\pi \otimes \overline{\pi}) < -1$.*

This theorem in turn follows from the following lemma.

Lemma 5.1.4. *Let V_λ and V_μ be irreducible representations of sl_2. Then*
(i) $def(V_\lambda \otimes \overline{V_\lambda}) = -(\lambda + 1)(\frac{\lambda}{2} + 1)$.
(ii) $def(V_\lambda \otimes \overline{V_\mu} \oplus V_\mu \otimes \overline{V_\lambda}) < 0$.

Proof.
(i) Follows from the fact that $V_\lambda \otimes \overline{V_\lambda} = \bigoplus_{i=0}^{\lambda} V_{2\lambda-2i}^{-1}$ and from Lemma 4.2.2.
(ii) Follows from Proposition 4.2.11. ∎

5.2. The pair $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$.

In this subsection prove that the pair $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$ is regular. For that it is enough to prove the following theorem.

Theorem 5.2.1. *Let V_1 and V_2 be quadratic spaces. Assume $\dim V_1 = \dim V_2$. Then the pair $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$ has negative normalized defect.*

This theorem immediately follows from the following one.

Theorem 5.2.2. *Let π be a (non-zero) graded representation of sl_2 such that $\dim \pi_0 = \dim \pi_1$ and $\pi \simeq \pi^*$. Then $\Lambda^2(\pi)$ has negative defect.*

For this theorem we will need the following lemma.

Lemma 5.2.3. *Let $V_{\lambda_1}^{w_1}$ and $V_{\lambda_2}^{w_2}$ be irreducible graded representations of sl_2. Then*
(i) $def(V_{\lambda_1}^{w_1} \otimes V_{\lambda_2}^{w_2}) =$

$$= -\frac{1}{2}\begin{cases} \min(\lambda_1,\lambda_2)+1-\frac{w_1w_2}{2}(\lambda_1+\lambda_2+1+(-1)^{\min(\lambda_1,\lambda_2)}(|\lambda_1-\lambda_2|-1)), & \lambda_1 \neq \lambda_2 \mod 2; \\ \min(\lambda_1,\lambda_2)+1-w_1w_2(\max(\lambda_1,\lambda_2)+1), & \lambda_1 \equiv \lambda_2 \equiv 0 \mod 2; \\ \min(\lambda_1,\lambda_2)+1-w_1w_2(\min(\lambda_1,\lambda_2)+1), & \lambda_1 \equiv \lambda_2 \equiv 1 \mod 2; \end{cases}$$

(ii) $def(\Lambda^2(V_{\lambda_1}^{w_1})) = -\frac{\lambda^2}{4} - \frac{\lambda}{2} - \frac{1+(-1)^{\lambda+1}}{8}$

Proof. This lemma follows by straightforward computations from Lemmas 4.1.4 and 4.2.2. □

Proof of Theorem 5.2.2. Since $\pi \simeq \pi^*$, π can be decomposed to a direct sum of irreducible graded representations in the following way

$$\pi = (\bigoplus_{i=1}^{l} V_{\lambda_i}^1) \oplus (\bigoplus_{j=1}^{m} V_{\mu_j}^{-1}) \oplus (\bigoplus_{k=1}^{n} V_{\nu_k}^1 \oplus V_{\nu_k}^{-1}).$$

Here, all λ_i and μ_j are even and ν_k are odd. Since $\dim \pi_0 = \dim \pi_1$, $l = m$.

By the last lemma, $def(V_{\lambda_i}^1 \otimes (V_{\nu_k}^1 \oplus V_{\nu_k}^{-1})) = -(\min(\lambda_1,\lambda_2)+1) < 0$. Similarly, $def(V_{\mu_j}^{-1} \otimes (V_{\nu_k}^1 \oplus V_{\nu_k}^{-1})) < 0$. Also, $def((V_{\nu_{k_1}}^1 \oplus V_{\nu_{k_1}}^{-1}) \otimes (V_{\nu_{k_2}}^1 \oplus V_{\nu_{k_2}}^{-1})) < 0$ and $def(\Lambda^2(V_\lambda^w)) \leq 0$ for all λ and w.

Hence if $l = 0$ we are done. Otherwise we can assume $n = 0$. Now,

$$def(\Lambda^2(\pi)) \leq \sum_{1 \leq i < j \leq l} |\lambda_i - \lambda_j| + \sum_{1 \leq i < j \leq l} |\mu_i - \mu_j| - \sum_{1 \leq i,j \leq l} (\lambda_i + \mu_j + 2) <$$

$$< \sum_{1 \leq i < j \leq l} (\lambda_i + \lambda_j) + \sum_{1 \leq i < j \leq l} (\mu_i + \mu_j) - \sum_{1 \leq i,j \leq l} (\lambda_i + \mu_j) = -\sum_{i=1}^{l} (\lambda_i + \mu_i) \leq 0.$$

□

5.3. The pairs $(GL(V), U(V))$, $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$ and $(U(V_D), O(V))$.

In this subsection prove that the pairs $(GL(V), U(V))$ and $(U(V_D), O(V))$ are weakly linearly tame and the pair $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$ is regular.

Let V be a hermitian space. Note that $(GL(V), U(V))$ is a form of $(GL(W) \times GL(W), \Delta GL(W))$ for some W and $(U(V \oplus V), U(V) \times U(V))$ is a form of $(GL(W \oplus W), GL(W)) \times GL(W))$ for some W.

Also, for any quadratic space V of dimension at least 2 and any quadratic extension D/F, the pair $(U(V_D), O(V))$ is a form of $(GL(W), O(W))$ for some quadratic space W.

Hence by Lemma 4.2.8 and Theorems 4.2.12, 4.2.13 and 5.1.1 those 3 pairs are of negative normalized defect and hence are weakly linearly tame. If $\dim V \leq 1$ then the pair $(U(V_D), O(V))$ is obviously linearly tame.

If V_1 and V_2 are non-isomorphic hermitian spaces then $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$ is regular by Proposition 2.2.22.

6. Computation of descendants

In this section we compute the descendants of the pairs we discussed before. For this we use a technique of computing centralizers of semisimple elements of orthogonal and unitary groups, which is described in [SpSt]. The proofs in this section are rather straightforward but technically involved. The most important things in this section are the formulations of the main theorems: Theorems 6.2.1, 6.3.1, 6.4.1, 6.5.1, 6.6.1. Those theorems are summarized graphically in subsection 6.7.

6.1. Preliminaries and notation for orthogonal and unitary groups.

6.1.1. *Orthogonal group.*

Notation 6.1.1. *Let V be a linear space over F. Let $x \in GL(V)$ be a semi-simple element and let $Q = \sum_{i=0}^{n} a_i \xi^i \in F[\xi]$ (where $a_n \neq 0$) be an irreducible polynomial.*

- *Denote $F_Q := F[\xi]/Q$*
- *Denote $inv(Q) := \sum_{i=0}^{n} a_{n-i} \xi^i$*
- *Denote $V_{Q,x}^0 := Ker(Q(x))$ and $V_{Q,x}^1 := Ker(inv(Q)(x))$*
  We define an F_Q-linear space structure on $V_{Q,x}^i$ by letting ξ act on $V_{Q,x}^0$ by x and on $V_{Q,x}^1$ by x^{-1}. We will consider $V_{Q,x}^i$ as linear spaces over F_Q.
- *In case Q is proportional to $inv(Q)$ we define an involution μ on F_Q by $\mu(P(\xi)) := P(\xi^{-1})$.*
- *For a linear space W over F_Q we can consider its dual space W^* over F_Q and the dual space of W over F which we denote by W_F^*. The space W_F^* has a canonical structure of a linear space over F_Q. The spaces W_F^* and W^* can be identified as linear spaces over F_Q. For this identification one has to choose an F-linear functional $\lambda : F_Q \to F$. We will fix such functional λ such that $\lambda(\mu(d)) = \lambda(d)$ if μ is defined.*
  From now on we will identify W_F^* and W^*.

The following two lemmas are straightforward.

Lemma 6.1.2. *Let V be a quadratic space over F. Let $x \in GL(V)$ and let $P, Q \in F[\xi]$ be irreducible polynomials. Suppose that either*
(i) $x = x^t$ and P is not proportional to Q or
(ii) $x \in O(V)$ and P is not proportional to $inv(Q)$
Then $Ker(Q(x))$ is orthogonal to $Ker(P(x))$.

Lemma 6.1.3. *Let (V, B) be a quadratic space over F. Let $x \in GL(V)$ be a semi-simple element and let $Q \in F[\xi]$ be an irreducible polynomial. Then*
(i) If $x = x^t$ then B defines an F_Q-linear isomorphism $V_{Q,x}^i \cong (V_{Q,x}^i)^*$.
(ii) If $x \in O(V)$ then B defines an F_Q-linear isomorphism $V_{Q,x}^i \cong (V_{Q,x}^{1-i})^*$.

6.1.2. *Unitary group.*
From now and till the end of the paper we fix a quadratic extension D of F and denote by τ the involution that fixes F.

Notation 6.1.4. *Let V be a hermitian space over (D, τ). Let $x \in GL(V)$ be a semi-simple element and let $Q = \sum_{i=0}^{n} a_i \xi^i \in D[\xi]$ (where $a_n \neq 0$) be an irreducible polynomial.*

- *Denote $D_Q := D[\xi]/Q$*
- *Denote*
$$inv(Q) := \sum_{i=0}^{n} a_{n-i} \xi^i, \quad Q^* := \tau(inv(Q))$$
- *Denote*
$$V_{Q,x}^{00} := Ker(Q(x)), \quad V_{Q,x}^{01} := Ker(Q^*(x)), \quad V_{Q,x}^{10} := Ker(inv(Q)(x)), \quad V_{Q,x}^{11} := Ker(\tau(Q)(x)).$$
  We twist the action of D on $V_{Q,x}^{i1}$ by τ. We define D_Q-linear space structure on $V_{Q,x}^{ij}$ by letting ξ act on $V_{Q,x}^{0j}$ by x and on $V_{Q,x}^{1j}$ by x^{-1}. We will consider $V_{Q,x}^{ij}$ as linear spaces over D_Q.
- *If Q is proportional to Q^* we define an involution μ^{01} on D_Q by $\mu^{01}(P(\xi)) := \tau(P)(\xi^{-1})$.*
  If Q is proportional to $inv(Q)$ we define an involution μ^{10} on D_Q by $\mu^{10}(P(\xi)) := P(\xi^{-1})$.
  If Q is proportional to $\tau(Q)$ we define an involution μ^{11} on D_Q by $\mu^{11}(P(\xi)) := \tau(P)(\xi)$.
- *For a linear space W over D_Q we can consider its dual space W^* over D_Q and the dual space of W over D which we denote by W_D^*. The space W_D^* has a canonical structure of a linear space over D_Q. The spaces W_D^* and W^* can be identified as linear spaces over D_Q. For this identification one has to choose a D-linear functional $\lambda : D_Q \to D$. We will fix such functional λ such that*
$$\lambda(\mu^{ij}(d)) = \tau^j(\lambda(d)) \quad \textit{if } \mu^{ij} \textit{ is defined.}$$

From now on we will identify W_D^* and W^*.

The following two lemmas are straightforward.

Lemma 6.1.5. *Let V be a hermitian space over (D,τ). Let $x \in GL(V)$ and let $P,Q \in D[\xi]$ be irreducible polynomials. Suppose that either*
(i) $x = x^*$ and P is not proportional to $\tau(Q)$ or
(ii) $x \in U(V)$ and P is not proportional to Q^*
Then $Ker(Q(x))$ is orthogonal to $Ker(P(x))$.

Lemma 6.1.6. *Let (V,B) be a hermitian space over (D,τ). Let $x \in GL(V)$ be a semi-simple element and let $Q \in D[\xi]$ be an irreducible polynomial. Then*
(i) If $x = x^*$ then B defines a D_Q-linear isomorphism $V_{Q,x}^{ij} \cong (V_{Q,x}^{1-i,1-j})^*$.
(ii) If $x \in U(V)$ then B defines a D_Q-linear isomorphism $V_{Q,x}^{ij} \cong (V_{Q,x}^{i,1-j})^*$.

6.2. **The pair** $(GL(V), O(V))$**.**

Theorem 6.2.1. *Let V be a quadratic space over F. Then all the descendants of the pair $(GL(V), O(V))$ are products of pairs of the type $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F.*

Proof. Note that in this case the anti-involution σ is given by $\sigma(x) = x^t$. Let $x \in GL(V)^\sigma$ be a semi-simple element. Let P be the minimal polynomial of x. We will now discuss a special case and then deduce the general case from it.

Case 1. P is irreducible over F.
Clearly $GL(V)_x \cong GL(V_{P,x}^0)$. The isomorphism $V_{P,x}^0 \cong (V_{P,x}^0)^*$ gives a quadratic structure on $V_{P,x}^0$. Now $O(V)_x \cong O(V_{P,x}^0)$.

Case 2. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible polynomials. Clearly $V = \bigoplus V_{P_i,x}^0$ and $V_{P_i,x}^0$ are orthogonal to each other. Hence the pair $(GL(V)_x, O(V)_x)$ is a product of pairs from Case 1. ☐

6.3. **The pair** $(GL(V), U(V))$**.**

Theorem 6.3.1. *Let (V,B) be a hermitian space over (D,τ). Then all the descendants of the pair $(GL(V), U(V))$ are products of pairs of the types*
(i) $(GL(W) \times GL(W), \Delta GL(W))$ for some linear space W over some field D' that extends D
(ii) $(GL(W), U(W))$ for some hermitian space W over some (D',τ') that extends (D,τ).

Proof. Note that in this case the anti-involution σ is given by $\sigma(x) = x^*$. Let $x \in GL(V)^\sigma$ be a semi-simple element. Let P be the minimal polynomial of x. Note that $\tau(P)$ is proportional to P. We will now discuss 2 special cases and then deduce the general case from them.

Case 1. $P = Q\tau(Q)$ where Q is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{Q,x}^{00}) \times GL(V_{Q,x}^{11})$. Recall that B gives a non-degenerate pairing between $V_{Q,x}^{00}$ and $V_{Q,x}^{11}$, and the spaces $V_{Q,x}^{ii}$ are isotropic. Therefore

$$GL(V_{Q,x}^{00}) \cong GL(V_{Q,x}^{11}), \quad GL(V)_x \cong GL(V_{Q,x}^{00})^2 \text{ and } U(V)_x \cong \Delta GL(V_{Q,x}^{00}) < GL(V_{Q,x}^{00})^2.$$

Case 2. P is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{P,x}^{00})$ and $V_{P,x}^{00}$ is identical to $V_{P,x}^{11}$ as F-linear spaces but the actions of D_P differ by a twist by μ^{11}. Hence the isomorphism $V_{P,x}^{00} \cong (V_{P,x}^{11})^*$ gives a hermitian structure on $V_{P,x}^{00}$ over (D_P, μ^{11}). Now $U(V)_x \cong U(V_{P,x}^{00}) < GL(V_{P,x}^{00})$.

Case 3. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible polynomials. Then $\tau(P_i)$ is proportional to

$P_{s(i)}$ where s is some permutation of I of order ≤ 2. Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of s. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V = \bigoplus V_\alpha$ and V_α are orthogonal to each other. Hence the pair $(GL(V)_x, U(V)_x)$ is a product of pairs from the first 2 cases. □

6.4. **The pair** $(U(V_D), O(V))$**.**

Theorem 6.4.1. *Let (V, B) be a quadratic space over F. Let $V_D := V \otimes_F D$ be its extension of scalars with the corresponding hermitian structure.*

Then all the descendants of the pair $(U(V_D), O(V))$ are products of pairs of the types
(i) $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F.
(ii) $(U(W_{D'}), O(W))$ for some extension (D', τ') of (D, τ) and some quadratic space W over $D'^{\tau'}$.

For the proof of this theorem we will need the following notation and lemma.

Notation 6.4.2. *Let (V, B) be a quadratic space over F. The involution τ defines an involution $\widetilde{\tau}$ on V_D. The form B defines a quadratic form B_D on V_D and a hermitian form B_D^τ on V_D.*

Lemma 6.4.3. *Let (V, B) be a quadratic space over F. Let P be an irreducible polynomial. Let $x \in U(V_D)$ be a semi-simple element such that $x = x^t$ (where x^t is defined by B_D). Then the involution $\widetilde{\tau}$ gives a D_P-linear isomorphism $V_{P,x}^{ij} \cong V_{P,x}^{i,1-j}$.*

Proof. We will show that $\widetilde{\tau}$ maps $V_{P,x}^{00}$ to $V_{P,x}^{11}$, and the other cases are done similarly. Let $v \in V_{P,x}^{ij}$. We have

$$P^*(x)(\widetilde{\tau}(v)) = \widetilde{\tau}(inv(P)(x^*)(v)) = \widetilde{\tau}(inv(P)(x^{-1})(v)) = \widetilde{\tau}(x^{-degP} P(x)(v)) = 0.$$

□

Proof of Theorem 6.4.1. Note that in this case the anti-involution σ is given by $\sigma(x) = x^t$. Let $x \in U(V_D)^\sigma$ be a semi-simple element. Let P be the minimal polynomial of x. Then P is proportional to P^*. We will now discuss 2 special cases and then deduce the general case from them.

Case 1. $P = QQ^*$ where Q is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{Q,x}^{00}) \times GL(V_{Q,x}^{01})$. Recall that B_D^τ gives a non-degenerate pairing between $V_{Q,x}^{00}$ and $V_{Q,x}^{01}$, and the spaces $V_{Q,x}^{0i}$ are isotropic. Therefore

$$GL(V_{Q,x}^{00}) \cong GL(V_{Q,x}^{01}), \quad GL(V)_x \cong GL(V_{Q,x}^{00})^2, \quad U(V)_x \cong \Delta GL(V_{Q,x}^{00}) < GL(V_{Q,x}^{00})^2$$

Compose the isomorphism $V_{Q,x}^{00} \cong V_{Q,x}^{01}$ given by $\widetilde{\tau}$ with the isomorphism $V_{Q,x}^{01} \cong (V_{Q,x}^{00})^*$ given by B_D^τ. This gives a quadratic structure on $V_{Q,x}^{00}$. Now

$$O(V)_x \cong \Delta O(V_{Q,x}^{00}) < \Delta GL(V_{Q,x}^{00}).$$

Case 2. P is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{P,x}^{00})$ and $V_{P,x}^{00}$ is identical to $V_{P,x}^{01}$ as F-linear spaces but the actions of D_P on them differ by a twist by μ^{01}. Hence the isomorphism $V_{P,x}^{00} \cong (V_{P,x}^{01})^*$ given by B_D^τ gives a hermitian structure on $V_{P,x}^{00}$ over (D_P, μ^{01}) and the isomorphism $V_{P,x}^{00} \cong V_{P,x}^{01}$ given by $\widetilde{\tau}$ gives an antilinear involution of $V_{P,x}^{00}$. Now

$$U(V)_x \cong U(V_{P,x}^{00}) < GL(V_{P,x}^{00}) \text{ and } O(V)_x \cong O(V_{P,x}^{00}) < U(V_{P,x}^{00}).$$

Case 3. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible polynomials. Then P_i^* is proportional to $P_{s(i)}$ where s is some permutation of I of order ≤ 2. Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of s. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V_D = \bigoplus V_\alpha$, V_α are orthogonal to each other and each V_α is invariant with respect to $\widetilde{\tau}$. Hence the pair $(GL(V)_x, U(V)_x)$ is a product of pairs from the first 2 cases. □

6.5. **The pair** $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$**.**

Theorem 6.5.1. *Let* (V, B) *be a quadratic space over* F*.*
Let $\varepsilon \in O(V)$ *be an element of order 2. Then all the descendants of the pair* $(O(V), O(V)_\varepsilon)$ *are products of pairs of the types*
(i) $(GL(W), O(W))$ *for some quadratic space* W *over some field* F' *that extends* F
(ii) $(U(W_E), O(W))$ *for some quadratic space* W *over some field* F' *that extends* F*, and some quadratic extension* E *of* F'*.*
(iii) $(O(W), O(W)_\varepsilon)$ *for some quadratic space* W *over* F*.*

For the proof of this theorem we will need the following straightforward lemma.

Lemma 6.5.2. *Let* (V, B) *be a quadratic space over* F*.*
Let $\varepsilon \in O(V)$ *be an element of order 2. Let* $x \in O(V)$ *such that* $\varepsilon x \varepsilon = x^{-1}$*. Let* Q *be an irreducible polynomial. Then* ε *gives an* F_Q*- linear isomorphism* $V^i_{Q,x} \cong V^{1-i}_{Q,x}$*.*

Proof of Theorem 6.5.1. Note that the involution σ on $O(V)$ is given by $x \mapsto \varepsilon x^{-1} \varepsilon$. Let $x \in O(V)^\sigma$ be a semi-simple element and let P be its minimal polynomial.

Note that the minimal polynomial of x^{-1} is $inv(P)$ and hence P is proportional to $inv(P)$. We will now discuss 3 special cases and then deduce the general case from them.

Case 1. $P = Qinv(Q)$, where Q is an irreducible polynomial.
Note that $GL(V)_x \cong \prod_i GL(V^i_{Q,x})$.

Since B defines a non-degenerate pairing $V^0_{Q,x} \cong (V^1_{Q,x})^*$, and $V^i_{Q,x}$ are isotropic, we have

$$O(V)_x \cong \Delta GL(V^0_{Q,x}) < GL(V^0_{Q,x})^2.$$

Now, compose the isomorphism $V^i_{Q,x} \cong V^{1-i}_{Q,x}$ given by ε with the isomorphism $V^{1-i}_{Q,x} \cong (V^i_{Q,x})^*$. This gives a quadratic structure on $V^0_{Q,x}$. Clearly, ε gives an isomorphism $V^0_{Q,x} \cong V^1_{Q,x}$ as quadratic spaces and hence

$$(O(V)_\varepsilon)_x \cong \Delta O(V^0_{Q,x}) < \Delta GL(V^0_{Q,x}).$$

Case 2. P is irreducible and $x \neq x^{-1}$
In this case $GL(V)_x \cong GL(V^0_{P,x})$. Also, $V^0_{P,x}$ and $V^1_{P,x}$ are identical as F-vector spaces but the action of F_P on them differs by a twist by μ. Therefore the isomorphism $V^0_{P,x} \cong (V^1_{P,x})^*$ gives a hermitian structure on $V^0_{P,x}$ over (F_P, μ) and ε gives an (F_P, μ)-antilinear automorphism of $V^0_{P,x}$. Now

$$O(V)_x \cong U(V^0_{P,x}).$$

Denote $W := (V^0_{P,x})^\varepsilon$. It is a linear space over $(F_P)^\mu$. It has a quadratic structure. Now

$$(O(V)_\varepsilon)_x \cong O(W) < U(V^0_{P,x}).$$

Case 3. P is irreducible and $x = x^{-1}$.
Again, $GL(V)_x \cong GL(V^0_{P,x})$. However, in this case $F_P = F$ and $V^0_{P,x} = V$. Also $O(V)_x \cong O(V^0_{P,x})$. Now, ε commutes with x and hence $\varepsilon \in O(V)_x \cong O(V^0_{P,x})$. Hence

$$(O(V)_\varepsilon)_x \cong (O(V^0_{P,x}))_\varepsilon < O(V^0_{P,x}).$$

Case 4. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible multiples. Since P is proportional to $inv(P)$, every P_i is proportional to $P_{s(i)}$ where s is some permutation of I of order ≤ 2.

Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of s. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V = \bigoplus V_\alpha$ and V_α are orthogonal to each other and ε-invariant. Hence the pair $(O(V)_x, (O(V)_\varepsilon)_x)$ is a product of pairs from the first 3 cases.

□

6.6. **The pair** $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$**.**

Theorem 6.6.1. *Let (V, B) be a hermitian space over (D, τ).*
Let $\varepsilon \in U(V)$ be an element of order 2. Then all the descendants of the pair $(U(V), U(V)_\varepsilon)$ are products of pairs of the types
(i) $(GL(W) \times GL(W), \Delta GL(W))$ for some linear space W over some field D' that extends D
(ii) $(U(W) \times U(W), \Delta U(W))$ for some hermitian space W over some extension (D', τ') of (D, τ)
(iii) $(GL(W), U(W))$ for some hermitian space W over some extension (D', τ') of (D, τ)
(iv) $(GL(W_{D'}), GL(W))$ where F' is a field extension of D, D'/F' is a quadratic extension, W is a linear space over F' and $W_{D'} := W \otimes_{F'} D'$ is its extension of scalars to D'
(v) $(GL(W), GL(W)_\varepsilon)$ where W is a linear space over D and $\varepsilon \in GL(W)$ is an element of order ≤ 2.
(vi) $(U(W_E), U(W))$ where W is a hermitian space over some extension (D', τ') of (D, τ), (E, τ'') is some quadratic extension of (D', τ') and $W_E = W \otimes_{D'} E$ is an extension of scalars with the corresponding hermitian structure.
(vii) $(U(W), U(W)_\varepsilon)$ where W is a hermitian space over (D, τ).

For the proof of this theorem we will need the following straightforward lemma.

Lemma 6.6.2. *Let (V, B) be a hermitian space over (D, τ).*
Let $\varepsilon \in U(V)$ be an element of order 2. Let $x \in U(V)$ such that $\varepsilon x \varepsilon = x^{-1}$. Let Q be an irreducible polynomial. Then ε gives an D_Q- linear isomorphism $V^{ij}_{Q,x} \cong V^{1-i,j}_{Q,x}$.

Proof of Theorem 6.6.1. Let $x \in U(V)^\sigma$ be a semi-simple element and let P be its minimal polynomial.

Note that the minimal polynomial of x^* is P^* and hence P^* is proportional to P. Since $x \in U(V)^\sigma$, we have $x^{-1} = \varepsilon x \varepsilon$ and hence its minimal polynomial is P. Hence P is proportional to $inv(P)$. We will now discuss 7 special cases and then deduce the general case from them.

Case 1. $P = QQ^* inv(Q)\tau(Q)$, where Q is an irreducible polynomial.
Note that $GL(V)_x \cong \prod_{ij} GL(V^{ij}_{Q,x}) \cong GL(V^{00}_{Q,x})^4$. This identifies $U(V)_x$ with a diagonal $\Delta GL(V^{00}_{Q,x})^2 < GL(V^{00}_{Q,x})^4$ and $(U(V)_\varepsilon)_x$ with a diagonal $\Delta GL(V^{00}_{Q,x}) < GL(V^{00}_{Q,x})^4$.

Case 2. $P = Qinv(Q)$, where Q is an irreducible polynomial and $Q^* = Q$.
Note that $GL(V)_x \cong \prod_i GL(V^{i0}_{Q,x}) \cong GL(V^{00}_{Q,x})^2$. Note also that in this case $V^{i0}_{Q,x}$ and $V^{i1}_{Q,x}$ are identical as sets and F-vector spaces but the actions of D_Q on them differ by a twist by μ^{01}. Now the isomorphism $V^{i0}_{Q,x} \cong (V^{i1}_{Q,x})^*$ gives a (D_Q, μ^{01})-hermitian structure on $V^{i0}_{Q,x}$. Therefore, $U(V)_x \cong U(V^{00}_{Q,x}) \times U(V^{10}_{Q,x})$. Note that ε gives an isomorphism of (D_Q, μ^{01})-hermitian spaces between $V^{00}_{Q,x}$ and $V^{01}_{Q,x}$. Hence

$$U(V)_x \cong U(V^{00}_{Q,x})^2 \text{ and } (U(V)_\varepsilon)_x \cong \Delta U(V^{00}_{Q,x}) < U(V^{00}_{Q,x})^2.$$

Case 3. $P = Qinv(Q)$, where Q is an irreducible polynomial and $Q^* = inv(Q)$.
Note that $GL(V)_x \cong \prod_i GL(V^{i0}_{Q,x}) \cong \prod_j GL(V^{0j}_{Q,x})$.

Since B defines a non-degenerate pairing $V^{00}_{Q,x} \cong (V^{01}_{Q,x})^*$ and $V^{0i}_{Q,x}$ are isotropic, we have

$$U(V)_x \cong \Delta GL(V^{00}_{Q,x}) < (GL(V^{00}_{Q,x}))^2.$$

Note that in this case $V^{ij}_{Q,x}$ and $V^{1-i,1-j}_{Q,x}$ are identical as sets and as F-vector spaces but the action of D_Q on them differs by a twist by μ^{11}.

Now, compose the isomorphism $V^{00}_{Q,x} \cong V^{10}_{Q,x}$ given by ε with the isomorphism $V^{10}_{Q,x} \cong (V^{11}_{Q,x})^*$. This gives a (D_Q, μ^{11}) unitary structure on $V^{00}_{Q,x}$. Similarly we get a unitary structure on $V^{10}_{Q,x}$. Finally, ε gives an isomorphism $V^{00}_{Q,x} \cong V^{10}_{Q,x}$ as unitary spaces and hence

$$(U(V)_\varepsilon)_x \cong \Delta U(V^{00}_{Q,x}) < \Delta GL(V^{00}_{Q,x}).$$

Case 4. $P = QQ^*$, where Q is an irreducible polynomial, $Q = inv(Q)$ and $x \neq x^{-1}$.

Note that $GL(V)_x \cong \prod_j GL(V^{0j}_{Q,x})$ and as before

$$U(V)_x \cong \Delta GL(V^{00}_{Q,x}) < (GL(V^{00}_{Q,x}))^2.$$

In this case $V^{0j}_{Q,x}$ and $V^{1j}_{Q,x}$ are identical as sets and as F-vector spaces but the action of D_Q on them differs by a twist by μ^{10}. Hence ε gives a (D_Q, μ^{10}) anti-linear automorphism of $V^{0j}_{Q,x}$. Let $W_j := (V^{0j}_{Q,x})^{\varepsilon}$. This is a linear space over $(D_Q)^{\mu^{10}}$. Therefore,

$$(U(V)_{\varepsilon})_x \cong \Delta GL(W_0) < \Delta GL(V^{00}_{Q,x}).$$

Case 5. $P = QQ^*$, where Q is an irreducible polynomial, $Q = inv(Q)$ and $x = x^{-1}$.
As in the previous case,

$$GL(V)_x \cong \prod_j GL(V^{0j}_{Q,x}) \text{ and } U(V)_x \cong \Delta GL(V^{00}_{Q,x}) < (GL(V^{00}_{Q,x}))^2.$$

In this case $D_Q = D$ and μ^{10} is trivial. Hence $V^{0j}_{Q,x}$ and $V^{1j}_{Q,x}$ are identical as D_Q-linear spaces.

Also, ε gives a D_Q-linear automorphism of $V^{0j}_{Q,x}$. So we can interpret ε as an element in $GL(V^{00}_{Q,x})$. Therefore,

$$(U(V)_{\varepsilon})_x \cong \Delta(GL(V^{00}_{Q,x}))_{\varepsilon} < \Delta GL(V^{00}_{Q,x}).$$

Case 6. P is irreducible and $x \neq x^{-1}$
In this case $GL(V)_x \cong GL(V^{00}_{P,x})$. Also, $V^{00}_{P,x}$ and $V^{01}_{P,x}$ are identical as F-vector spaces but the action of D_P on them differs by a twist by μ^{01}. Again, the isomorphism $V^{00}_{P,x} \cong (V^{01}_{P,x})^*$ gives a (D_P, μ^{01}) hermitian structure on $V^{00}_{P,x}$ and

$$U(V)_x \cong U(V^{00}_{P,x}).$$

Note that $V^{00}_{P,x}$ and $V^{10}_{P,x}$ are identical as F-vector spaces but the action of D_P on them differs by a twist by μ^{10}. Hence, ε gives a (D_P, μ^{10}) anti-linear automorphism of $V^{00}_{P,x}$. Denote $W := (V^{00}_{P,x})^{\varepsilon}$. It is a linear space over $(D_P)^{\mu^{10}}$. It has a $((D_P)^{\mu^{10}}, \mu^{01}|_{(D_P)^{\mu^{10}}})$ hermitian structure. Now

$$(U(V)_{\varepsilon})_x \cong U(W) < U(V^{00}_{P,x}).$$

Case 7. P is irreducible and $x = x^{-1}$.
Again,

$$GL(V)_x \cong GL(V^{00}_{P,x}) \text{ and } U(V)_x \cong U(V^{00}_{P,x}).$$

In this case $D_P = D$ and $\mu^{01} = \tau$. Also, ε commutes with x and hence $\varepsilon \in U(V)_x \cong U(V^{00}_{P,x})$. Hence

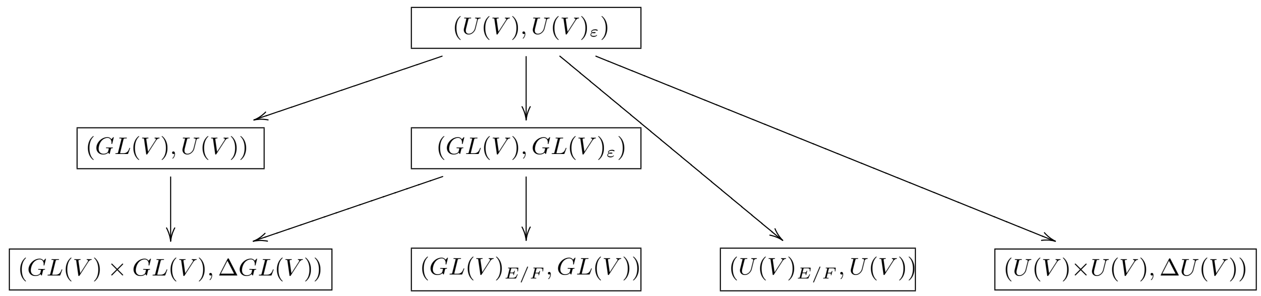
$$(U(V)_{\varepsilon})_x \cong U(V^{00}_{P,x})_{\varepsilon} < U(V^{00}_{P,x}).$$

Case 8. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible multiples. Since P is proportional to $inv(P)$, every P_i is proportional to $P_{s_1(i)}$ for some permutation s_1 of I of order ≤ 2. Since P is proportional to P^*, every P_i is proportional to some $P_{s_2(i)}$. This gives rise to an action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on I.

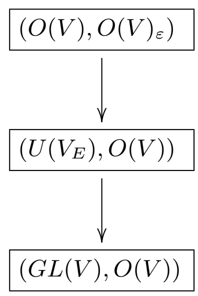
Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of this action. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V = \bigoplus V_\alpha$ and V_α are orthogonal to each other and ε-invariant. Hence the pair $(U(V)_x, (U(V)_{\varepsilon})_x)$ is a product of pairs from the first 7 cases. $\square$

6.7. **Genealogical trees of the symmetric pairs considered in this paper.**
The following diagrams sum up the results of this section.

An arrow "$(G_1, H_1) \to (G_2, H_2)$" means that pairs of type (G_1, H_1) may have descendants with factor of the type (G_2, H_2). We will not draw the obvious arrows "$(G, H) \to (G, H)$" and when we draw "$(G_1, H_1) \to (G_2, H_2) \to (G_3, H_3)$" we mean also "$(G_1, H_1) \to (G_3, H_3)$".



Here V is a linear or hermitian space over some finite field extension of F and E is some quadratic extension of F.



Here V is a quadratic space over some finite field extension of F and E is some quadratic extension of F.

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A PARTIAL ANALOG OF THE INTEGRABILITY THEOREM FOR DISTRIBUTIONS ON P-ADIC SPACES AND APPLICATIONS

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Abstract. Let X be a smooth real algebraic variety. Let ξ be a distribution on it. One can define the singular support of ξ to be the singular support of the D_X-module generated by ξ (some times it is also called the characteristic variety). A powerful property of the singular support is that it is a coisotropic subvariety of T^*X. This is the integrability theorem (see [KKS, Mal, Gab]). This theorem turned out to be useful in representation theory of real reductive groups (see e.g. [AG4, AS, Say]).

The aim of this paper is to give an analog of this theorem to the non-Archimedean case. The theory of D-modules is not available to us so we need a different definition of the singular support. We use the notion wave front set from [Hef] and define the singular support to be its Zariski closure. Then we prove that the singular support satisfies some property that we call weakly coisotropic, which is weaker than being coisotropic but is enough for some applications. We also prove some other properties of the singular support that were trivial in the Archimedean case (using the algebraic definition) but not obvious in the non-Archimedean case.

We provide two applications of those results:

- a non-Archimedean analog of the results of [Say] concerning Gelfand property of nice symmetric pairs
- a proof of Multiplicity one Theorems for GL_n which is uniform for all local fields. This theorem was proven for the non-Archimedean case in [AGRS] and for the Archimedean case in [AG4] and [SZ].

Contents

Key words and phrases. singular support, waive front set, coistropic subvariety, integrability theorem, invariant distributions, multiplicity one, Gelfand pair, symmetric pair.

MSC Classes: 46F10, 20C99, 20G05, 20G25, 22E45.

1. Introduction

The theory of invariant distributions is widely used in representation theory of reductive algebraic groups over local fields. We can roughly divide this theory into two parts.

- Archimedean - distributions on smooth manifolds, Nash manifolds, real analytic manifolds, real algebraic manifolds, etc.
- Non-Archimedean - distributions on l-spaces, p-adic analytic manifolds, p-adic algebraic manifolds, etc.

In general the non-Archimedean case of the theory of invariant distributions is easier than the Archimedean one, but there is one significant tool that is available only in the Archimedean case. This tool is the theory of differential operators. One of the powerful tools coming from the use of differential operators is the notion of singular support (sometimes it is also called the characteristic variety). The singular support of a distribution ξ on a real algebraic manifold X is a subvariety of T^*X. A deep and important property of the singular support is the fact that it is coisotropic. This fact is the integrability theorem (see [KKS, Mal, Gab]). This theorem turned out to be useful in the representation theory of real reductive groups (see e.g. [AG4, AS, Say]).

The aim of this paper is to give an analog of this theorem to the non-Archimedean case. Though we didn't achieve a full analog of the integrability theorem, we managed to formulate and prove some partial analog of it. Namely we prove that the singular support satisfies some property that we call weakly coisotropic, which is weaker than being coisotropic but enough for some applications. We also prove some other properties of the singular support that were trivial in the Archimedean case but not obvious in the non-Archimedean case.

We provide two applications of those results.

- We give a non-Archimedean analog of the results of [Say] concerning Gelfand property of nice symmetric pairs.
- We give a proof of Multiplicity one Theorems for GL_n which is uniform for all local fields. This theorem was proven for the non-Archimedean case in [AGRS] and for the non-Archimedean case in [AG4] and [SZ].

The results of this paper are also applied in [Sun] where multiplicity one theorems for Fourier-Jacobi models are established.

1.1. **The singular support and the wave front set.**

The theory of D-modules is not available to us so we need a different definition of singular support.

We use the notion of wave front set from [Hef] and define the singular support to be its Zariski closure. Unlike the algebraic definition of the singular support, the definition of the wave front set is analytic and uses Fourier transform instead of differential operators, this is what makes it available for the non-Archimedean case.

Surprisingly, the fact that in the non-Archimedean case the singular support is weakly coisotropic quite easily follows from the basic properties of the wave front set developed in [Hef]. However another important property of the the singular support that was trivial in the Archimedean case is not obvious in the non-Archimedean case. Namely in presence of a group action one can exhibit some restriction on the singular support of invariant distribution. We also provide a non-Archimedean analog of this property.

In general our results are based on the work [Hef] where the theory of the wave front set is developed for the non-Archimedean case.

1.2. **Structure of the paper.**
In section 2 we give notations that will be used throughout the paper and give some preliminaries on distributions, including some results from [Hef] on the wave front set.

In section 3 we introduce the notion of coistropic variety and weakly coistropic variety and discuss some properties of them.

In section 4 we prove the main results on singular support and the wave front set. We sum up the properties of singular support in subsection 4.2. In subsection 4.3 we apply those properties to get some technical results that will be useful for proving Gelfand property.

In section 5 we generalize the results of [Say] to arbitrary local fields of characteristic 0.

In subsection 5.1 we give the necessary preliminaries for section 5. In subsubsection 5.1.1 we provide basic preliminaries on Gelfand pairs. In subsubsection 5.1.2 we review a technique from [AG2] for proving that a given pair is a Gelfand pair. In subsubsections 5.1.3-5.1.7 we review a technique from [AG2] and [AG3] for proving that a given symmetric pair is a Gelfand pair.

In section 6 we indicate a proof of Multiplicity one Theorems for GL_n which is uniform for all local fields of characteristic 0. This theorem was proven for the non-Archimidian case in [AGRS] and for the non-Archimidian case in [AG4] and [SZ].

1.3. **Acknowledgements.**
I wish to thank **Dmitry Gourevitch**, **Anthony Joseph** and **Eitan Sayag** for fruitful discussions. Also I cordially thank **Dmitry Gourevitch** for his careful proof reading.

2. Notations and preliminaries

- Throughout the paper F is a local field of characteristic zero.
- All the algebraic varieties, analytic varieties and algebraic groups that we will consider will be defined over F.
- By a reductive group we mean an algebraic reductive group.
- Let E be an extension of F. Let G be an algebraic group defined over F. We denote by $G_{E/F}$ the canonical algebraic group defined over F such that $G_{E/F}(F) = G(E)$.
- By Sp_{2n} we mean the symplectic group of $2n \times 2n$ matrixes.
- The word manifold will always mean that the object is smooth (e.g. by algebraic manifold we mean smooth algebraic variety).
- For a group G acting on a set X and a point $x \in X$ we denote by Gx or by $G(x)$ the orbit of x and by G_x the stabilizer of x. we also denote by X^G the set of G invariant elements and for an element $g \in G$ denote by X^g the set of g invariant elements

- An action of a Lie algebra $\mathfrak{g}$ on a (smooth, algebraic, etc) manifold M is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Note that an action of a (Lie, algebraic, etc) group on M defines an action of its Lie algebra on M.
- For a Lie algebra $\mathfrak{g}$ acting on M, an element $\alpha \in \mathfrak{g}$ and a point $x \in M$ we denote by $\alpha(x) \in T_xM$ t he value at point x of the vector field corresponding to α. We denote by $\mathfrak{g}x \subset T_xM$ or by $\mathfrak{g}(x) \subset T_xM$ the image of the map $\alpha \mapsto \alpha(x)$ and by $\mathfrak{g}_x \subset \mathfrak{g}$ its kernel. We denote $M^{\mathfrak{g}} := \{x \in M | \mathfrak{g}x = 0\}$ and $M^{\alpha} := \{x \in M | \alpha(x) = 0\}$, analogously to the group case.
- For manifolds $L \subset M$ we denote by $N_L^M := (T_M|_L)/T_L$ the normal bundle to L in M.
- Denote by $CN_L^M := (N_L^M)^*$ the conormal bundle.
- For a point $y \in L$ we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space.
- Let M, N be (smooth, algebraic, etc) manifolds. Let E be a bundle over N. Let $\phi : M \to N$ be a morphism. We denote by $\phi^*(E)$ to be the pullback of E.
- Let M, N be (smooth, algebraic, etc) manifolds. Let $S \subset (T^*(N))$. Let $\phi : M \to N$ be a morphism. We denote $\phi^*(S) := d(\phi)^*(S \times_N M)$.
- Let M, N be topological spaces. Let E be a over N. Let $\phi : M \to N$ be a morphism. We denote by $\phi^*(E)$ to be the pullback of E.
- Let V be a linear space. For a point $x = (v, \phi) \in V \times V^*$ we denote $\widehat{x} = (\phi, -v) \in V^* \times V$, similarly for subset $X \subset V \times V^*$ we define $\widehat{X}$. for a (smooth, algebraic, etc) manifold and a subset $X \subset T^*(M \times V)$ we denote $\widehat{X}_V \subset T^*(M \times V^*)$ in a similar way.
- Let B be a non-degenerate bilinear form on V. This gives an identification between V and V^* and therefore, by the previous notation, maps $F_B : V \times V \to V \times V$ and $F_B : T^*M \times V \times V \to T^*M \times V \times V$. If there is no ambiguity we will denote it by F_V.

2.1. **Distributions.**

In this paper we will refer to distributions on algebraic varieties over archimedean and non-archimedean fields. In the non-archimedean case we mean the notion of distributions on l-spaces from [BZ], that is linear functionals on the space of locally constant compactly supported functions.

We will use the following notations.

Notation 2.1.1. *Let X be an l-space.*

- *Denote by $\mathcal{S}(X)$ the space of Schwartz functions on X (i.e. locally constant compactly supported functions) Denote $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ to be the dual space to $\mathcal{S}(X)$.*
- *For any locally constant sheaf E over X we denote by $\mathcal{S}(X, E)$ the space of compactly supported sections of E and by $\mathcal{S}^*(X, E)$ its dual space.*
- *For any finite dimensional complex vector space V we denote $\mathcal{S}(X, V) := \mathcal{S}(X, X \times V)$ and $\mathcal{S}^*(X, V) := \mathcal{S}^*(X, X \times V)$, where $X \times V$ is a constant sheaf.*
- *Let $Z \subset X$ be a closed subset. We denote*
$$\mathcal{S}_X^*(Z) := \{\xi \in \mathcal{S}^*(X) | \mathrm{Supp}(\xi) \subset Z\}.$$
For a locally closed subset $Y \subset X$ we denote $\mathcal{S}_X^*(Y) := \mathcal{S}_{X \setminus (\overline{Y} \setminus Y)}^*(Y)$. In the same way, for any locally constant sheaf E on X we define $\mathcal{S}_X^*(Y, E)$.
- *Suppose that X is an analytic variety over a non-Archimedean field F. Then we define D_X to be the sheaf of locally constant measures on X (i.e. measures that locally are restriction of Haar measure on F^n). We denote $\mathcal{G}(X) := \mathcal{S}^*(X, D_X)$ and $\mathcal{G}(X, E) := \mathcal{S}^*(X, D_X \otimes E^*)$.*
- *For an analytic map $\phi : X \to Y$ of analytic manifolds over non-Archimedean field we denote by $\phi^* : \mathcal{G}(Y) \to \mathcal{G}(X)$ the pullback, similarly we denote $\phi^* : \mathcal{G}(Y, E) \to \mathcal{G}(X, \phi^*(E))$ for any locally constant sheaf E.*

In the Archimedean case we will use the theory of Schwartz functions and distributions as developed in [AG1]. This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic*.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

Notation 2.1.2. *Let X be a Nash manifold.*

Denote by $\mathcal{S}(X)$ the space of Schwartz functions on X. Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the dual space to $\mathcal{S}(X)$. We define D_X to be the bundle of densities on X for any Nash bundle E on X we define $\mathcal{S}^*(X,E), \mathcal{S}^*_X(Y), \mathcal{G}(X), \phi^*$, etc analogously to the non-Archimedean case.

2.1.1. *Invariant distributions.*

Proposition 2.1.3. *Let an l-group G act on l-space X. Let $Z \subset X$ be a closed subset.*

Let $Z = \bigcup_{i=0}^{l} Z_i$ be a G-invariant stratification of Z. Let χ be a character of G. Suppose that for any $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i)^{G,\chi} = 0$. Then $\mathcal{S}^*_X(Z)^{G,\chi} = 0$.

This proposition immediately follows from [BZ, section 1.2].

Proposition 2.1.4. *Let a Nash group G act on a Nash manifold X. Let $Z \subset X$ be a closed subset.*

Let $Z = \bigcup_{i=0}^{l} Z_i$ be a Nash G-invariant stratification of Z. Let χ be a character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i, Sym^k(CN_{Z_i}^X))^{G,\chi} = 0$. Then $\mathcal{S}^*_X(Z)^{G,\chi} = 0$.

This proposition immediately follows from [AGS, Corollary 7.2.6].

Theorem 2.1.5 (Frobenius reciprocity)**.** *Let an l-group (respectively Nash group) G act transitively on an l-space (respectively Nash manifold) Z. Let $\varphi : X \to Z$ be a G-equivariant map. Let $z \in Z$. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z,\chi \cdot \Delta_G|_{G_z} \cdot \Delta_{G_z}^{-1}}$ where Δ denotes the modular character.*

For a proof see [Ber, section 1.5] for the non-Archimedean case and [AG2, Theorem 2.3.8] for the non-Archimedean case.

2.1.2. *Fourier transform.*
From now till the end of the paper we fix an additive character κ of F. If F is Archimedean we fix κ to be defined by $\kappa(x) := e^{2\pi i \operatorname{Re}(x)}$.

Notation 2.1.6. *Let V be a vector space over F. For any distribution $\xi \in \mathcal{S}^*(V)$ we define $\widehat{\xi} \in \mathcal{G}(V^*)$ to be its Fourier transform.*

For a space X (an l-space or a Nash manifold depending on F), for any distribution $\xi \in \mathcal{S}^*(X \times V)$ we define $\widehat{\xi}_V \in \mathcal{G}(X \times V^*)$ to be its partial Fourier transform

Let B be a non-degenerate bilinear form on V. Then B identifies $\mathcal{G}(V^*)$ with $\mathcal{S}^*(V)$. We denote by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ and $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the corresponding Fourier transforms.

If there is no ambiguity, we will write $\mathcal{F}_V$, and sometimes just $\mathcal{F}$, instead of $\mathcal{F}_B$.

We will use the following trivial observation.

Lemma 2.1.7. *Let V be a finite dimensional vector space over F. Let a Nash group G act linearly on V. Let B be a G-invariant non-degenerate symmetric bilinear form on V. Let $\xi \in \mathcal{S}^*(V)$ be a G-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also G-invariant.*

Notation 2.1.8. *Let V be a vector space over F. Consider the homothety action of $F^\times$ on V by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$.*

Also, for any $\lambda \in F^\times$ denote $|\lambda| := \frac{dx}{\rho(\lambda)dx}$, where dx denotes the Haar measure on F.

Notation 2.1.9. *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. We denote*

$$Z(B) := \{x \in V(F) | B(x,x) = 0\}.$$

Theorem 2.1.10 (Homogeneity Theorem)**.** *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a space(an l-space or a Nash manifold depending on F). Let $L \subset \mathcal{S}^*_{V(F)\times M}(Z(B) \times M)$ be a non-zero subspace such that $\forall \xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).*

Then there exist a non-zero distribution $\xi \in L$ and a unitary character u of $F^\times$ such that either $\rho(\lambda)\xi = ||\lambda||^{\frac{dimV}{2}} u(\lambda)\xi$ for any $\lambda \in F^\times$ or $\rho(\lambda)\xi = |\lambda|^{\frac{dimV}{2}+1} u(\lambda)\xi$ for any $\lambda \in F^\times$.

For a proof see [AG2, Theorem 5.1.7].

2.1.3. *The wave front set.*
In this subsubsection F is a non-Archimedean field. We will use the notion of the wave front set of a distribution on analytic space from [Hef]. First we will remind it for a distribution on an open subset of F^n.

Definition 2.1.11. *Let $U \subset F^n$ be an open subset and $\xi \in \mathcal{S}^*(U)$ be a distribution. We say that ξ is smooth at $(x_0, v_0) \in T^*U$ if there are open neighborhoods A of x_0 and B of v_0 such that for any $\phi \in \mathcal{S}(A)$ there is an $N_\phi > 0$ for which for any $\lambda \in F$ satisfying $\lambda > N_\phi$ we have $\widehat{(\phi\xi)}|_{\lambda B} = 0$. The complement in T^*U of the set of smooth pairs (x_0, v_0) of ξ is called the wave front set of ξ and denoted by $WF(\xi)$.*

Remark 2.1.12. *This notion appears in* [Hef] *with two differences.*

1) The notion in [Hef] *is more general and depends on some subgroup $\Lambda \subset F$, in our case $\Lambda = F$.*

2) The notion in [Hef] *defines the wave front set of ξ to be a subset in $T^*U - U \times 0$. In our notation this subset will be $WF(\xi) - U \times 0$.*

The following lemmas are trivial

Lemma 2.1.13. *Let $U \subset F^n$ be an open subset and $\xi \in \mathcal{S}^*(U)$ be a distribution. Then $WF(\xi)$ is closed, invariant with respect to the homothety $(x, v) \mapsto (x, \lambda v)$ and*

$$p_U(WF(\xi)) = WF(\xi) \cap (U \times 0) = \mathrm{Supp}(\xi).$$

Lemma 2.1.14. *Let $V \subset U \subset F^n$ be open subsets and $\xi \in \mathcal{S}^*(U)$ then $WF(\xi|_V) = WF(\xi) \cap p_U^{-1}(V)$.*

Lemma 2.1.15. *Let $U \subset F^n$ be an open subset, $\xi_1, \xi_2 \in \mathcal{S}^*(X)$ be distributions and f_1, f_2 be locally constant functions on X. Then $WF(f_1\xi_1 + f_2\xi_2) \subset WF(\xi_1) \cup WF(\xi_2)$.*

Corollary 2.1.16. *For any locally constant sheaf E on U we can define the wave front set of any element in $\mathcal{S}^*(U, E)$ and $\mathcal{G}(U, E)$.*

We will use the following theorem from [Hef], see Theorem 2.8.

Theorem 2.1.17. *Let $U \subset F^m$ and $V \subset F^n$ be open subsets, and suppose that $f : U \to V$ is an analytic submersion. Then for any $\xi \in \mathcal{G}(V)$ we have $WF(f^*(\xi)) \subset f^*(WF(\xi))$.*

Corollary 2.1.18. *Let $V, U \subset F^n$ be open subsets and $f : V \to U$ be an analytic isomorphism. Then for any $\xi \in \mathcal{G}(V)$ we have $WF(f^*(\xi)) = f^*(WF(\xi))$.*

Corollary 2.1.19. *Let X be an analytic manifold, E be a locally constant sheaf on X. We can define the the wave front set of any element in $\mathcal{S}^*(X, E)$ and $\mathcal{G}(X, E)$. Moreover, Theorem 2.1.17 holds for submersions between analytic manifolds.*

3. Coisotropic varieties

Definition 3.0.1. *Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M-**coisotropic** if one of the following equivalent conditions holds.*
(i) The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.
(ii) At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^{\perp}$. Here, $(T_zZ)^{\perp}$ denotes the orthogonal complement to T_zZ in T_zM with respect to ω.
(iii) For a generic smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^{\perp}$.
If there is no ambiguity, we will call Z a coisotropic variety.

Note that every non-empty M-coisotropic variety is of dimension at least $\frac{1}{2}\dim M$.

Notation 3.0.2. *For a smooth algebraic variety X we always consider the standard symplectic form on T^*X. Also, we denote by $p_X : T^*X \to X$ the standard projection.*

Definition 3.0.3. *Let (V, ω) be a symplectic vector space with a fixed Lagrangian subspace $L \subset V$. Let $p : V \to V/L$ be the standard projection. Let $Z \subset V$ be a linear subspace. We call it V-weakly coisotropic with respect to L if one of the following equivalent conditions holds.*
(i) $p(Z) \supset p(Z^{\perp})$. Here, $Z^{\perp}$ denotes the orthogonal complement with respect to ω.
(ii) $p(Z)^{\perp} \subset Z \cap L$. Here, $p(Z)^{\perp}$ denotes the orthogonal complement in L under the identification $L \cong (V/L)^*$.

Definition 3.0.4. *Let X be a smooth algebraic variety. Let $Z \subset T^*X$ be an algebraic subvariety. We call it T^*X-**weakly coisotropic** if one of the following equivalent conditions holds.*
(i) At every smooth point $z \in Z$ the space $T_z(Z)$ is $T_z(T^*(X))$ -weakly coisotropic with respect to $Ker(dp_X)$.
(ii)For a generic smooth point $z \in Z$ the space $T_z(Z)$ is $T_z(T^*(X))$ -weakly coisotropic with respect to $Ker(dp_X)$.
(iii) For any smooth point $x \in Z$ and for a generic smooth point $y \in p_X^{-1}(x) \cap Z$ we have $CN^X_{p_X(Z),x} \subset T_y(p_X^{-1}(x) \cap Z)$.
(iv) For any smooth point $x \in p_X(Z)$ the fiber $p_X^{-1}(x) \cap Z$ is locally invariant with respect to shifts by $CN^X_{p_X(Z),x}$ i.e. for any point $y \in p_X^{-1}(x)$ the intersection $(y + CN^X_{p_X(Z),x}) \cap (p_X^{-1}(x) \cap Z)$ is Zariski open in $y + CN_{p_X(Z)}$.
If there is no ambiguity, we will call Z a weakly coisotropic variety.

Note that every non-empty T^*X-weakly coisotropic variety is of dimension at least $\dim X$.
The following lemma is straightforward.

Lemma 3.0.5. *Any T^*X-coisotropic variety is T^*X-weakly coisotropic.*

Proposition 3.0.6. *Let X be a smooth algebraic variety with a symplectic form on it. Let $R \subset T^*X$ be an algebraic subvariety. Then there exists a maximal T^*X-weakly coisotropic subvariety of R i.e. a T^*X-weakly coisotropic subvariety $T \subset R$ that includes all T^*X-weakly coisotropic subvarieties of R.*

Proof. Let T' be the union of all smooth T^*X-weakly coisotropic subvarieties of R. Let T be the Zariski closure of T' in R. It is easy to see that T is the maximal T^*X-weakly coisotropic subvariety of R. ☐

The following lemma is trivial.

Lemma 3.0.7. *Let X be a smooth algebraic variety. Let a group G act on X this induces an action on T^*X. Let $S \subset T^*X$ be a G-invariant subvariety. Then the maximal T^*X-weakly coisotropic subvariety of S is also G-invariant.*

Notation 3.0.8. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be any subvariety. We define* **the restriction** $R|_Z \subset T^*Z$ **of** R **to** Z *by* $R|_Z := i^*(R)$, *where $i : Z \to Y$ is the embedding.*

Lemma 3.0.9. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be a weakly coisotropic subvariety. Assume that any smooth point $z \in Z \cap p_Y(R)$ is also a smooth point of $p_Y(R)$ and we have $T_z(Z \cap p_Y(R)) = T_z(Z) \cap T_z(p_Y(R))$.*

Then $R|_Z$ is T^*Z-weakly coisotropic.

Proof. Let $x \in Z$, let $M := p_Y^{-1}(x) \cap R \subset p_Y^{-1}(x)$ and $L := CN^Y_{p_Y(R),x} \subset p_Y^{-1}(x)$. We know that M is locally invariant with respect to shifts in L. Let $M' := p_Z^{-1}(x) \cap R|_Z \subset p_Z^{-1}(x)$ and $L' := CN^Y_{p_Z(R|_Z),x} \subset p_Z^{-1}(x)$. We want to show that M' is locally invariant with respect to shifts in L'. Let $q : p_Y^{-1}(x) \to p_Z^{-1}(x)$ be the standard projection. Note that $M' = q(M)$ and $L' = q(L)$. Now clearly M' is locally invariant with respect to shifts in L'. ☐

Corollary 3.0.10. *Let Y be a smooth algebraic variety. Let an algebraic group H act on Y. Let $q : Y \to B$ be an H-equivariant morphism. Let $O \subset B$ be an orbit. Consider the natural action of G on T^*Y and let $R \subset T^*Y$ be an H-invariant subvariety. Suppose that $p_Y(R) \subset q^{-1}(O)$. Let $x \in O$. Denote $Y_x := q^{-1}(x)$. Then*

- *if R is T^*Y-weakly coisotropic then $R|_{Y_x}$ is $T^*(Y_x)$-weakly coisotropic.*

Corollary 3.0.11. *In the notation of the previous corollary, if $R|_{Y_x}$ has no (non-empty) $T^*(Y_x)$-weakly coisotropic subvarieties then R has no (non-empty) $T^*(Y)$-weakly coisotropic subvarieties.*

Remark 3.0.12. *The results on weakly coistropic varieties that we presented here have versions for coistropic varieties, see* [AG4, section 5.1].

4. Properties of singular support and the wave front set

4.1. **The wave front set.**

In this subsection F is a non-Archimedean field.

Theorem 4.1.1. *Let $Y \subset X$ be algebraic varieties, let $y \in Y(F)$ and suppose that X is smooth and Y is smooth at y. Let $\xi \in \mathcal{S}^*(X(F), E)$ and suppose that* $\mathrm{Supp}(\xi) \subset Y(F)$. *Then $WF(\xi) \cap p_X^{-1}(y)(F)$ is invariant with respect to shifts by $CN^X_{Y,y}(F)$.*

This theorem immediately follows from the following one

Theorem 4.1.2. *Let $Y \subset X$ be analytic manifolds and let $y \in Y$. Let $\xi \in \mathcal{S}^*_X(Y)$ and suppose that* $\mathrm{Supp}(\xi) \subset Y$. *Then $WF(\xi) \cap p_X^{-1}(y)$ is invariant with respect to shifts by $CN^X_{Y,y}$.*

In order to prove this theorem we will need the following standard lemma which is a version of the implicit function theorem.

Lemma 4.1.3. *Let $Y \subset X$ be analytic manifolds. Let $n := \dim(X)$ and $k := \dim(Y)$. Let $y \in Y$. Then there exist a open neighborhood $y \in U \subset X$ and an analytic isomophism $\phi : U \to W$, where W is open subset of F^n such that $\phi(Y \cap U) = W \cap F^k$, where $F^k \subset F^n$ is a coordinate subspace.*

Proof of theorem 4.1.2.

Case 1: $X = F^n$, $Y = F^k$.
in this case the theorem follows from the fact that if a distribution on F^n is supported on F^k then its Fourier transform is invariant with respect to shifts by the orthogonal complement to F^k.

Case 2: $X = U \subset F^n$, $Y = F^k \cap U$, where $U \subset F^n$ is open.
Follows immediately from the previous case.

Case 3: the general case.
Follows from the previous case using the lemma and theorem 2.1.18. ☐

Theorem 4.1.4. *Let an algebraic group G act on a smooth algebraic variety X. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X)^G$. Then $WF(\xi) \subset \{(x,v) \in T^*X(F) | v(\mathfrak{g}x) = 0\}$.*

We will prove a slightly more general theorem.

Theorem 4.1.5. *Let an analytic group G act on an analytic manifold X. Let E be a G-equivariant locally constant sheaf on X. Let $\xi \in \mathcal{G}(X,E)^G$. Then $WF(\xi) \subset \{(x,v) \in T^*X(F) | v(\mathfrak{g}x) = 0\}$.*

In order to prove this theorem we will need the following easy lemma.

Lemma 4.1.6. *Let X, Y be analytic manifolds. Let E be a locally constant sheaf on X. Let $\xi \in \mathcal{G}(X,E)$. Let $p : X \times Y \to X$ be the projection. Then $WF(p^*(\xi)) = p^*(WF(\xi))$.*

Proof of theorem 4.1.5. Consider the action map $m : G \times X \to X$ and the projection $p : G \times X \to X$. Let $S := WF(\xi)$. We are given an isomorphism $p^*(E) \cong m^*(E)$ and we know that under this identification $p^*(\xi) = m^*(\xi)$. Therefore $WF(p^*(\xi)) = WF(m^*(\xi))$. By the lemma we have $WF(p^*(\xi)) = p^*(S)$. by theorem 2.1.17 we have $WF(m^*(\xi)) \subset m^*(S)$. Thus we got $p^*(S) \subset m^*(S)$ which implies the requested inclusion. ☐

4.2. **Singular support.**

Definition 4.2.1. *Let X be a smooth algebraic variety let $\xi \in \mathcal{S}^*(X(F))$. We will now define the singular support of ξ, it is an algebraic subvariety of T^*X and we will denote it by $SS(\xi)$.*

In the case when F is non-Archimedean we define it to be the Zariski closure of $WF(\xi)$. In the case when F is Archimedean we define it to be the singular support of the D_X-module generated by ξ (as in [AG4]*).*

In [AG4, section 2.3] the following list of properties of the singular support for the Archimedean case was introduced:

Let X be a smooth algebraic variety.

(1) Let $\xi \in \mathcal{S}^*(X(F))$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))(F)$, where $\overline{\mathrm{Supp}(\xi)}_{Zar}$ denotes the Zariski closure of $\mathrm{Supp}(\xi)$.
(2) Let an algebraic group G act on X. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X(F))^{G(F)}$. Then
$$SS(\xi) \subset \{(x,\phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g}\, \phi(\alpha(x)) = 0\}.$$
(3) Let V be a linear space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that $\mathrm{Supp}(\xi) \subset Z(F)$. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p^{-1}_{X \times V}(Z))$.
(4) Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(F))$. Then $SS(\xi)$ is coisotropic.

Remark 4.2.2. *Property 4 is a corollary of the integrability theorem (see* [KKS, Mal, Gab]*).*

The result of the last subsection implies the following theorem

Theorem 4.2.3. *The properties above satisfied for the non-Archimedean case with the following modification, property 4 should be replaced by the following weaker one:*

(4') Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(F))$. Then $SS(\xi)$ is weakly coisotropic.

We conjecture that property 4 holds for the non-Archimedean case without modification.

4.3. **Distributions on non distinguished nilpotent orbits.**

In this subsection we deduce from the properties of singular support some technical results that are useful for proving Gelfand property.

Notation 4.3.1. *Let V be an algebraic finite dimensional representation over F of a reductive group G. We denote*
$$Q(V) := (V/V^G)(F).$$

Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V(F)$. We also denote

$$\Gamma(V) = \{y \in V(F) \,|\, \overline{G(F)}y \ni 0\}.$$

Note that $\Gamma(V) \subset Q(V)$. We denote also $R(V) := Q(V) - \Gamma(V)$.

Definition 4.3.2. *Let V be an algebraic finite dimensional representation over F of a reductive group G. Suppose that there is a finite number of G orbits in $\Gamma(V)$. Let $x \in \Gamma(V)$. We will call it G-distinguished, if $CN_{Gx,x}^{Q(V)} \subset \Gamma(V^*)$. We will call a G orbit G-distinguished if all (or equivalently one of) its elements are G- distinguished.*

If there is no ambiguity we will omit the "G-".

Example 4.3.3. *For the case of a semi-simple group acting on its Lie algebra, the notion of G-distinguished element coincides with the standard notion of distinguished nilpotent element. In particular, in the case when $G = SL_n$ and $V = sl_n$ the set of G-distinguished elements is exactly the set of regular nilpotent elements.*

Proposition 4.3.4. *Let V be an algebraic finite dimensional representation over F of a reductive group G. Suppose that there is a finite number of G orbits on $\Gamma(V)$. Let $W := Q(V)$, let A be the set of non-distinguished elements in $\Gamma(V)$. Then there are no non-empty $W \times W^*$-weakly coisotropic subvarieties of $A \times \Gamma(V^*)$.*

The proof is clear.

Corollary 4.3.5. *Let $\xi \in \mathcal{S}^*(W)$ and suppose that* $\mathrm{Supp}(\xi) \subset \Gamma(V)$ *and supp*$(\widehat{\xi}) \subset \Gamma(V^*)$. *Then the set of distinguished elements in* $\mathrm{Supp}(\xi)$ *is dense in* $\mathrm{Supp}(\xi)$

Remark 4.3.6. *In the same way one can prove an analogous result for distributions on $W \times M(F)$ for any algebraic variety M.*

5. Applications towards Gelfand properties of symmetric pairs

In this section we will use the property of singular support to generate the results of [Say] for any local field of characteristic 0. Namely we prove that a big class of symmetric pairs are *regular*. The property of regularity of symmetric pair was introduced in [AG2] and was shown to be useful for proving Gelfand property. We will give more details on the regularity property and its connections with Gelfand property in subsubsections 5.1.3-5.1.7.

5.1. **Preliminaries.**

In this subsection we give the necessary preliminaries for section 5.

5.1.1. *Gelfand pairs.*
In this subsubsection we recall a technique due to Gelfand and Kazhdan (see [GK]) which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS, section 2].

Definition 5.1.1. *Let G be a reductive group. By an* ***admissible representation of*** *G we mean an admissible representation of $G(F)$ if F is non-Archimedean (see* [BZ]*) and admissible smooth Fréchet representation of $G(F)$ if F is Archimedean.*

We now introduce three notions of Gelfand pair.

Definition 5.1.2. *Let $H \subset G$ be a pair of reductive groups.*

- *We say that (G, H) satisfy* **GP1** *if for any irreducible admissible representation (π, E) of G we have*

$$\dim Hom_{H(F)}(E, \mathbb{C}) \leq 1$$

- *We say that (G,H) satisfy* **GP2** *if for any irreducible admissible representation (π,E) of G we have*
$$\dim Hom_{H(F)}(E,\mathbb{C})\cdot \dim Hom_{H(F)}(\widetilde{E},\mathbb{C})\leq 1$$
- *We say that (G,H) satisfy* **GP3** *if for any irreducible* **unitary** *representation $(\pi,\mathcal{H})$ of $G(F)$ on a Hilbert space $\mathcal{H}$ we have*
$$\dim Hom_{H(F)}(\mathcal{H}^{\infty},\mathbb{C})\leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vD, BvD]).

We have the following straightforward proposition.

Proposition 5.1.3. $GP1 \Rightarrow GP2 \Rightarrow GP3$.

We will use the following theorem from [AGS] which is a version of a classical theorem of Gelfand and Kazhdan.

Theorem 5.1.4. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H)=H$. Suppose $\tau(\xi)=\xi$ for all bi $H(F)$-invariant distributions ξ on $G(F)$. Then (G,H) satisfies GP2.*

Remark 5.1.5. *In many cases it turns out that GP2 is equivalent to GP1.*

5.1.2. *Tame actions.*
In this subsubsection we review some tools developed in [AG2] for solving problems of the following type. A reductive group G acts on a smooth affine variety X, and τ is an automorphism of X which normalizes the action of G. We want to check whether any $G(F)$-invariant Schwartz distribution on $X(F)$ is also τ-invariant.

Definition 5.1.6. *Let π be an action of a reductive group G on a smooth affine variety X. We say that an algebraic automorphism τ of X is G-**admissible** if*
(i) $\pi(G(F))$ is of index ≤ 2 in the group of automorphisms of X generated by $\pi(G(F))$ and τ.
(ii) For any closed $G(F)$ orbit $O \subset X(F)$, we have $\tau(O)=O$.

Definition 5.1.7. *We call an action of a reductive group G on a smooth affine variety X **tame** if for any G-admissible $\tau : X \to X$, we have $\mathcal{S}^*(X(F))^{G(F)} \subset \mathcal{S}^*(X(F))^{\tau}$.*

Definition 5.1.8. *We call an algebraic representation of a reductive group G on a finite dimensional linear space V over F **linearly tame** if for any G-admissible linear map $\tau : V \to V$, we have $\mathcal{S}^*(V(F))^{G(F)} \subset \mathcal{S}^*(V(F))^{\tau}$.*

*We call a representation **weakly linearly tame** if for any G-admissible linear map $\tau : V \to V$, such that $\mathcal{S}^*(R(V))^{G(F)} \subset \mathcal{S}^*(R(V))^{\tau}$ we have $\mathcal{S}^*(Q(V))^{G(F)} \subset \mathcal{S}^*(Q(V))^{\tau}$.*

Theorem 5.1.9. *Let a reductive group G act on a smooth affine variety X. Suppose that for any G-semisimple $x \in X(F)$, the action of G_x on $N^X_{Gx,x}$ is weakly linearly tame. Then the action of G on X is tame.*

For a proof see [AG2, Theorem 6.0.5].

Definition 5.1.10. *We call an algebraic representation of a reductive group G on a finite dimensional linear space V over F **special** if for any $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F)}$ such that for any G-invariant decomposition $Q(V)=W_1 \oplus W_2$ and any two G-invariant symmetric non-degenerate bilinear forms B_i on W_i the Fourier transforms $\mathcal{F}_{B_i}(\xi)$ are also supported in $\Gamma(V)$, we have $\xi=0$.*

Proposition 5.1.11. *Every special algebraic representation V of a reductive group G is weakly linearly tame.*

For a proof see [AG2, Proposition 6.0.7].

5.1.3. *Symmetric pairs.*
In the coming 4 subsubsections we review some tools developed in [AG2] that enable to prove that a symmetric pair is a Gelfand pair.

Definition 5.1.12. *A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^{\theta}$. We call a symmetric pair **connected** if G/H is connected.*

*For a symmetric pair (G, H, θ) we define an anti-involution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$, denote $\mathfrak{g} := LieG$, $\mathfrak{h} := LieH$, $\mathfrak{g}^{\sigma} := \{a \in \mathfrak{g} | \theta(a) = -a\}$. Note that H acts on $\mathfrak{g}^{\sigma}$ by the adjoint action. Denote also $G^{\sigma} := \{g \in G | \sigma(g) = g\}$ and define a **symmetrization map** $s : G \to G^{\sigma}$ by $s(g) := g\sigma(g)$.*

In case when the involution is obvious we will omit it.

Remark 5.1.13. *Let (G, H, θ) be a symmetric pair. Then $\mathfrak{g}$ has a $\mathbb{Z}/2\mathbb{Z}$ grading given by θ.*

Definition 5.1.14. *Let (G_1, H_1, θ_1) and (G_2, H_2, θ_2) be symmetric pairs. We define their **product** to be the symmetric pair $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$.*

Definition 5.1.15. *We call a symmetric pair (G, H, θ) **good** if for any closed $H(F) \times H(F)$ orbit $O \subset G(F)$, we have $\sigma(O) = O$.*

Proposition 5.1.16. *Every connected symmetric pair over $\mathbb{C}$ is good.*

For a proof see e.g. [AG2, Corollary 7.1.7].

Definition 5.1.17. *We say that a symmetric pair (G, H, θ) is a **GK pair** if any $H(F) \times H(F)$-invariant distribution on $G(F)$ is σ-invariant.*

Remark 5.1.18. *Theorem 5.1.4 implies that any GK pair satisfies GP2.*

5.1.4. *Descendants of symmetric pairs.*

Proposition 5.1.19. *Let (G, H, θ) be a symmetric pair. Let $g \in G(F)$ such that HgH is closed. Let $x = s(g)$. Then x is a semisimple element of G.*

For a proof see e.g. [AG2, Proposition 7.2.1].

Definition 5.1.20. *In the notations of the previous proposition we will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a **descendant** of (G, H, θ).*

5.1.5. *Tame symmetric pairs.*

Definition 5.1.21.

- *We call a symmetric pair (G, H, θ) **tame** if the action of $H \times H$ on G is tame*
- *We call a symmetric pair (G, H, θ) **linearly tame** if the action of H on $\mathfrak{g}^{\sigma}$ is linearly tame*
- *We call a symmetric pair (G, H, θ) **weakly linearly tame** if the action of H on $\mathfrak{g}^{\sigma}$ is weakly linearly tame*
- *We call a symmetric pair (G, H, θ) **special** if the action of H on $\mathfrak{g}^{\sigma}$ is special*

Remark 5.1.22. *Evidently, any good tame symmetric pair is a GK pair.*

Theorem 5.1.23. *Let (G, H, θ) be a symmetric pair. Suppose that all its descendants (including itself) are weakly linearly tame. Then (G, H, θ) is tame.*

For a proof see [AG2, Theorem 7.3.3].

5.1.6. *Regular symmetric pairs.*

Definition 5.1.24. *Let (G,H,θ) be a symmetric pair. We call an element $g \in G(F)$* ***admissible*** *if*
(i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
(ii) $Ad(g)|_{\mathfrak{g}^\sigma}$ is H-admissible.

Definition 5.1.25. *We call a symmetric pair (G,H,θ)* ***regular*** *if for any admissible $g \in G(F)$ such that every $H(F)$-invariant distribution on $R_{G,H}$ is also $Ad(g)$-invariant, we have*
(*) every $H(F)$-invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also $Ad(g)$-invariant.

The following two propositions are evident.

Proposition 5.1.26. *Let (G,H,θ) be symmetric pair. Suppose that any $g \in G(F)$ satisfying $\sigma(g)g \in Z(G(F))$ lies in $Z(G(F))H(F)$. Then (G,H,θ) is regular. In particular if the normalizer of $H(F)$ lies inside $Z(G(F))H(F)$ then (G,H,θ) is regular.*

Proposition 5.1.27.
(i) Any weakly linearly tame pair is regular.
(ii) A product of regular pairs is regular (see [AG2, Proposition 7.4.4]*).*

The importance of the notion of regular pair is demonstrated by the following theorem.

Theorem 5.1.28. *Let (G,H,θ) be a good symmetric pair such that all its descendants (including itself) are regular. Then it is a GK pair.*

For a proof see [AG2, Theorem 7.4.5].

5.1.7. *Defects of symmetric pairs.*
In this subsection we review some tools developed in [AG2] and [AG3] that enable to prove that a symmetric pair is special.

Definition 5.1.29. *We fix standard basis e,h,f of $sl_2(F)$. We fix a grading on $sl_2(F)$ given by $h \in sl_2(F)_0$ and $e,f \in sl_2(F)_1$. A* ***graded representation of*** *sl_2 is a representation of sl_2 on a graded vector space $V = V_0 \oplus V_1$ such that $sl_2(F)_i(V_j) \subset V_{i+j}$ where $i,j \in \mathbb{Z}/2\mathbb{Z}$.*

The following lemma is standard.

Lemma 5.1.30.
(i) Every graded representation of sl_2 which is irreducible as a graded representation is irreducible just as a representation.
(ii) Every irreducible representation V of sl_2 admits exactly two gradings. In one highest weight vector lies in V_0 and in the other in V_1.

Definition 5.1.31. *We denote by V_λ^w the irreducible graded representation of sl_2 with highest weight λ and highest weight vector of parity p where $w = (-1)^p$.*

The following lemma is straightforward.

Lemma 5.1.32. $(V_\lambda^w)^* = V_\lambda^{w(-1)^\lambda}$.

Definition 5.1.33. *Let π be a graded representation of sl_2. We define the* ***defect*** *of π to be*

$$def(\pi) = \mathrm{Tr}(h|_{(\pi^e)_0}) - \dim(\pi_1).$$

The following lemma is straightforward

Lemma 5.1.34.

$$def(\pi \oplus \tau) = def(\pi) + def(\tau) \quad (1)$$

$$def(V_\lambda^w) = \frac{1}{2}(\lambda w + w(\frac{1+(-1)^\lambda}{2}) - 1) = \frac{1}{2}\begin{cases} \lambda w + w - 1 & \lambda \text{ is even} \\ \lambda w - 1 & \lambda \text{ is odd} \end{cases} \quad (2)$$

Lemma 5.1.35. *Let $\mathfrak{g}$ be a $(\mathbb{Z}/2\mathbb{Z})$ graded Lie algebra. Let $x \in \mathfrak{g}_1$ be a nilpotent element. Then there exists a graded homomorphism $\pi_x : sl_2 \to \mathfrak{g}$ such that $\pi_x(e) = x$.*

For a proof see e.g. [AG2, Lemma 7.1.11].

Lemma 5.1.36. *The morphism π_x is unique up to the exponentiated adjoint action of $(\mathfrak{g}_0)_x(\overline{F})$.*

For a proof see e.g. [KR, Proposition 4].

Remark 5.1.37. *In fact, the proof in* [KR] *also shows that π_x is unique up to the exponentiated adjoint action of $(\mathfrak{g}_0)_x(F)$.*

Definition 5.1.38. *Let $\mathfrak{g}$ be a $(\mathbb{Z}/2\mathbb{Z})$ graded Lie algebra. Let $x \in \mathfrak{g}_1$. We define the defect of x by*

$$def(x) = def(ad \circ \pi_x).$$

Lemma 5.1.36 implies that $def(x)$ is well defined.

Lemma 5.1.39. *Let (G, H, θ) be a symmetric pair. Then there exists a G-invariant θ-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}$. In particular, $B|_{\mathfrak{h}}$ and $B|_{\mathfrak{g}^\sigma}$ are also non-degenerate and $\mathfrak{h}$ is orthogonal to $\mathfrak{g}^\sigma$.*

For a proof see e.g. [AG2, Lemma 7.1.9].
From now on we will fix such B and identify $\mathfrak{g}^\sigma$ with $(\mathfrak{g}^\sigma)^*$.

Lemma 5.1.40. *let (G, H, θ) be a symmetric pair. Assume that $\mathfrak{g}$ is semi-simple. Then*
(i) for any $x \in \mathfrak{g}^\sigma$ we have $CN_{Hx,x}^{\mathfrak{g}^\sigma} = (\mathfrak{g}^\sigma)^x$
(ii) $Q(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$.

Proof.
(i) is trivial.
(ii) assume the contrary: there exist $0 \neq x \in \mathfrak{g}^\sigma$ such that $Hx = x$. Then $\dim(CN_{Hx,x}^{\mathfrak{g}^\sigma}) = \dim \mathfrak{g}^\sigma$, hence $CN_{Hx,x}^{\mathfrak{g}^\sigma} = \mathfrak{g}^\sigma$ which means, $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma)^x$. therefor x lies in the center of $\mathfrak{g}$ which is impossible. ☐

Proposition 5.1.41. *Let (G, H, θ) be a symmetric pair. Let $\xi \in \mathcal{S}^*(Q(\mathfrak{g}^\sigma))$. Suppose that both ξ and $\mathcal{F}(\xi)$ are supported on $\Gamma(\mathfrak{g}^\sigma)$. Then the set of elements in* Supp(ξ) *which have non-negative defect is dense in* Supp(ξ)

The proof is the same as the proof of [AG2, Proposition 7.3.7].

5.2. **All the nice symmetric pairs are regular.**

Definition 5.2.1. *let (G, H, θ) be a symmetric pair Let $x \in \Gamma(\mathfrak{g}^\sigma)$ be a nilpotent element. we will call it distinguished if it is distinguished with respect to the action of H on $\mathfrak{g}^\sigma$.*

Lemma 5.2.2. *Our definition of distinguished element coincides with the one in* [Sek]. *Namely an element $x \in \Gamma(\mathfrak{g}^\sigma)$ is distinguished iff $((\mathfrak{g}_s)^\sigma)^x$ does not contain semi-simple elements. Here $\mathfrak{g}_s$ is the semi-simple part of $\mathfrak{g}$.*

This lemma follows immediately from 5.1.40.

Definition 5.2.3. *We will call a symmetric pair* (G, H, θ) *a pair of negative distinguished defect if all the distinguished elements in* $\Gamma(\mathfrak{g}^\sigma)$ *have negative defect.*

Theorem 5.2.4. *Let* (G, H, θ) *be a symmetric pair of negative distinguished defect. Then it is special.*

Proof. Let $\xi \in \mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{H(F)}$ such that both ξ and $\mathcal{F}(\xi)$ are supported in $\Gamma(\mathfrak{g}^\sigma)$. Choose stratification

$$\Gamma(\mathfrak{g}^\sigma) = X_n \supset X_{n-1} \supset X_0 = 0 \supset X_{-1} = \emptyset$$

such that $X_i - X_{i-1}$ is an H-orbit which is open in X_i. We will prove by descending induction that ξ is supported on X_i. So we fix i and assume that ξ is supported on X_i, our aim is to prove that ξ is supported on X_{i-1}. Suppose that $X_i - X_{i-1}$ is non-distinguished. Then by Corollary 4.3.5 we have $\mathrm{Supp}(\xi) \subset X_{i-1}$. Now suppose that $X_i - X_{i-1}$ is distinguished. Then by Proposition 5.1.41 we have $\mathrm{Supp}(\xi) \subset X_{i-1}$. ☐

We will use the notion of nice symmetric pair from [LS]. We will use the following definition.

Definition 5.2.5. *A symmetric pair* (G, H, θ) *is called nice iff the semi simple part of the pair* $(\mathfrak{g}, \mathfrak{h})$ *decomposes, over the algebraic closure, to a product of pairs of the following types:*

- $(g_1 \oplus g_1, g_1)$, *where* g_1 *is a simple Lie algebra*
- (sl_m, so_m)
- $(sl_{2m}, sl_m \oplus sl_m \oplus \mathfrak{g}_a)$, *where* $\mathfrak{g}_a$ *is the one dimensional Lie algebra.*
- $(sp_{2m}, sl_m \oplus \mathfrak{g}_a)$
- $(so_{2m+k}, so_{m+k} \oplus so_m)$, *for* $k = 0, 1, 2$
- (e_6, sp_8)
- $(e_6, sl_6 \oplus sl_2)$
- (e_7, sl_8)
- (e_8, so_{16})
- $(f_4, sp_6 \oplus sl_2)$
- $(g_2, sl_2 \oplus sl_2)$

This notion is motivated by [Sek], where the following theorem is proven (see Theorem 6.3).

Theorem 5.2.6. *Let* (G, H, θ) *be a nice symmetric pair. Let* $\pi : sl_2 \to \mathfrak{g}$ *be a graded homomorphism such that* $\pi(e)$ *is distinguished. Consider* $\mathfrak{g}$ *as a graded representation of* sl_2, *decompose it to irreducible representations by* $\mathfrak{g} = \bigoplus V_{\lambda_i}^{\omega_i}$. *Then*

$$\sum_{i \text{ s.t. } \omega_i(-1)^{\lambda_i} = -1} (\lambda_i + 2) - \dim(\mathfrak{g}^\sigma) > 0.$$

Corollary 5.2.7. *Any nice symmetric pair is of negative distinguished defect. Thus by Theorem 5.2.4 it is special and hence weakly linearly tame and regular.*

This corollary follows immediately from the theorem using the following lemma and the fact that $\mathfrak{g} \cong \mathfrak{g}^*$ as a graded representation of sl_2

Lemma 5.2.8. *Let* V *be a graded representation of* sl_2. *Decompose it to irreducible representations by* $V = \bigoplus V_{\lambda_i}^{\omega_i}$. *Denote*

$$\delta(V) := \sum_{i \text{ s.t. } \omega_i(-1)^{\lambda_i} = -1} (\lambda_i + 2) - \dim(V_1).$$

Then

$$\delta(V) + \delta(V^*) + def(V) + def(V^*) = 0$$

Proof. This lemma is straightforward computation using Lemma 5.1.34 and Lemma 5.1.32. ☐

6. A uniform proof of Multiplicity One Theorems for GL_n

In this section we indicate a proof of Multiplicity one Theorems for GL_n which is uniform for all local fields of characteristic 0. This theorem was proven for the non-Archimedean case in [AGRS] and for the Archimedean case in [AG4] and [SZ]. We will not give all the details since this theorem was proven before. We will indicate the main steps and will give the details in the parts which are more essential. The proof that we present here is based on the ideas from the previous proofs and uses our partial analog of the integrability theorem.

Let us first formulate the Multiplicity one Theorems for GL_n.

Theorem 6.0.1. *Consider the standard imbedding* $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. *We consider the action of* $\mathrm{GL}_n(F)$ *on* $\mathrm{GL}_{n+1}(F)$ *by conjugation. Then any* $\mathrm{GL}_n(F)$*-invariant distribution on* $\mathrm{GL}_{n+1}(F)$ *is invariant with respect to transposition.*

It has the following corollary in representation theory.

Theorem 6.0.2. *Let* π *be an irreducible admissible smooth Fréchet representation of* $\mathrm{GL}_{n+1}(F)$ *and* τ *be an irreducible admissible smooth Fréchet representation of* $\mathrm{GL}_n(F)$. *Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1. \tag{3}$$

6.1. Notation.

- Let $V := V_n$ be the standard n-dimensional linear space defined over F.
- Let $\mathrm{sl}(V)$ denote the Lie algebra of operators with zero trace.
- Denote $X := X_n := \mathrm{sl}(V_n) \times V_n \times V_n^*$.
- Denote $G := G_n := \mathrm{GL}(V_n)$.
- Denote $\mathfrak{g} := \mathfrak{g}_n := \mathrm{Lie}(G_n) = \mathrm{gl}(V_n)$.
- Let G_n act on G_{n+1}, $\mathfrak{g}_{n+1}$ and on $\mathrm{sl}(V_n)$ by $g(A) := gAg^{-1}$.
- Let G act on $V \times V^*$ by $g(v, \phi) := (gv, (g^{-1})^*\phi)$. This gives rise to an action of G on X.
- Let $\sigma : X \to X$ be given by $\sigma(A, v, \phi) = A^t, \phi^t, v^t$.
- We fix the standard trace form on $\mathrm{sl}(V)$ and the standard form on $V \times V^*$.
- Denote $S := \{(A, v, \phi) \in X_n | A^n = 0 \text{ and } \phi(A^i v) = 0 \text{ for any } 0 \leq i \leq n\}$.
- Note that $S \supset \Gamma(X)$.
- Denote $S' := \{(A, v, \phi) \in S | A^{n-1}v = (A^*)^{n-1}\phi = 0\}$.
- Denote

$$\check{S} := \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \,|\, \forall i, j \in \{1, 2\} \\ (A_i, v_j, \phi_j) \in S \text{ and } \forall \alpha \in \mathrm{gl}(V), \alpha(A_1, v_1, \phi_1) \perp (A_2, v_2, \phi_2)\}.$$

- Note that

$$\check{S} = \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \,|\, \forall i, j \in \{1, 2\} \\ (A_i, v_j, \phi_j) \in S \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

- Denote

$$\check{S}' := \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in \check{S} |\, \forall i, j \in \{1, 2\} (A_i, v_j, \phi_j) \in S'\}.$$

6.2. Reformulation.

A standard use of the Harish-Chandra descent method shows that it is enough to show that any $G(F)$ invariant distribution on $X(F)$ is invariant with respect to σ, moreover it is enough to show this under the assumption that this is true for distributions on $(X - S)(F)$. So it is enough to prove the following theorem

Theorem 6.2.1. *The action of G on X is special (and hence weakly linearly tame).*

Remark 6.2.2. *One can show that this implies that the action of G_n on G_{n+1} is tame.*

6.3. **Proof of Theorem 6.2.1.** It is enough to show that any distribution $\xi \in \mathcal{S}^*(X(F))^{G(F)}$, such that ξ, $\mathcal{F}_{V\times V^*}(\xi)$, $\mathcal{F}_{sl(V)}(\xi)$ and $\mathcal{F}_X(\xi)$ are supported on $S(F)$, is zero.

Lemma 6.3.1. *Let $\xi \in \mathcal{S}^*(X(F))^{G(F)}$ such that both ξ and $\mathcal{F}_{V\times V^*}(\xi)$ are supported on $S(F)$. Then ξ is supported on $S'(F)$.*

Proof. This is a direct computation using Propositions 2.1.3, 2.1.4 , Theorem 2.1.5 and Theorem 2.1.10, and the fact that $S - S' \subset sl(V) \times (V \times 0 \cup 0 \times V^*)$. ☐

Corollary 6.3.2. *Let $\xi \in \mathcal{S}^*(X(F))^{G(F)}$ such that $\xi, \mathcal{F}_{V\times V^*}(\xi), \mathcal{F}_{sl(V)}(\xi)$ and $\mathcal{F}_X(\xi)$ are supported on $S(F)$ then $SS(\xi) \subset \check{S}'$.*

Now the following geometric statement implies Theorem 6.2.1.

Theorem 6.3.3 (The geometric statement)**.** *There are no non-empty $X \times X$-weakly coisotropic subvarieties of $\check{S}'$.*

6.4. **Proof of the geometric statement.**

Notation 6.4.1. *Denote $\check{S}'' := \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in \check{S}' | A_1^{n-1} = 0\}$.*

By Theorem 4.3.4 (and Example 4.3.3) there are no non-empty $X \times X$-weakly coisotropic subvarieties of $\check{S}''$. Therefore it is enough to prove the following Key proposition.

Proposition 6.4.2 (Key proposition)**.** *There are no non-empty $X \times X$-weakly coisotropic subvarieties of $\check{S}' - \check{S}''$.*

Notation 6.4.3. *Let $A \in sl(V)$ be a nilpotent Jordan block. Denote*

$$R_A := (\check{S}' - \check{S}'')|_{\{A\}\times V\times V^*}.$$

By Proposition 3.0.11 the Key proposition follows from the following Key Lemma.

Lemma 6.4.4 (Key Lemma)**.** *There are no non-empty $V \times V^* \times V \times V^*$-weakly coisotropic subvarieties of R_A.*

Proof. Denote $Q_A = \bigcup_{i=1}^{n-1}(KerA^i) \times (Ker(A^*)^{n-i})$. It is easy to see that $R_A \subset Q_A \times Q_A$ and

$$Q_A \times Q_A = \bigcup_{i,j=0}^{n} (KerA^i) \times (Ker(A^*)^{n-i}) \times (KerA^j) \times (Ker(A^*)^{n-j}).$$

Denote $L_{ij} := (KerA^i) \times (Ker(A^*)^{n-i}) \times (KerA^j) \times (Ker(A^*)^{n-j})$.
It is easy to see that any weakly coisotropic subvariety of $Q_A \times Q_A$ is contained in $\bigcup_{i=1}^{n-1} L_{ii}$.
Hence it is enough to show that for any $0 < i < n$, we have $\dim R_A \cap L_{ii} < 2n$.
Let $f \in \mathcal{O}(L_{ii})$ be the polynomial defined by

$$f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1},$$

where $(\cdot)_i$ means the i-th coordinate. It is enough to show that $f(R_A \cap L_{ii}) = \{0\}$.
Let $(v_1, \phi_1, v_2, \phi_2) \in L_{ii}$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$. Clearly, M is of the form

$$M = \begin{pmatrix} 0_{i\times i} & * \\ 0_{(n-i)\times i} & 0_{(n-i)\times(n-i)} \end{pmatrix}.$$

Note also that $M_{i,i+1} = f(v_1, \phi_1, v_2, \phi_2)$.

It is easy to see that any B satisfying $[A, B] = M$ is upper triangular. On the other hand, we know that there exists a nilpotent B satisfying $[A, B] = M$. Hence this B is upper nilpotent, which implies $M_{i,i+1} = 0$ and hence $f(v_1, \phi_1, v_2, \phi_2) = 0$.
To sum up, we have shown that $f(R_A \cap L_{ii}) = \{0\}$, hence $dim(R_A \cap L_{ii}) < 2n$. Hence every coisotropic subvariety of R_A has dimension less than $2n$ and therefore is empty. □

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INVARIANT DISTRIBUTIONS ON NON-DISTINGUISHED NILPOTENT ORBITS WITH APPLICATION TO THE GELFAND PROPERTY OF $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$

AVRAHAM AIZENBUD AND EITAN SAYAG

Abstract. We study invariant distributions on the tangent space to a symmetric space. We prove that an invariant distribution with the property that both its support and the support of its Fourier transform are contained in the set of non-distinguished nilpotent orbits, must vanish. We deduce, using recent developments in the theory of invariant distributions on symmetric spaces that the symmetric pair $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$ is a Gelfand pair. More precisely, we show that for any irreducible smooth admissible Fréchet representation (π, E) of $GL_{2n}(\mathbb{R})$ the ring of continuous functionals $Hom_{Sp_{2n}(\mathbb{R})}(E, \mathbb{C})$ is at most one dimensional. Such a result was previously proven for p-adic fields in [HR] and for $\mathbb{C}$ in [Say1].

Contents

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1. Introduction

Let (V,ω) be a symplectic vector space over $\mathbb{R}$. Consider the standard imbedding $Sp(V) \subset GL(V)$ and the natural action of $Sp(V) \times Sp(V)$ on $GL(V)$. In this paper we prove the following theorem:

Theorem A. *Any $Sp(V) \times Sp(V)$ - invariant distribution on $GL(V)$ is invariant with respect to transposition.*

It has the following corollary in representation theory:

Theorem B. *Let (V,ω) be a symplectic vector space and let E be an irreducible admissible smooth Fréchet representation of $GL(V)$. Then*

$$dimHom_{Sp(V)}(E,\mathbb{C}) \leq 1$$

In the language of [AGS], Theorem B means that the pair $(GL(V), Sp(V))$ is a Gelfand pair, more precisely satisfies GP1. In particular, Theorem B implies that the spectral decomposition of the unitary representation $L^2(GL(V)/Sp(V))$ is multiplicity free (see e.g. [Lip]).

Theorem B is deduced from Theorem A using the Gelfand-Kazhdan method (adapted to the archimedean case in [AGS]).

The analogue of Theorem A and Theorem B for non-archimedean fields were proven in [HR] using the method of Gelfand and Kazhdan. A simple argument over finite fields is explained in [GG] and using this a simpler proof of the non-archimedean case was written in [OS3]. Recently, one of us, using the ideas of [AG2] extended the result to the case $F = \mathbb{C}$ (see [Say1]).

Our proof of Theorem A is based on the methods of [AG2]. In that work the notion of regular symmetric pair was introduced and shown to be a useful tool in the verification of the Gelfand property . Thus, the main result of the present work is the *regularity* of the symmetric pair $(GL(V), Sp(V))$. In previous works the proof of regularity of symmetric pairs was based either on some simple considerations or on a criterion that requires negativity of certain eigenvalues (this was implicit in [JR], [RR] and was explicated in [AG2], [AG3], [AG4], [Say1]).

The pair $(GL(V), Sp(V))$ does not satisfy the above mentioned criterion and requires new techniques.

1.1. Main ingredients of the proof.

To show regularity we study distributions on the space $\mathfrak{g}^\sigma = \{X \in gl_{2n} : JX = XJ\}$ where $J = \begin{pmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{pmatrix}$. More precisely, we are interested in those distributions that are invariant with respect to the conjugation action of Sp_{2n} and supported on the nilpotent cone. To classify the nilpotent orbits of the action we use the method of [GG] to identify these orbits with nilpotent orbits of the adjoint action of GL_n on its Lie algebra. This allows us to show that there exists a unique *distinguished* nilpotent orbit $\mathcal{O}$ and that this orbit is open in the nilpotent cone. Next, we use the theory of D-modules, as in [AG5], to prove that there are no distributions supported on non-distinguished orbits whose Fourier transform is also supported on non-distinguished orbits (see Theorem 3.0.11).

1.2. **Related works.**
The problem of identifying symmetric pairs that are Gelfand pairs was studied by various authors. In the case of symmetric spaces of rank one this problem was studied extensively in [RR], [vD], [BvD] both in the archimedean and non-archimedean case. Recently, cases of symmetric spaces of high rank were studied in [AGS], [AG2], [AG3], [AG4], [Say2]. However, as hinted above, all these works could treat a restricted class of symmetric pairs, first introduced in [Sek] that are now commonly called *nice* symmetric pairs.

The pair $(GL(V), Sp(V))$ is *not* a nice symmetric pair and additional methods are needed to study invariant distributions on the corresponding symmetric space. For that, we use the theory of D-modules as in [AG5] and analysis of the nilpotent cone of the pair in question, in order to prove the Gelfand property.

In the non-archimedean case, the pair (GL_{2n}, Sp_{2n}) is a part of a list (GL_{2n}, H_k, ψ_k), $k = 0, 1, ..., n$, of twisted Gelfand pairs that provide a model in the sense of [BGG] to the unitary representations of GL_{2n}. Namely, every irreducible unitarizable representation of GL_{2n} appears exactly once in $\bigoplus_{k=0}^{n} Ind_{H_k}^{GL_{2n}}(\psi_k)$ (see [OS1],[OS2],[OS3]). Considering the strategy taken in those works, a major first step in transferring these results to the archimedean case is taken in the present paper.

1.3. **Structure of the paper.**
In section 2 we give some preliminaries on distributions, symmetric pairs and Gelfand pairs. We introduce the notion of regular symmetric pairs and show that Theorem 7.4.5 of [AG2] and the results of [Say1] allow us to reduce the Gelfand property of the pair in question to proving that the pair is regular. In section 3 we prove the main technical result on distributions, Theorem 3.0.11. It states that under certain conditions there are no distributions supported on non-distinguished nilpotent orbits. The proof is based on the theory of D-modules. In section 4 we use Theorem 3.0.11 to prove that the pair $(GL(V), Sp(V))$ is regular.

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2. Preliminaries

2.1. Notations on invariant distributions.

2.1.1. *Schwartz distributions on Nash manifolds.*
We will use the theory of Schwartz functions and distributions as developed in [AG1]. This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic*.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

Notation 2.1.1. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.*

Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the space of Schwartz distributions on X.

For any Nash vector bundle E over X we denote by $\mathcal{S}(X,E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X,E)$ its dual space.

Notation 2.1.2. *Let X be a smooth manifold and let $Z \subset X$ be a closed subset. We denote $\mathcal{S}^*_X(Z) := \{\xi \in \mathcal{S}^*(X) | \mathrm{Supp}(\xi) \subset Z\}$.*

For a locally closed subset $Y \subset X$ we denote $\mathcal{S}^*_X(Y) := \mathcal{S}^*_{X \setminus (\overline{Y} \setminus Y)}(Y)$. In the same way, for any bundle E on X we define $\mathcal{S}^*_X(Y,E)$.

Remark 2.1.3. *Schwartz distributions have the following two advantages over general distributions:*
(i) For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence

$$0 \to \mathcal{S}^*_X(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

(ii) Fourier transform defines an isomorphism $\mathcal{F} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.

2.1.2. *Basic tools.*
We present here some basic tools on equivariant distributions that we will use in this paper.

Proposition 2.1.4. *Let a Nash group G act on a Nash manifold X. Let $Z \subset X$ be a closed subset.*

Let $Z = \bigcup_{i=0}^{l} Z_i$ be a Nash G-invariant stratification of Z. Let χ be a character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i, Sym^k(CN_{Z_i}^X))^{G,\chi} = 0$. Then $\mathcal{S}^*_X(Z)^{G,\chi} = 0$.

This proposition immediately follows from Corollary 7.2.6 in [AGS].

Theorem 2.1.5 (Frobenius reciprocity)**.** *Let a Nash group G act transitively on a Nash manifold Z. Let $\varphi : X \to Z$ be a G-equivariant Nash map. Let $z \in Z$. Let G_z be its stabilizer. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z, \chi \cdot \Delta_G|_{G_z} \cdot \Delta_{G_z}^{-1}}$ where Δ denotes the modular character.*

For proof see [AG2], Theorem 2.3.8.

2.1.3. *Fourier transform.*
From now till the end of the paper we fix an additive character κ of $\mathbb{R}$ given by $\kappa(x) := e^{2\pi i x}$.

Notation 2.1.6. *Let V be a vector space over $\mathbb{R}$. Let B be a non-degenerate bilinear form on V. Then B defines Fourier transform with respect to the self-dual Haar measure on V. We denote it by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$.*

For any Nash manifold M we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the partial Fourier transform.

If there is no ambiguity, we will write $\mathcal{F}_V$, and sometimes just $\mathcal{F}$, instead of $\mathcal{F}_B$.

We will use the following trivial observation.

Lemma 2.1.7. *Let V be a finite dimensional vector space over $\mathbb{R}$. Let a Nash group G act linearly on V. Let B be a G-invariant non-degenerate symmetric bilinear form on V. Let M be a Nash manifold with an action of G. Let $\xi \in \mathcal{S}^*(V \times M)$ be a G-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also G-invariant.*

2.2. **Gelfand pairs and invariant distributions.**
In this section we recall a technique due to Gelfand and Kazhdan (see [GK]) which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS], section 2.

Definition 2.2.1. *Let G be a reductive group. By an* ***admissible representation of*** *G we mean an admissible smooth Fréchet representation of $G(\mathbb{R})$.*

We now introduce three notions of Gelfand pair.

Definition 2.2.2. *Let $H \subset G$ be a pair of reductive groups.*

- *We say that (G, H) satisfy* **GP1** *if for any irreducible admissible smooth Fréchet representation (π, E) of G we have*
$$\dim Hom_{H(\mathbb{R})}(E, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy* **GP2** *if for any irreducible admissible smooth Fréchet representation (π, E) of G we have*
$$\dim Hom_{H(\mathbb{R})}(E, \mathbb{C}) \cdot \dim Hom_{H(\mathbb{R})}(\widetilde{E}, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy* **GP3** *if for any irreducible* **unitary** *representation $(\pi, \mathcal{H})$ of $G(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ we have*
$$\dim Hom_{H(\mathbb{R})}(\mathcal{H}^\infty, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g.[vD, BvD]).

We have the following straightforward proposition.

Proposition 2.2.3. $GP1 \Rightarrow GP2 \Rightarrow GP3$.

We will use the following theorem from [AGS] which is a version of a classical theorem of Gelfand and Kazhdan.

Theorem 2.2.4. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(\mathbb{R})$-invariant distributions ξ on $G(\mathbb{R})$. Then (G, H) satisfies GP2.*

In our case GP2 is equivalent to GP1 by the following proposition.

Proposition 2.2.5. *Suppose $H \subset \mathrm{GL}_n$ is transpose invariant subgroup. Then $GP1$ is equivalent to $GP2$ for the pair (GL_n, H).*

For proof see [AGS], proposition 2.4.1.

Corollary 2.2.6. *Theorem A implies Theorem B.*

2.3. **Symmetric pairs.**
In this subsection we review some tools developed in [AG2] that enable to prove that, granting certain hypothesis, a symmetric pair is a Gelfand pair.

Definition 2.3.1. *A* ***symmetric pair*** *is a triple* (G, H, θ) *where* $H \subset G$ *are reductive groups, and* θ *is an involution of* G *such that* $H = G^{\theta}$*. In cases when there is no ambiguity we will omit* θ

For a symmetric pair (G, H, θ) *we define an anti-involution* $\sigma : G \to G$ *by* $\sigma(g) := \theta(g^{-1})$*, denote* $\mathfrak{g} := LieG$*,* $\mathfrak{h} := LieH$*,* $\mathfrak{g}^{\sigma} := \{a \in \mathfrak{g} | \theta(a) = -a\}$*. Note that* H *acts on* $\mathfrak{g}^{\sigma}$ *by the adjoint action. Denote also* $G^{\sigma} := \{g \in G | \sigma(g) = g\}$ *and define a* ***symmetrization map*** $s : G(\mathbb{R}) \to G^{\sigma}(\mathbb{R})$ *by* $s(g) := g\sigma(g)$*.*

The following lemma is standard:

Lemma 2.3.2. *The symmetrization map* $s : G \to G^{\sigma}$ *is submersive and hence open.*

Definition 2.3.3. *Let* (G_1, H_1, θ_1) *and* (G_2, H_2, θ_2) *be symmetric pairs. We define their* ***product*** *to be the symmetric pair* $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$*.*

Definition 2.3.4. *We call a symmetric pair* (G, H, θ) ***good*** *if for any closed* $H(\mathbb{R}) \times H(\mathbb{R})$ *orbit* $O \subset G(\mathbb{R})$*, we have* $\sigma(O) = O$*.*

Definition 2.3.5. *We say that a symmetric pair* (G, H, θ) *is a* ***GK pair*** *if any* $H(\mathbb{R}) \times H(\mathbb{R})$ *- invariant distribution on* $G(\mathbb{R})$ *is* σ *- invariant.*

Definition 2.3.6. *We define an involution* $\theta : GL_{2n} \to GL_{2n}$ *by* $\theta(x) = Jx^tJ^{-1}$ *where* $J = \begin{pmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{pmatrix}$*. Note that* $(GL_{2n}, Sp_{2n}, \theta)$ *is a symmetric pair.*

Theorem A can be rephrased in the following way:

Theorem A'. *The pair* (GL_{2n}, Sp_{2n}) *defined over* $\mathbb{R}$ *is a GK pair.*

2.3.1. *Descendants of symmetric pairs.*

Proposition 2.3.7. *Let* (G, H, θ) *be a symmetric pair. Let* $g \in G(\mathbb{R})$ *such that* HgH *is closed. Let* $x = s(g)$*. Then* x *is semisimple.*

For proof see e.g. [AG2], Proposition 7.2.1.

Definition 2.3.8. *In the notations of the previous proposition we will say that the pair* $(G_x, H_x, \theta|_{G_x})$ *is a descendant of* (G, H, θ)*. Here* G_x *(and similarly for* H*) denotes the stabilizer of* x *in* G*.*

2.3.2. *Regular symmetric pairs.*

Notation 2.3.9. *Let* V *be an algebraic finite dimensional representation over* $\mathbb{R}$ *of a reductive group* G*. Denote* $Q(V) := V/V^G$*. Since* G *is reductive, there is a canonical embedding* $Q(V) \hookrightarrow V$*.*

Notation 2.3.10. *Let* (G, H, θ) *be a symmetric pair. We denote by* $\mathcal{N}_{G,H}$ *the subset of all the nilpotent elements in* $Q(\mathfrak{g}^{\sigma})$*. Denote* $R_{G,H} := Q(\mathfrak{g}^{\sigma}) - \mathcal{N}_{G,H}$*.*

Our notion of $R_{G,H}$ coincides with the notion $R(\mathfrak{g}^{\sigma})$ used in [AG2], Notation 2.1.10. This follows from Lemma 7.1.11 in [AG2].

Definition 2.3.11. *Let π be an action of a real reductive group G on a smooth affine variety X. We say that an algebraic automorphism τ of X is G-**admissible** if*
(i) $\pi(G(\mathbb{R}))$ is of index at most 2 in the group of automorphisms of X generated by $\pi(G(\mathbb{R}))$ and τ.
(ii) For any closed $G(\mathbb{R})$ orbit $O \subset X(\mathbb{R})$, we have $\tau(O) = O$.

Definition 2.3.12. *Let (G, H, θ) be a symmetric pair. We call an element $g \in G(\mathbb{R})$ **admissible** if*
(i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
(ii) $Ad(g)|_{\mathfrak{g}^\sigma}$ is H-admissible.

Definition 2.3.13. *We call a symmetric pair (G, H, θ) **regular** if for any admissible $g \in G(\mathbb{R})$ such that every $H(\mathbb{R})$-invariant distribution on $R_{G,H}$ is also $Ad(g)$-invariant, we have*
(*) every $H(\mathbb{R})$-invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also $Ad(g)$-invariant.

Clearly, the product of regular pairs is regular (see [AG2], Proposition 7.4.4).
We will deduce Theorem A' (and hence Theorem A) from the following Theorem:

Theorem C. *The pair (GL_{2n}, Sp_{2n}) defined over $\mathbb{R}$ is regular.*

The deduction is based on the following theorem (see [AG2], Theorem 7.4.5.):

Theorem 2.3.14. *Let (G, H, θ) be a good symmetric pair such that all its descendants (including itself) are regular. Then it is a GK pair.*

Corollary 2.3.15. *Theorem C implies Theorem A.*

Proof. The pair (GL_{2n}, Sp_{2n}) is good by Corollary 3.1.3 of [Say1]. In [Say1] it is shown that all the descendance of the pair (GL_{2n}, Sp_{2n}) are products of pairs of the form (GL_{2m}, Sp_{2m}) and $((GL_{2m})_{\mathbb{C}/\mathbb{R}}, (Sp_{2m})_{\mathbb{C}/\mathbb{R}})$, here $G_{\mathbb{C}/\mathbb{R}}$ denotes the restriction of scalars (in particular $G_{\mathbb{C}/\mathbb{R}}(\mathbb{R}) = G(\mathbb{C})$). By Corollary 3.3.1. of [Say1] the pair $((GL_{2m})_{\mathbb{C}/\mathbb{R}}, (Sp_{2m})_{\mathbb{C}/\mathbb{R}})$ is regular. Now clearly Theorem C implies Theorem A' and hence Theorem A. □

We will also need the following Proposition, whose proof we include for completeness.

Proposition 2.3.16. *Let $\pi : \mathfrak{g}^\sigma \to Spec(\mathcal{O}(\mathfrak{g}^\sigma))^H$ be the projection, where $\mathcal{O}(\mathfrak{g}^\sigma)$ denote the space of regular functions on the algebraic variety $\mathfrak{g}^\sigma$.*
Let $x \in \mathcal{N}_{G,H}$ be a smooth point. Then π submersive at x.

Proof. Let $\mathcal{J} = \{f \in \mathcal{O}(\mathfrak{g}^\sigma)^H : f(0) = 0\}$. By Theorem 14 of [KR], $\mathcal{J}$ is a radical ideal. Using the Nullstellensatz, this implies that $Ker(d_x\pi) = T_x(\mathcal{N}_{G,H})$. This proves that π is submersive. □

2.4. **Singular support of distributions.**
In this subsection we introduce the notion *Singular Support* of a distribution ξ and list some of its properties. In the literature this notion is sometimes also called *Characteristic Variety.* For more details see [AG5].

Notation 2.4.1. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Let M_ξ be the D_X submodule of $\mathcal{S}^*(X(\mathbb{R}))$ generated by ξ. We denote by $SS(\xi) \subset T^*X$ the singular support of M_ξ (for the definition see* [Bor]*). We will call it the* **singular support** *of ξ.*

Remark 2.4.2.
(i) A similar, but not equivalent notion is sometimes called in the literature a 'wave front of ξ'.
(ii) In some of the literature, singular support of a distribution is a subset of X not to be confused with our $SS(\xi)$ which is a subset of T^*X. We use terminology from the theory of D-modules where the set $SS(\xi)$ is called both the characteristic variety and the singular support of the D-module M_ξ.

Notation 2.4.3. *Let (V, B) be a quadratic space. Let X be a smooth algebraic variety. Consider B as a map $B : V \to V^*$. Identify $T^*(X \times V)$ with $T^*X \times V \times V^*$. We define $F_V : T^*(X \times V) \to T^*(X \times V)$ by $F_V(\alpha, v, \phi) := (\alpha, -B^{-1}\phi, Bv)$.*

Definition 2.4.4. *Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M-**co-isotropic** if one of the following equivalent conditions holds.*

1. *The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.*
2. *At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$. Here, $(T_zZ)^\perp$ denotes the orthogonal complement to (T_zZ) in (T_zM) with respect to ω.*
3. *For a generic smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$.*

If there is no ambiguity, we will call Z a co-isotropic variety.

Note that every non-empty M-co-isotropic variety is of dimension at least $\frac{1}{2} dim M$.

Notation 2.4.5. *For a smooth algebraic variety X we always consider the standard symplectic form on T^*X. Also, we denote by $p_X : T^*X \to X$ the standard projection.*

Let X be a smooth algebraic variety. Below is a list of properties of the Singular support. Proofs can be found in [AG5] section 2.3 and Appendix B.

Property 2.4.6. *Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))(\mathbb{R})$, where $\overline{\mathrm{Supp}(\xi)}_{Zar}$ denotes the Zariski closure of* $\mathrm{Supp}(\xi)$.

Property 2.4.7.
Let an algebraic group G act on X. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))^{G(\mathbb{R})}$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \mid \forall \alpha \in \mathfrak{g}\, \phi(\alpha(x)) = 0\}.$$

Property 2.4.8. *Let (V, B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that* $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$. *Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X\times V}^{-1}(Z))$.*

Finally, the following is a corollary of the integrability theorem ([KKS], [Mal], [Gab]):

Property 2.4.9. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $SS(\xi)$ is co-isotropic.*

3. Invariant distributions supported on non-distinguished nilpotent orbits in symmetric pairs

For this section we fix a symmetric pair (G, H, θ).

Definition 3.0.10. *We say that a nilpotent element $x \in \mathfrak{g}^\sigma$ is distinguished if*

$$\mathfrak{g}_x \cap Q(\mathfrak{g}^\sigma) \subset \mathcal{N}_{G,H}$$

Theorem 3.0.11. *Let $A \subset \mathcal{N}_{G,H}$ be an H invariant closed subset and assume that all elements of A are non-distinguished. Let $W = \mathcal{S}^*_{\mathfrak{g}^\sigma}(A)^H$. Then $W \cap \mathcal{F}(W) = 0$.*

Remark 3.0.12. *We believe that the methods of* [SZ] *allow to show the same result without the assumption of H-invariance.*

The proof is based on the following proposition:

Proposition 3.0.13. *Let $A \subset \mathcal{N}_{G,H}$ be an H invariant closed subset and assume that all elements of A are non-distinguished. Denote by*

$$B = \{(\alpha, \beta) \in A \times A : [\alpha, \beta] = 0\} \subset Q(\mathfrak{g}^\sigma) \times Q(\mathfrak{g}^\sigma).$$

Identify $T^*(Q(\mathfrak{g}^\sigma))$ with $Q(\mathfrak{g}^\sigma) \times Q(\mathfrak{g}^\sigma)$. Then there is no non-empty $T^*(Q(\mathfrak{g}^\sigma))$-co-isotropic subvariety of B.

Proof. Stratify A by its orbits $\mathcal{O}_1, ..., \mathcal{O}_r$. Let $p : A \times A \to A$ be the projection onto the first factor. By inductive argument it is enough to show that, for any orbit $\mathcal{O}$, $p^{-1}(\mathcal{O}) \cap B$ does not include a non empty co-isotropic subvariety. Consider the set

$$C_\mathcal{O} = \{(a, b) : a \in \mathcal{O}, b \in Q(\mathfrak{g}^\sigma), [a, b] = 0\}.$$

Then $dim(C_\mathcal{O}) = dim(Q(\mathfrak{g}^\sigma))$. Since $\mathcal{O}$ is not distinguished, $p^{-1}(\mathcal{O}) \cap B$ is a closed subvariety of $C_\mathcal{O}$ which does not include any of the irreducible components of $C_\mathcal{O}$. This finishes the proof. ☐

Proof of Theorem 3.0.11. Let $\xi \in W \cap \mathcal{F}(W)$ and let B be as in proposition 3.0.13. By properties 2.4.6, 2.4.7, 2.4.8 we conclude that $SS(\xi) \subset B$. But by Property 2.4.9 it is co-isotropic and hence by Proposition 3.0.13 it is empty. Thus $\xi = 0$. ☐

4. Regularity

In this section we prove the main result of the paper:

Theorem C. *The pair (GL_{2n}, Sp_{2n}) defined over $\mathbb{R}$ is regular.*

For the rest of this section we let (G, H) to be the symmetric pair $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$.

4.1. H orbits on $\mathfrak{g}^\sigma$.

Proposition 4.1.1. *There exists a unique distinguished H-orbit in $\mathcal{N}_{G,H}(\mathbb{R})$. This orbit is open in $\mathcal{N}_{G,H}(\mathbb{R})$ and invariant with respect to any admissible $g \in G$.*

For the proof we will use the following Proposition (this is Proposition 2.1 of [GG]):

Proposition 4.1.2. *Let F be an arbitrary field. For $x \in GL_n(F)$ define*

$$\gamma(x) = \begin{pmatrix} x & 0 \\ 0 & I_n \end{pmatrix}$$

Then γ induces a bijection between the set of conjugacy classes in $GL_n(F)$ and the set of orbits of $Sp_{2n}(F) \times Sp_{2n}(F)$ in $GL_{2n}(F)$.

Corollary 4.1.3. *Let $d : \mathrm{gl}_n \to \mathfrak{g}^\sigma$ be defined by*

$$d(X) = \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix}.$$

Then d induces a bijection between nilpotent conjugacy classes in gl_n and H orbits in $\mathcal{N}_{G,H}$.

Proof. Let $s : GL_{2n} \to GL_{2n}^\sigma$ be given by $s(g) = g\sigma(g)$. Let $W = s(GL_{2n}(\mathbb{R}))$. By Proposition 4.1.2, the map $s \circ \gamma$ induces a bijection between conjugacy classes in $GL_n(\mathbb{R})$ and H orbits on W.

Let $e : \mathcal{N} \to GL_n$ be given by $e(X) = 1 + X$ where $\mathcal{N}$ is the cone of nilpotent elements in gl_n. Let $\ell : W \to \mathfrak{g}^\sigma$ given by $\ell(w) = w - 1$.

Then, it is easy to see that the map $d|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}_{G,H}$ coincides with the composition $\ell \circ s \circ \gamma \circ e$.

To finish the proof of the Proposition it is enough to show that $\ell(W)$ contains all nilpotent elements. Indeed, by lemma 2.3.2 the set $W = s(GL_{2n}(\mathbb{R}))$ is open and thus $\ell(W)$ is open and hence contains all nilpotent elements. □

We are now ready to prove the proposition.

Proof of Proposition 4.1.1. It is easy to see that if X is non regular nilpotent then $d(X)$ is not distinguished. Also, a simple verification shows that if $X = J_n$ is a standard Jordan block then $d(J_n)$ is distinguished. Thus we only need to show that $C = Ad(H)d(J_n)$ is open in $\mathcal{N}_{G,H}$. For this we will show that C is dense in $\mathcal{N}_{G,H}$. Indeed, $\bar{C} \supset d(\overline{Ad(GL_n)J_n}) = d(\mathcal{N})$, where $\mathcal{N}$ is the set of nilpotent elements in gl_n. But C is $Ad(H)$-invariant and this implies that $\bar{C} = \mathcal{N}_{G,H}$ □

4.2. **Proof of Theorem C.**
Theorem C follows from Theorem 3.0.11 and the next Proposition:

Proposition 4.2.1. *Let $g \in G$ be an admissible element. Let A be the union of all non-distinguished elements. Note that A is closed. Let ξ be any H-invariant distribution on $\mathfrak{g}^\sigma$ which is anti-invariant with respect to $Ad(g)$. Then $Supp(\xi) \subset A$.*

Proof. Let $O_0 \subset \mathcal{N}_{G,H}$ be the distinguished orbit. Let $\widetilde{H} = \langle Ad(H), Ad(g) \rangle$ be the group of automorphisms of $\mathfrak{g}^\sigma$ generated by the adjoint action of H and g. Let χ be the character of $\widetilde{H}$ defined by $\chi(\widetilde{H} - H) = -1$. We need to show

$$\mathcal{S}^*_{Q(g^\sigma)}(O_0)^{\widetilde{H},\chi} = 0$$

By Proposition 2.1.4 it is enough to show

$$\mathcal{S}^*(O_0, Sym^k(CN_{O_0}^{Q(\mathfrak{g}^\sigma)}))^{\widetilde{H},\chi} = 0$$

Notice that $\widetilde{H}$ acts trivially on $Spec(O(\mathfrak{g}^\sigma))^H$. Hence, by Proposition 2.3.16 the bundle $N_{O_0}^{Q(\mathfrak{g}^\sigma)}$ is trivial as a $\widetilde{H}$ bundle. This completes the proof. □

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MULTIPLICITY ONE THEOREMS

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Abstract. In the local, characteristic 0, non-archimedean case, we consider distributions on $GL(n+1)$ which are invariant under conjugation by $GL(n)$. We prove that such distributions are invariant by transposition. This implies multiplicity at most one for restrictions from $GL(n+1)$ to $GL(n)$.
Similar Theorems are obtained for orthogonal or unitary groups.

Introduction

Let $\mathbb{F}$ be a non-archimedean local field of characteristic 0. Let W be a vector space over $\mathbb{F}$ of finite dimension $n+1 \geqslant 1$ and let $W = V \oplus U$ be a direct sum decomposition with $\dim V = n$. Then we have an imbedding of $GL(V)$ into $GL(W)$. Our goal is to prove the following Theorem:

Theorem (1)**.** *If π (resp. ρ) is an irreducible admissible representation of $GL(W)$ (resp. of $GL(V)$) then*

$$\dim\left(\mathrm{Hom}_{GL(V)}(\pi_{|GL(V)}\,,\,\rho)\right) \leqslant 1.$$

We choose a basis of V and a non-zero vector in U thus getting a basis of W. We can identify $GL(W)$ with $GL(n+1,\mathbb{F})$ and $GL(V)$ with $GL(n,\mathbb{F})$. The transposition map is an involutive anti-automorphism of $GL(n+1,\mathbb{F})$ which leaves $GL(n,\mathbb{F})$ stable. It acts on the space of distributions on $GL(n+1,\mathbb{F})$.

Theorem 1 is a Corollary of

Theorem (2)**.** *A distribution on $GL(W)$ which is invariant under conjugation by $G = GL(V)$ is invariant by transposition.*

One can raise a similar question for orthogonal and unitary groups. Let $\mathbb{D}$ be either $\mathbb{F}$ or a quadratic extension of $\mathbb{F}$. If $x \in \mathbb{D}$ then $\overline{x}$ is the conjugate of x if $\mathbb{D} \neq \mathbb{F}$ and is equal to x if $\mathbb{D} = \mathbb{F}$.

Let W be a vector space over $\mathbb{D}$ of finite dimension $n+1 \geqslant 1$. Let $\langle .,. \rangle$ be a non-degenerate hermitian form on W. This form is bi-additive and

$$\langle dw, d'w' \rangle = d\,\overline{d'}\langle w, w' \rangle, \quad \langle w', w \rangle = \overline{\langle w, w' \rangle}.$$

Given a $\mathbb{D}$-linear map u from W into itself, its adjoint u^* is defined by the usual formula

$$\langle u(w), w' \rangle = \langle w, u^*(w') \rangle.$$

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Choose a vector e in W such that $\langle e,e\rangle \neq 0$; let $U = \mathbb{D}e$ and $V = U^{\perp}$ the orthogonal complement. Then V has dimension n and the restriction of the hermitian form to V is non-degenerate.

Let M be the unitary group of W that is to say the group of all $\mathbb{D}$-linear maps m of W into itself which preserve the hermitian form or equivalently such that $mm^* = 1$. Let G be the unitary group of V. With the p-adic topology both groups are of type lctd (locally compact, totally discontinuous) and countable at infinity. They are reductive groups of classical type.

The group G is naturally imbedded into M.

Theorem (1'). *If π (resp ρ) is an irreducible admissible representation of M (resp of G) then*

$$\dim\big(\mathrm{Hom}_G(\pi_{|G},\rho)\big) \leq 1.$$

Choose a basis $e_1, \ldots e_n$ of V such that $\langle e_i, e_j\rangle \in \mathbb{F}$. For

$$w = x_0 e + \sum_1^n x_i e_i$$

put

$$\overline{w} = \overline{x}_0 e + \sum_1^n \overline{x_i}\, e_i.$$

If u is a $\mathbb{D}$-linear map from W into itself, let $\overline{u}$ be defined by

$$\overline{u}(w) = \overline{u(\overline{w})}.$$

Let σ be the anti-involution $\sigma(m) = \overline{m}^{-1}$ of M; Theorem 1' is a consequence of

Theorem (2'). *A distribution on M which is invariant under conjugation by G is invariant under σ.*

The structure of our proof. Let us describe briefly our proof. In section 1 we recall why Theorem 2 (2') implies Theorem 1(1'). This idea goes to back Gelfand-Kazhdan ([GK75]).

For the proofs of Theorems 2 and 2' we systematically use two classical results : Bernstein's localization principle and a variant of Frobenius reciprocity which we call Frobenius descent. For the convenience of the reader they are both recalled in section 2.

Then we proceed with $\mathrm{GL}(n)$. The proof is by induction on n; the case $n = 0$ is trivial. In general we first linearize the problem by replacing the action of $\mathrm{GL}(V)$ on $\mathrm{GL}(W)$ by the action on the Lie algebra of $\mathrm{GL}(W)$. As a $\mathrm{GL}(V)$-module this Lie algebra is isomorphic to a direct sum $\mathfrak{g} \oplus V \oplus V^* \oplus \mathbb{F}$ with $\mathfrak{g}$ the Lie algebra of $G = \mathrm{GL}(V)$ and V^* the dual space of V. The group G acts trivially on $\mathbb{F}$, by the adjoint action on its Lie algebra and the natural actions on V and V^*. The component $\mathbb{F}$ plays no role. Let u be a linear bijection of V onto V^* which transforms some basis of V into its dual basis. The involution may be taken as

$$(X, v, v^*) \mapsto (u^{-1}\,{}^tX\, u, u^{-1}(v^*), u(v)).$$

We have to show that a distribution T on $\mathfrak{g} \oplus V \oplus V^*$ which is invariant under G and skew relative to the involution is 0.

In section 3 we prove that the support of such a distribution is contained in the set of singular elements. On the $\mathfrak{g}$ side, using Harish-Chandra descent we get that the support of T must be contained in $(\mathfrak{z} + \mathcal{N}) \times (V \oplus V^*)$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathcal{N}$ the cone of nilpotent elements in $\mathfrak{g}$. On the $V \oplus V^*$ side we show that the support must be contained in $\mathfrak{g} \times \Gamma$ where Γ is the cone $\langle v, v^* \rangle = 0$ in $V \oplus V^*$. On $\mathfrak{z}$ the action is trivial so we are reduced to the case of a distribution on $\mathcal{N} \times \Gamma$.

In section 4 we consider such distributions. The end of the proof is based on two facts. First, viewing the distribution as a distribution on $\mathcal{N} \times (V \oplus V^*)$ its partial Fourier transform relative to $V \oplus V^*$ has the same invariance properties and hence must also be supported on $\mathcal{N} \times \Gamma$. This implies in particular a homogeneity condition on $V \oplus V^*$. The idea of using Fourier transform in this kind of situation goes back at least to Harish-Chandra ([HC99]) and is conveniently expressed using a particular case of the Weil or oscillator representation.

For $(v, v^*) \in \Gamma$, let X_{v,v^*} be the map $x \mapsto \langle x, v^* \rangle v$ of V into itself. The second fact is that the one parameter group of transformations

$$(X, v, v^*) \mapsto (X + \lambda X_{v,v^*}, v, v^*)$$

is a group of (non-linear) homeomorphisms of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ which commute with G and the involution. It follows that the image of the support of our distribution must also be singular. This allows us to replace the condition $\langle v, v^* \rangle = 0$ by the stricter condition $X_{v,v^*} \in \mathrm{Im}\ \mathrm{ad}\, X$.

Using the stratification of $\mathcal{N}$ we proceed one nilpotent orbit at a time, transferring the problem to $V \oplus V^*$ and a fixed nilpotent matrix X. The support condition turns out to be compatible with direct sum so that it is enough to consider the case of a principal nilpotent element. In this last situation the key is the homogeneity condition coupled with an easy induction.

The orthogonal and unitary cases are proved in a similar vein. In section 5 we reduce the support to the singular set. Here the main difference is that we use Harish-Chandra descent directly on the group. Note that the Levi subgroups have components of type GL so that Theorem 2 has to be used. Finally in section 6 we consider the case of a distribution whose support is contained in the set of singular elements; the proof is along the same lines as in section 4.

Remarks. As for the archimedean case, partial analogs of the results of this paper were obtained in [AGS08a, AGS08b, vD08]. Recently, the full analogs were obtained in [AG08] and [SZ08].

Let us add some comments on the Theorems themselves. First note that Theorem 2 gives an independent proof of a well known theorem of Bernstein: choose a basis $e_1, \ldots, e_n$ of V, add some vector e_0 of W to obtain a basis of W and let P be the isotropy of e_0 in $\mathrm{GL}(W)$. Then Theorem B of [Ber84] says that a distribution on $\mathrm{GL}(W)$ which is invariant under the action of P is invariant under the action of $\mathrm{GL}(W)$. Now, by Theorem 2 such a distribution is invariant under conjugation by the transpose of P and the group of inner automorphisms is generated by the images of P and its transpose. This Theorem implies Kirillov's conjecture which states that any unitary irreducible representation of $\mathrm{GL}(W)$ remains irreducible when restricted to P.

The occurrence of involutions in multiplicity at most one problems is of course nothing new. The situation is fairly simple when all the orbits are stable by the involution thanks to Bernstein's localization principle and constructivity theorem ([BZ76, GK75]). In our case this is not true : only generic orbits are stable. Non-stable orbits may carry invariant measures but they do not extend to the ambient space (a similar situation is already present in [Ber84]).

An illustrative example is the case $n = 1$ for GL. It reduces to $\mathbb{F}^*$ acting on $\mathbb{F}^2$ as $(x, y) \mapsto (tx, t^{-1}y)$. On the x axis the measure $d^*x = dx/|x|$ is invariant but does not extend invariantly. However the symmetric measure

$$f \mapsto \int_{\mathbb{F}^*} f(x, 0)d^*x + \int_{\mathbb{F}^*} f(0, y)d^*y$$

does extend.

As in similar cases (for example [JR96]) our proof does not give a simple explanation of why all invariant distributions are symmetric. The situation would be much better if we had some kind of density theorem. For example in the GL case let us say that an element (X, v, v^*) of $\mathfrak{g} \oplus V \oplus V^*$ is regular if $(v, Xv, \ldots X^{n-1}v)$ is a basis of V and $(v^*, \ldots, {}^tX^{n-1}v^*)$ is basis of V^*. The set of regular elements is a non-empty Zariski open subset; regular elements have trivial isotropy subgroups. The regular orbits are the orbits of the regular elements; they are closed, separated by the invariant polynomials and stable by the involution (see [RS07]). In particular they carry invariant measures which, the orbits being closed, do extend and are invariant by the involution. It is tempting to conjecture that the subspace of the space of invariant distributions generated by these measures is weakly dense. This would provide a better understanding of Theorem 2. Unfortunately if true at all, such a density theorem is likely to be much harder to prove.

Assuming multiplicity at most one, a more difficult question is to find when it is one. Some partial results are known.

For the orthogonal group (in fact the special orthogonal group) this question has been studied by B. Gross and D. Prasad ([GP92, Pra93]) who formulated a precise conjecture. An up to date account is given by B. Gross and M. Reeder ([GR06]). In a different setup, in their work on "Shintani" functions A. Murase and T. Sugano obtained complete results for $GL(n)$ and the split orthogonal case but only for spherical representations ([Kat03, Mur96]). Finally we should mention, Hakim's publication [Hak03], which, at least for the discrete series, could perhaps lead to a different kind of proof.

Multiplicity one theorems have important applications to the relative trace formula, to automorphic descent, to local and global liftings of automorphic representations, and to determinations of L-functions. In particular, multiplicity at most one is used as a hypothesis in the work [GPSR97] on the study of automorphic L-functions on classical groups. At least for the last two authors, the original motivation for this work came in fact from [GPSR97].

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1. Theorem 2(2') implies Theorem 1(1')

A group of type lctd is a locally compact, totally disconnected group which is countable at infinity. We consider smooth representations of such groups. If (π, E_π) is such a representation then (π^*, E_π^*) is the smooth contragradient. Smooth induction is denoted by Ind and compact induction by ind. For any topological space T of type lctd, $\mathcal{S}(T)$ is the space of functions locally constant, complex valued, defined on T and with compact support. The space $\mathcal{S}'(T)$ of distributions on T is the dual space to $\mathcal{S}(T)$.

Proposition 1.1. *Let M be a lctd group and N a closed subgroup, both unimodular. Suppose that there exists an involutive anti-automorphism σ of M such that $\sigma(N) = N$ and such that any distribution on M, biinvariant under N, is fixed by σ. Then, for any irreducible admissible representation π of M*

$$\dim\left(\mathrm{Hom}_M(ind_N^M(1), \pi)\right) \times \dim\left(\mathrm{Hom}_M(ind_N^M(1), \pi^*)\right) \leq 1.$$

This is well known (see for example [Pra90]).

Remark. *There is a variant for the non-unimodular case; we will not need it.*

Corollary 1.1. *Let M be a lctd group and N a closed subgroup, both unimodular. Suppose that there exists an involutive anti-automorphism σ of M such that $\sigma(N) = N$ and such that any distribution on M, invariant under conjugation by N, is fixed by σ. Then, for any irreducible admissible representation π of M and any irreducible admissible representation ρ of N*

$$\dim\left(\mathrm{Hom}_N(\pi_{|N}, \rho^*)\right) \times \dim\left(\mathrm{Hom}_N((\pi^*)_{|N}, \rho)\right) \leq 1.$$

Proof. Let $M' = M \times N$ and N' be the closed subgroup of M' which is the image of the diagonal embedding of N in M'. The map $(m,n) \mapsto mn^{-1}$ of M' onto M defines a homeomorphism of M'/N' onto M. The inverse map is $m \mapsto (m,1)N'$. On M'/N' left translations by N' correspond to the action of N on M by conjugation. We have a bijection between the space of distributions T on M invariant under the action of N by conjugation and the space of distributions S on M' which are biinvariant under N'. Explicitly

$$\langle S, f(m,n)\rangle = \langle T, \int_N f(mn,n)dn\rangle.$$

Suppose that T is invariant under σ and consider the involutive anti-automorphism σ' of M' given by $\sigma'(m,n) = (\sigma(m), \sigma(n))$. Then

$$\langle S, f \circ \sigma'\rangle = \langle T, \int_N f(\sigma(n)\sigma(m), \sigma(n))dn\rangle.$$

Using the invariance under σ and for the conjugation action of N we get

$$\begin{aligned}\langle T, \int_N f(\sigma(n)\sigma(m), \sigma(n))dn\rangle &= \langle T, \int_N f(\sigma(n)m, \sigma(n))dn\rangle \\ &= \langle T, \int_N f(mn,n)dn\rangle \\ &= \langle S, f\rangle.\end{aligned}$$

Hence S is invariant under σ'. Conversely if S is invariant under σ' the same computation shows that T is invariant under σ. Under the assumption of the corollary we can now apply Proposition 1.1 and we obtain the inequality

$$\dim\Big(\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi \otimes \rho)\Big) \times \dim\Big(\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi^* \otimes \rho^*)\Big) \leq 1.$$

We know that $Ind_{N'}^{M'}(1)$ is the smooth contragredient representation of $ind_{N'}^{M'}(1)$; hence

$$\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi^* \otimes \rho^*) \approx \mathrm{Hom}_{M'}(\pi \otimes \rho, Ind_{N'}^{M'}(1)).$$

Frobenius reciprocity tells us that

$$\mathrm{Hom}_{M'}\big(\pi \otimes \rho, Ind_{N'}^{M'}(1)\big) \approx \mathrm{Hom}_{N'}\big((\pi \otimes \rho)_{|N'}, 1\big).$$

Clearly

$$\mathrm{Hom}_{N'}\big((\pi \otimes \rho)_{|N'}, 1\big) \approx \mathrm{Hom}_N\big(\rho, (\pi_{|N})^*\big) \approx \mathrm{Hom}_N(\pi_{|N}, \rho^*).$$

Using again Frobenius reciprocity we get

$$\mathrm{Hom}_N\big(\rho, (\pi_{|N})^*\big) \approx \mathrm{Hom}_M\big(ind_N^M(\rho), \pi^*\big).$$

In the above computations we may replace ρ by ρ^* and π by π^*. Finally

$$\begin{aligned}
\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi^* \otimes \rho^*) &\approx \mathrm{Hom}_N(\rho, (\pi_{|N})^*) \\
&\approx \mathrm{Hom}_N(\pi_{|N}, \rho^*) \\
&\approx \mathrm{Hom}_M(ind_N^M(\rho), \pi^*). \\
\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi \otimes \rho) &\approx \mathrm{Hom}_N(\rho^*, ((\pi^*)_{|N})^*) \\
&\approx \mathrm{Hom}_N((\pi^*)_{|N}, \rho) \\
&\approx \mathrm{Hom}_M(ind_N^M(\rho^*), \pi).
\end{aligned}$$

□

Consider the case $M = GL(W)$ and $N = GL(V)$ in the notation of the intrduction. In order to use Corollary 1.1 to infer Theorem 1 from Theorem 2 it remains to show that

(1) $$\mathrm{Hom}_N((\pi^*)_{|N}, \rho) \approx \mathrm{Hom}_N(\pi_{|N}, \rho^*)$$

Let E_π be the space of the representation π and let E_π^* be the smooth dual (relative to the action of $GL(W)$). Let E_ρ be the space of ρ and E_ρ^* be the smooth dual for the action of $GL(V)$. We know, [BZ76, section 7] that the contragredient representation π^* in E_π^* is isomorphic to the representation $g \mapsto \pi({}^t g^{-1})$ in E_π. The same is true for ρ^*. Therefore an element of $\mathrm{Hom}_N(\pi_{|N}, \rho^*)$ may be described as a linear map A from E_π into E_ρ such that, for $g \in N$

$$A\pi(g) = \rho({}^t g^{-1})A.$$

An element of $\mathrm{Hom}_N((\pi^*)_{|N}, \rho)$ may be described as a linear map A' from E_π into E_ρ such that, for $g \in N$

$$A'\pi({}^t g^{-1}) = \rho(g)A'.$$

This yields (1).

Similarly, we prove that Theorem 2' implies Theorem 1'. With the notation of the introduction, this would follow from Corollary 1.1 provided that

(2) $$\mathrm{Hom}\left(\pi^*_{|G}, \rho\right) \approx \mathrm{Hom}\left(\pi_{|G}, \rho^*\right).$$

To show (2) we use the following result of [MVW87, Chapter 4]. Choose $\delta \in GL_{\mathbb{F}}(W)$ such that $\langle \delta w, \delta w' \rangle = \langle w', w \rangle$. If π is an irreducible admissible representation of M, let π^* be its smooth contragredient and define π^δ by

$$\pi^\delta(x) = \pi(\delta x \delta^{-1}).$$

Then π^δ and π^* are equivalent. We choose $\delta = 1$ in the orthogonal case $\mathbb{D} = \mathbb{F}$. In the unitary case, fix an orthogonal basis of W, say $e_1, \ldots, e_{n+1}$, such that $e_2, \ldots, e_{n+1}$ is a basis of V; put $\langle e_i, e_i \rangle = a_i$. Then

$$\langle \sum x_i e_i, \sum y_j e_j \rangle = \sum a_i x_i \overline{y_i}.$$

Define δ by

$$\delta\left(\sum x_i e_i\right) = \sum \overline{x_i} e_i.$$

Note that $\delta^2 = 1$.

Let E_π be the space of π. Then, up to equivalence, π^* is the representation $m \mapsto \pi(\delta m \delta^{-1})$. If ρ is an admissible irreducible representation of G in a vector space E_ρ then an element A of $\mathrm{Hom}\left(\pi^*_{|G}, \rho\right)$ is a linear map from E_π into E_ρ such that

$$A\pi(\delta g \delta^{-1}) = \pi(g)A, \quad g \in G.$$

In turn the contragredient ρ^* of ρ is equivalent to the representation $g \mapsto \rho(\delta g \delta^{-1})$ in E_ρ. Then an element B of $\mathrm{Hom}\left(\pi_{|G}, \rho^*\right)$ is a linear map from E_π into E_ρ such that

$$B\pi(g) = \rho(\delta g \delta^{-1})B, \quad g \in G.$$

As $\delta^2 = 1$ the conditions on A and B are the same. Thus (2) follows.

From now on we concentrate on Theorems 2 and 2'.

2. Some tools

We shall state two theorems which are systematically used in our proof.

If X is a Hausdorff totally disconnected locally compact topological space (lctd space in short) we denote by $\mathcal{S}(X)$ the vector space of locally constant functions with compact support of X into the field of complex numbers $\mathbb{C}$. The dual space $\mathcal{S}'(X)$ of $\mathcal{S}(X)$ is the space of distributions on X with the weak topology. All the lctd spaces we introduce are countable at infinity.

If a lctd topological group G acts continuously on a lctd space X then it acts on $\mathcal{S}(X)$ by

$$(gf)(x) = f(g^{-1}x)$$

and on distributions by

$$(gT)(f) = T(g^{-1}f)$$

The space of invariant distributions is denoted by $\mathcal{S}'(X)^G$. More generally, if χ is a character of G we denote by $\mathcal{S}'(X)^{G,\chi}$ the space of distributions T which transform according to χ that is to say $gT = \chi(g)T$.

The following result is due to Bernstein [Ber84], section 1.4.

Theorem 2.1 (Localization principle)**.** *Let $q : Z \to T$ be a continuous map between two topological spaces of type lctd. Denote $Z_t := q^{-1}(t)$. Consider $\mathcal{S}'(Z)$ as $\mathcal{S}(T)$-module. Let M be a closed subspace of $\mathcal{S}'(Z)$ which is an $\mathcal{S}(T)$-submodule. Then $M = \overline{\bigoplus_{t\in T}(M \cap \mathcal{S}'(Z_t))}$.*

Corollary 2.1. *Let $q : Z \to T$ be a continuous map between topological spaces of type lctd. Let a lctd group H act on Z preserving the fibers of q. Let μ be a character of H. Suppose that for any $t \in T$, $\mathcal{S}'(q^{-1}(t))^{H,\mu} = 0$. Then $\mathcal{S}'(Z)^{H,\mu} = 0$.*

The second theorem is a variant of Frobenius reciprocity ([Ber84, section 1.5] and [BZ76, sections 2.21-2.36]).

Theorem 2.2 (Frobenius descent)**.** *Suppose that a unimodular lctd topological group H act transitively on a lctd topological space Z. Let $\varphi : E \to Z$ be an H-equivariant map of* *lctd topological spaces. Let $x \in Z$. Assume that the stabilizer $Stab_H(x)$ is unimodular. Let $W = \varphi^{-1}(x)$ be the fiber of x. Let χ be a character of H. Then*

(1) *There exists a canonical isomorphism* $\mathrm{Fr} : \mathcal{S}'(E)^{H,\chi} \to \mathcal{S}'(W)^{Stab_H(x),\chi}$ *given by*

$$\langle \mathrm{Fr}(\xi), f \rangle = \int_Z \chi(g_z)\langle \xi, g_z f\rangle dz,$$

where dz denotes the Haar measure on Z, and $g_z \in H$ is an element such that $g_z z = x$.

(2) *For any distribution $\xi \in \mathcal{S}'(E)^{H,\chi}$,* $\mathrm{Supp}(\mathrm{Fr}(\xi)) = \mathrm{Supp}(\xi) \cap W$.

In particular, consider the case where H acts transitively on Z and W is a finite dimensional vector space over $\mathbb{F}$ with a nondegenerate bilinear form B. Assume that H acts on W linearly preserving B. Let $\mathrm{Fr} : \mathcal{S}'(Z \times W)^{H,\chi} \to \mathcal{S}'(W)^{Stab_H(x)}$ be the Frobenius isomorphism with respect to the projection map $Z \times W \to Z$. Let $\mathcal{F}_B$ be the Fourier transform in the W-coordinate. We have

Proposition 2.1. *For any $\xi \in \mathcal{S}'(Z \times W)^{H,\chi}$, we have $\mathcal{F}_B(\mathrm{Fr}(\xi)) = \mathrm{Fr}(\mathcal{F}_B(\xi))$*

This Proposition will be used in sections 4 and 6.

Finally as $\mathbb{F}$ is non-archimedean, a distribution which is 0 on some open set may be identified with a distribution on the (closed) complement. This will be used throughout this work.

3. Reduction to the singular set : the GL(n) case

Consider the case of the general linear group. From the decomposition $W = V \oplus \mathbb{F}e$ we get, with obvious identifications

$$\mathrm{End}(W) = \mathrm{End}(V) \oplus V \oplus V^* \oplus \mathbb{F}.$$

Note that $\mathrm{End}(V)$ is the Lie algebra $\mathfrak{g}$ of G. The group G acts on $\mathrm{End}(W)$ by $g(X, v, v^*, t) = (gXg^{-1}, gv, {}^tg^{-1}v^*, t)$. As before choose a basis $(e_1, \dots, e_n)$ of V and let $(e_1^*, \dots, e_n^*)$ be the dual basis of V^*. Define an isomorphism u of V onto V^* by $u(e_i) = e_i^*$. On $GL(W)$ the involution σ is $h \mapsto u^{-1t}h^{-1}u$. It depends upon the choice of the basis but the action on the space of invariant distributions does not depend upon this choice.

It will be convenient to introduce an extension $\widetilde{G}$ of G of degree two. Let $\mathrm{Iso}(V, V^*)$ be the set of isomorphisms of V onto V^*. We define $\widetilde{G} = G \cup \mathrm{Iso}(V, V^*)$. The group law, for $g, g' \in G$ and $u, u' \in \mathrm{Iso}(V, V^*)$ is

$$g \times g' = gg', \;\; u \times g = ug, \;\; g \times u = {}^tg^{-1}u, \;\; u \times u' = {}^tu^{-1}u'.$$

Now from $W = V \oplus \mathbb{F}e$ we obtain an identification of the dual space W^* with $V^* \oplus \mathbb{F}e^*$ with $\langle e^*, V\rangle = (0)$ and $\langle e^*, e\rangle = 1$. Any u as above extends to an isomorphism of W onto W^* by defining $u(e) = e^*$. The group $\widetilde{G}$ acts on $GL(W)$:

$$h \mapsto ghg^{-1}, \;\; h \mapsto {}^t(uhu^{-1}), \quad g \in G,\, h \in GL(W),\; u \in \mathrm{Iso}(V, V^*)$$

and also on $\mathrm{End}(W)$ with the same formulas.

Let χ be the character of $\widetilde{G}$ which is 1 on G and -1 on $\mathrm{Iso}(V,V^*)$. Our goal is to prove that $\mathcal{S}'(GL(W))^{\widetilde{G},\chi} = (0)$.

Proposition 3.1. *If* $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$ *then* $\mathcal{S}'(GL(W))^{\widetilde{G},\chi} = (0)$.

Proof. We have $\mathrm{End}(W) = \big(\mathrm{End}(V) \oplus V \oplus V^*\big) \oplus \mathbb{F}$ and the action of $\widetilde{G}$ on $\mathbb{F}$ is trivial. Thus $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$ implies that $\mathcal{S}'(\mathrm{End}(W))^{\widetilde{G},\chi} = (0)$. Let $T \in \mathcal{S}'(GL(W))^{\widetilde{G},\chi}$. Let $h \in GL(W)$ and choose a compact open neighborhood K of Det h such that $0 \notin K$. For $x \in \mathrm{End}(W)$ define $\varphi(x) = 1$ if $\mathrm{Det}\, x \in K$ and $\varphi(x) = 0$ otherwise. Then φ is a locally constant function. The distribution $(\varphi_{|GL(W)})T$ has a support which is closed in $\mathrm{End}(W)$ hence may be viewed as a distribution on $\mathrm{End}(W)$. This distribution belongs to $\mathcal{S}'(\mathrm{End}(W))^{\widetilde{G},\chi}$ so it must be equal to 0. It follows that T is 0 in the neighborhood of h. As h is arbitrary we conclude that $T = 0$. ☐

Our task is now to prove that $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$. We shall use induction on the dimension n of V. The action of $\widetilde{G}$ is, for $X \in \mathfrak{g}$, $v \in V$, $v^* \in V^*$, $g \in G$, $u \in \mathrm{Iso}(V,V^*)$

$$(X, v, v^*) \mapsto (gXg^{-1}, gv, {}^tg^{-1}v^*), \;\; (X, v, v^*) \mapsto ({}^t(uXu^{-1}), {}^tu^{-1}v^*, uv).$$

The case $n = 0$ is trivial.

We suppose that V is of dimension $n \geq 1$, assuming the result up to dimension $n-1$ and for all $\mathbb{F}$. If $T \in \mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi}$ we are going to show that its support is contained in the "singular set". This will be done in two stages.

On $V \oplus V^*$ let Γ be the cone $\langle v^*, v\rangle = 0$. It is stable under $\widetilde{G}$.

Lemma 3.1. *The support of* T *is contained in* $\mathfrak{g} \times \Gamma$.

Proof. For $(X, v, v^*) \in \mathfrak{g} \oplus V \oplus V^*$ put $q(X, v, v^*) = \langle v^*, v\rangle$. Let Ω be the open subset $q \neq 0$. We have to show that $\mathcal{S}'(\Omega)^{\widetilde{G},\chi} = (0)$. By Bernstein's localization principle (Corollary 2.1) it is enough to prove that, for any fiber $\Omega_t = q^{-1}(t)$, $t \neq 0$, one has $\mathcal{S}'(\Omega_t)^{\widetilde{G},\chi} = (0)$.

G acts transitively on the quadric $\langle v^*, v\rangle = t$. Fix a decomposition $V = \mathbb{F}\varepsilon \oplus V_1$ and identify $V^* = \mathbb{F}\varepsilon^* \oplus V_1^*$ with $\langle \varepsilon^*, \varepsilon\rangle = 1$. Then $(X, \varepsilon, t\varepsilon^*) \in \Omega_t$ and the isotropy subgroup of $(\varepsilon, t\varepsilon^*)$ in $\widetilde{G}$ is, with an obvious notation $\widetilde{G}_{n-1}$. By Frobenius descent (Theorem 2.2) there is a linear bijection between $\mathcal{S}'(\Omega_t)^{\widetilde{G},\chi}$ and the space $\mathcal{S}'(\mathfrak{g})^{\widetilde{G}_1,\chi_1}$ and this last space is (0) by induction. ☐

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ that is to say the space of scalar matrices. Let $\mathcal{N} \subseteq [\mathfrak{g}, \mathfrak{g}]$ be the nilpotent cone in $\mathfrak{g}$.

Proposition 3.2. *If* $T \in \mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi}$ *then the support of* T *is contained in* $(\mathfrak{z} + \mathcal{N}) \times \Gamma$. *If* $\mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi} = (0)$ *then* $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$.

Proof. Let us prove that the support of such a distribution T is contained in $(\mathfrak{z} + \mathcal{N}) \times (V \oplus V^*)$. We use Harish-Chandra's descent method. For $X \in \mathfrak{g}$ let $X = X_s + X_n$ be the Jordan decomposition of X with X_s semisimple and X_n nilpotent. This decomposition commutes with the action of $\widetilde{G}$. The centralizer $Z_G(X)$ of an element $X \in \mathfrak{g}$ is unimodular

([SS70, page 235]) and there exists an isomorphism u of V onto V^* such that ${}^tX = uXu^{-1}$ (any matrix is conjugate to its transpose). It follows that the centralizer $Z_{\widetilde{G}}(X)$ of X in $\widetilde{G}$, a semi direct product of $Z_G(X)$ and S_2, is also unimodular.

Let E be the space of monic polynomials of degree n with coefficients in $\mathbb{F}$. For $p \in E$, let $\mathfrak{g}_p$ be the set of all $X \in \mathfrak{g}$ with characteristic polynomial p. Note that $\mathfrak{g}_p$ is fixed by $\widetilde{G}$. By Bernstein localization principle (Theorem 2.1) it is enough to prove that if p is not $(T-\lambda)^n$ for some λ then $\mathcal{S}'(\mathfrak{g}_p \times V \times V^*)^{\widetilde{G},\chi} = (0)$.

Fix p. We claim that the map $X \mapsto X_s$ restricted to $\mathfrak{g}_p$ is continuous. Indeed let $\widetilde{\mathbb{F}}$ be a finite Galois extension of $\mathbb{F}$ containing all the roots of p. Let

$$p(\xi) = \prod_1^s (\xi - \lambda_i)^{n_i}$$

be the decomposition of p. Recall that if $X \in \mathfrak{g}_p$ and $V_i = \operatorname{Ker}(X-\lambda_i)^{n_i}$ then $V = \oplus V_i$ and the restriction of X_s to V_i is the multiplication by λ_i. Then choose a polynomial R, with coefficients in $\widetilde{\mathbb{F}}$ such that for all i, R is congruent to λ_i modulo $(\xi - \lambda_i)^{n_i}$ and $R(0) = 0$. Clearly $X_s = R(X)$. As the Galois group of $\widetilde{\mathbb{F}}$ over $\mathbb{F}$ permutes the λ_i we may even choose $R \in \mathbb{F}[\xi]$. This implies the required continuity.

There is only one semi-simple orbit γ_p in $\mathfrak{g}_p$ and it is closed. We use Frobenius descent (Theorem 2.2) for the map $(X, v, v^*) \mapsto X_s$ from $\mathfrak{g}_p \times V \times V^*$ to γ_p.

Fix $a \in \gamma_p$; its fiber is the product of $V \oplus V^*$ by the set of nilpotent elements which commute with a. It is a closed subset of the centralizer $\mathfrak{m} = \mathfrak{z}_{\mathfrak{g}}(a)$ of a in $\mathfrak{g}$. Let $M = Z_G(a)$ and $\widetilde{M} = Z_{\widetilde{G}}(a)$.

Following [SS70] let us describe these centralizers. Let P be the minimal polynomial of a ; all its roots are simple. Let $P = P_1 \dots P_r$ be the decomposition of P into (distinct) irreducible factors, over $\mathbb{F}$. If $V_i = \operatorname{Ker} P_i(a)$, then $V = \oplus V_i$ and $V^* = \oplus V_i^*$. An element x of G which commutes with a is given by a family $\{x_1, \dots, x_r\}$ where each x_i is a linear map from V_i to V_i, commuting with the restriction of a to V_i. Now $\mathbb{F}[\xi]$ acts on V_i, by specializing ξ to $a_{|V_i}$ and P_i acts trivially so that, if $\mathbb{F}_i = \mathbb{F}[\xi]/(P_i)$, then V_i becomes a vector space over $\mathbb{F}_i$. The $\mathbb{F}$-linear map x_i commutes with a if and only if it is $\mathbb{F}_i$-linear.

Fix i. Let ℓ be a non-zero $\mathbb{F}$-linear form on $\mathbb{F}_i$. If $v_i \in V_i$ and $v_i' \in V_i^*$ then $\lambda \mapsto \langle \lambda v_i, v_i' \rangle$ is an $\mathbb{F}$-linear form on $\mathbb{F}_i$, hence there exists a unique element $S(v_i, v_i')$ of $\mathbb{F}_i$ such that $\langle \lambda v_i, v_i' \rangle = \ell(\lambda S(v_i, v_i'))$. One checks trivially that S is $\mathbb{F}_i$-linear with respect to each variable and defines a non degenerate duality, over $\mathbb{F}_i$ between V_i and V_i^*. Here $\mathbb{F}_i$ acts on V_i^* by transposition, relative to the $\mathbb{F}$-duality $\langle ., . \rangle$,of the action on V_i. Finally if $x_i \in \operatorname{End}_{\mathbb{F}_i} V_i$, its transpose, relative to the duality $S(., .)$ is the same as its transpose relative to the duality $\langle ., . \rangle$.

Thus M is a product of linear groups and the situation (M, V, V^*) is a composite case, each component being a linear case (over various extensions of $\mathbb{F}$).

Let u be an isomorphism of V onto V^* such that ${}^ta = uau^{-1}$ and that, for each i, $u(V_i) = V_i^*$. Then $u \in \widetilde{M}$ and $\widetilde{M} = M \cup uM$.

Suppose that a does not belong to the center of $\mathfrak{g}$. Then each V_i has dimension strictly smaller than n and we can use the inductive assumption. Therefore $\mathcal{S}'(\mathfrak{m} \oplus V \oplus V^*)^{\widetilde{M},\chi} = (0)$. However the nilpotent cone $\mathcal{N}_{\mathfrak{m}}$ in $\mathfrak{m}$ is a closed subset so $\mathcal{S}'(\mathcal{N}_{\mathfrak{m}} \times V \times V^*)^{\widetilde{M},\chi} = (0)$.

Together with Lemma 3.1 this proves the first assertion of the Proposition.

If a belongs to the center then $\widetilde{M} = \widetilde{G}$ and the fiber is $(a + \mathcal{N}) \times V \times V^*$. This implies the second assertion. □

Remark 1. *Strictly speaking the singular set is defined as the set of all (X, v, v^*) such that for any polynomial P invariant under $\widetilde{G}$ one has $P(X, v, v^*) = P(0)$. Thus, in principle, we also need to consider the polynomials $P(X, v, v^*) = \langle v^*, X^p v\rangle$ for $p > 0$. In fact, one can show that the support of the distribution T is contained in the singular set in the strict sense (i.e., the above polynomials vanish on the support). As this is not needed in the sequel we omit the proof.*

4. End of the proof for GL(n)

In this section we consider a distribution $T \in \mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ and prove that $T = 0$. The following observation will play a crucial role.

Choose a non-trivial additive character ψ of $\mathbb{F}$. On $V \oplus V^*$ we have the bilinear form

$$\big((v_1, v_1^*), (v_2, v_2^*)\big) \mapsto \langle v_1^*, v_2\rangle + \langle v_2^*, v_1\rangle$$

Define the Fourier transform of a function φ on $V \oplus V^*$ by

$$\widehat{\varphi}(v_2, v_2^*) = \int_{V \oplus V^*} \varphi(v_1, v_1^*)\, \psi(\langle v_1^*, v_2\rangle + \langle v_2^*, v_1\rangle)\, dv_1 dv_1^*$$

where $dv_1 dv_1^*$ is the self-dual Haar measure.

This Fourier transform commutes with the action of $\widetilde{G}$; hence the (partial) Fourier transform $\widehat{T}$ of our distribution T has the same invariance properties and the same support conditions as T itself.

Let $\mathcal{N}_i$ be the union of nilpotent orbits of dimension at most i. We will prove, by descending induction on i, that the support of any $(\widetilde{G}, \chi)$-equivariant distribution on $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ must be contained in $\mathcal{N}_i \times \Gamma$. Suppose we already know that, for some i, the support must be contained in $\mathcal{N}_i \times \Gamma$. We must show that, for any nilpotent orbit $\mathcal{O}$ of dimension i, the restriction of the distribution to $\mathcal{O} \times \Gamma$ is 0.

If $v \in V$ and $v^* \in V^*$ we call X_{v,v^*} the rank one map $x \mapsto \langle v^*, x\rangle v$. Let

$$\nu_\lambda(X, v, v^*) = (X + \lambda X_{v,v^*}, v, v^*), \quad (X, v, v^*) \in \mathfrak{g} \times \Gamma,\ \ \lambda \in \mathbb{F}.$$

Then ν_λ is a one parameter group of homeomorphisms of $\mathfrak{g} \times \Gamma$ and note that $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ is invariant. The key observation is that ν_λ *commutes with the action of* $\widetilde{G}$. Therefore the image of T by ν_λ transforms according to the character χ of $\widetilde{G}$. Its support is contained in $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ and hence must be contained in $\mathcal{N} \times \Gamma$ and in fact in $\mathcal{N}_i \times \Gamma$. This means that if (X, v, v^*) belongs to the support of T then, for all λ, $(X + \lambda X_{v,v^*}, v, v^*)$ must belong to $\mathcal{N}_i \times \Gamma$.

The orbit $\mathcal{O}$ is open in $\mathcal{N}_i$. Thus if $X \in \mathcal{O}$ the condition $X + \lambda X_{v,v^*} \in \mathcal{N}_i$ implies that, at least for $|\lambda|$ small enough, $X + \lambda X_{v,v^*} \in \mathcal{O}$. It follows that X_{v,v^*} belongs to the tangent space to $\mathcal{O}$ at the point X ; this tangent space is the image of $\operatorname{ad} X$.

Define $Q(X)$ to be the set of all pairs (v, v^*) such $X_{v,v^*} \in \operatorname{Im} \operatorname{ad} X$.

By the discussion above, it is enough to prove the following Lemma:

Lemma 4.1. *Let $T \in \mathcal{S}'(\mathcal{O} \times V \times V^*)^{\widetilde{G},\chi}$. Suppose that the support of T and of $\widehat{T}$ are contained in the set of triplets (X, v, v^*) such that $(v, v^*) \in Q(X)$. Then $T = 0$.*

Note that the trace of X_{v,v^*} is $\langle v^*, v\rangle$ and that $X_{v,v^*} \in \operatorname{Im} \operatorname{ad} X$ implies that its trace is 0. Therefore $Q(X)$ is contained in Γ.

We proceed in three steps. First we transfer the problem to $V \oplus V^*$ and a fixed nilpotent endomorphism X. Then we show that if Lemma 4.1 holds for (V_1, X_1) and (V_2, X_2) then it holds for the direct sum $(V_1 \oplus V_2, X_1 \oplus X_2)$. Finally, decomposing X into Jordan blocks we are left with the case of a principal nilpotent element for which we give a direct proof, using Weil representation.

Consider the map $(X, v, v^*) \mapsto X$ from $\mathcal{O} \times V \times V^*$ onto $\mathcal{O}$. Choose $X \in \mathcal{O}$ and let C (resp $\widetilde{C}$) be the stabilizer in G (resp. in $\widetilde{G}$) of an element X of $\mathcal{O}$; both groups are unimodular, hence we may use Frobenius descent (Theorem 2.2).

Now we have to deal with a distribution, which we still call T, which belongs to $\mathcal{S}'(V \oplus V^*)^{\widetilde{C},\chi}$ such that both T and its Fourier transform are supported by $Q(X)$ (Proposition 2.1). Let us say that X is *nice* if the only such distribution is 0. We want to prove that all nilpotent endomorphisms are nice.

Lemma 4.2. *Suppose that we have a decomposition $V = V_1 \oplus V_2$ such that $X(V_i) \subseteq V_i$. Let X_i be the restriction of X to V_i. Then if X_1 and X_2 are nice, so is X.*

Proof. Let $(v, v^*) \in Q(X)$ and choose $A \in \mathfrak{g}$ such that $X_{v,v^*} = [A, X]$. Decompose $v = v_1 + v_2,\ v^* = v_1^* + v_2^*$ and put

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

Writing X_{v,v^*} as a 2 by 2 matrix and looking at the diagonal blocks one gets that $X_{v_i,v_i^*} = [A_{i,i}, X_i]$. This means that

$$Q(X) \subseteq Q(X_1) \times Q(X_2).$$

For $i = 1, 2$ let C_i be the centralizer of X_i in $GL(V_i)$ and $\widetilde{C}_i$ the corresponding extension by S_2. Let T be a distribution as above and let $\varphi_2 \in \mathcal{S}(V_2 \oplus V_2^*)$. Let T_1 be the distribution on $V_1 \oplus V_1^*$ defined by $\varphi_1 \mapsto \langle T, \varphi_1 \otimes \varphi_2\rangle$. The support of T_1 is contained in $Q(X_1)$ and T_1 is invariant under the action of C_1 . We have

$$\langle \widehat{T}_1, \varphi_1\rangle = \langle T_1, \widehat{\varphi}_1\rangle = \langle T, \widehat{\varphi_1} \otimes \varphi_2\rangle = \langle \widehat{T}, \check{\varphi}_1 \otimes \widehat{\varphi_2}\rangle.$$

Here $\check{\varphi}_1(v_1, v_1^*) = \varphi_1(-v_1, -v_1^*)$. By assumption the support of $\widehat{T}$ is contained in $Q(X)$ so that the support of $\widehat{T_1}$ is supported in $-Q(X_1) = Q(X_1)$. Because (X_1) is nice this implies that T_1 in invariant under $\widetilde{C}_1$.

Extend the action of $\widetilde{C}_1$ to $V \oplus V^*$ trivially. We obtain that T is invariant with respect to $\widetilde{C}_1$. Similarly it is invariant under $\widetilde{C}_2$. Since the actions of $\widetilde{C}_1$ and $\widetilde{C}_2$ together with the action of C generate the action of $\widetilde{C}$ we obtain that T must be invariant under $\widetilde{C}$ and hence must be 0. $\square$

Decomposing X into Jordan blocks we still have to prove Lemma 4.1 for a principal nilpotent element. We need some preliminary results.

Lemma 4.3. *The distribution T satisfies the following homogeneity condition:*

$$\langle T, f(tv, tv^*)\rangle = |t|^{-n}\langle T, f(v, v^*)\rangle.$$

Proof. We use a particular case of Weil or oscillator representation. Let E be a vector space over $\mathbb{F}$ of finite dimension m. To simplify assume that m is even. Let q be a non-degenerate quadratic form on E and let b be the bilinear form

$$b(e, e') = q(e + e') - q(e) - q(e').$$

Fix a continuous non-trivial additive character ψ of $\mathbb{F}$. We define the Fourier transform on E by

$$\widehat{f}(e') = \int_E f(e)\psi(b(e, e'))de$$

where de is the self dual Haar measure.

There exists ([RS07]) a representation π of $SL(2, \mathbb{F})$ in $\mathcal{S}(E)$ such that:

$$\begin{aligned}
\pi\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f(e) &= \psi(uq(e))f(e) \\
\pi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} f(e) &= \frac{\gamma(q)}{\gamma(tq)}|t|^{m/2}f(te) \\
\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(e) &= \gamma(q)\widehat{f}(e)
\end{aligned}$$

where $\gamma(\cdot)$ is a certain roots of unity, which is 1 if (E, q) is a sum of hyperbolic planes.

We have a contragredient action in the dual space $\mathcal{S}'(E)$.

Suppose that T is a distribution on E such that T and $\widehat{T}$ are supported on the isotropic cone $q(e) = 0$. This means that

$$\langle T, \pi\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f\rangle = \langle T, f\rangle, \quad \langle \widehat{T}, \pi\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f\rangle = \langle \widehat{T}, f\rangle.$$

Using the relation

$$\langle \widehat{T}, \varphi\rangle = \langle T, \overline{\gamma(q)}\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\rangle$$

the second relation is equivalent to

$$\langle T, \pi\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} f\rangle = \langle T, f\rangle.$$

The matrices

$$\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}1 & 0\\ u & 1\end{pmatrix} \quad u \in \mathbb{F}$$

generate the group $SL(2,\mathbb{F})$. Therefore the distribution T is invariant by $SL(2,\mathbb{F})$. In particular

$$\langle T, f(te)\rangle = \frac{\gamma(tq)}{\gamma(q)}|t|^{-m/2}\langle T, f\rangle$$

and $T = \gamma(q)\widehat{T}$.

Remark 2. *Note that (for even m) $\gamma(tq)/\gamma(q)$ is a character of t and non-zero distributions which are invariant under $SL(2,\mathbb{F})$ do exist. In the case where m is odd one obtains a representation of the two-fold covering of $SL(2,\mathbb{F})$ and we obtain the same homogeneity condition. However $\gamma(tq)/\gamma(q)$ is not a character; hence no non-zero T can exist.*

In our situation we take $E = V \oplus V^*$ and $q(v, v^*) = \langle v^*, v\rangle$. Then

$$b\Big((v_1, v_1^*), (v_2, v_2^*)\Big) = \langle v_1^*, v_2\rangle + \langle v_2^*, v_1\rangle.$$

The Fourier transform commutes with the action of $\widetilde{G}$. Both T and $\widehat{T}$ are supported on $Q(X)$ which is contained in Γ. As $\gamma(tq) = 1$ for all t this proves the Lemma and also that $T = \widehat{T}$. □

Remark 3. *The same type of argument could have been used for the quadratic form $\mathrm{Tr}(XY)$ on $\mathfrak{sl}(V) = [\mathfrak{g}, \mathfrak{g}]$. This would have given a short proof for even n and a homogeneity condition for odd n.*

Now we find $Q(X)$.

Lemma 4.4. *If X is principal then $Q(X)$ is the set of pairs (v, v^*) such that for $0 \le k < n$, $\langle v^*, X^k v\rangle = 0$.*

Proof. Choose a basis $(e_1, \ldots, e_n)$ of V such that $Xe_1 = 0$ and $Xe_j = e_{j-1}$ for $j \ge 2$. Consider the map $A \mapsto XA - AX$ from the space of n by n matrices into itself. This map is anti-symmetric with respect to the Killing form and hence its image is the orthogonal complement to its kernel. A simple computation shows that the kernel of this map, that is to say the Lie algebra $\mathfrak{c}$ of the centralizer C, is the space of polynomials (of degree at most $n-1$) in X. Therefore

$$Q(X) = \{(v, v^*) | X_{v,v^*} \in \mathrm{Im}\,\mathrm{ad}\,X\} = \{(v, v^*) | \forall\, 0 \le k < n,\ \mathrm{Tr}(X_{v,v^*}X^k) = 0\} =$$
$$= \{(v, v^*) | \forall\, 0 \le k < n,\ \langle v^*, X^k v\rangle = 0\}.$$

□

End of the proof of Lemma 4.1 For a principal X, we proceed by induction on n. Keep the above notation. The centralizer C of X is the space of polynomials (of degree at most $n-1$) in X with non-zero constant term. In particular the orbit Ω of e_n is the open subset $x_n \neq 0$. We shall prove that the restriction of T to $\Omega \times V^*$ is 0. Note that the centralizer of

e_n in C is trivial. By Frobenius descent (Theorem 2.2), to the restriction of T corresponds a distribution R on V^* with support in the set of v^* such that $(e_n, v^*) \in Q(X)$. By the last Lemma this means that R is a multiple $a\delta$ of the Dirac measure at the origin. The distribution T satisfies the two conditions

$$\langle T, f(v, v^*)\rangle = \langle T, f(tv, t^{-1}v^*)\rangle = |t|^n \langle T, f(tv, tv^*)\rangle.$$

therefore

$$\langle T, f(v, t^2 v^*) = |t|^{-n}\langle T, f(v, v^*)\rangle.$$

Now T is recovered from R by the formula

$$\langle T, f(v, v^*) = \int_C \langle R, f(ce_n, {}^t c^{-1}v^*\rangle dc = a\int_C f(ce_n, 0)dc, \;\; f \in \mathcal{S}(\Omega \times V^*).$$

Unless $a = 0$ this is not compatible with this last homogeneity condition.

Exactly in the same way one proves that T is 0 on $V \times \Omega^*$ where Ω^* is the open orbit $x_1^* \neq 0$ of C in V^*. The same argument is valid for $\widehat{T}$ (which is even equal to T ...).

If $n = 1$ then T is obviously 0. If $n \geq 2$ then there exists a distribution T' on

$$\bigoplus_{1<j<n} \mathbb{F}e_j \oplus \mathbb{F}e_j^*$$

such that,

$$T = T' \otimes \delta_{x_n=0} \otimes dx_1 \otimes \delta_{x_1^*=0} \otimes dx_n^*.$$

Let u be the isomorphism of V onto V^* given by $u(e_j) = e^*_{n+1-j}$. Recall that it acts on $\mathfrak{g} \times V \times V^*$ by $(X, v, v^*) \mapsto ({}^t(uXu^{-1}), {}^t u^{-1}v^*, uv)$. It belongs to $\widetilde{C}$ but not to C so it must transform T into $-T$.

The case $n = 1$ has just been settled. If $n = 2$ in the above formula T' should be replaced by a constant. The constant must be 0 if we want $u(T) = -T$. If $n > 2$ let

$$V' = \left(\oplus_1^{n-1}\mathbb{F}e_i\right)/\mathbb{F}e_1$$

and let X' be the nilpotent endomorphism of V' defined by X. We may consider T' as a distribution on $V' \oplus V'^*$ and one easily checks that, with obvious notation, it transforms according to the character χ of the the centralizer $\widetilde{C}'$ of X' in $\widetilde{G}'$. By induction $T' = 0$, hence $T = 0$.

□

5. Reduction to the singular set: the orthogonal and unitary cases

We now turn our attention to the unitary case. We keep the notation of the introduction. In particular $W = V \oplus \mathbb{D}e$ is a vector space over $\mathbb{D}$ of dimension $n+1$ with a non-degenerate hermitian form $\langle ., .\rangle$ such that e is orthogonal to V. The unitary group G of V is embedded into the unitary group M of W.

Let A be the set of all bijective maps u from V to V such that

$$u(v_1 + v_2) = u(v_1) + u(v_2), \;\; u(\lambda v) = \overline{\lambda}u(v), \;\; \langle u(v_1), u(v_2)\rangle = \overline{\langle v_1, v_2\rangle}.$$

An example of such a map is obtained by choosing a basis $e_1, \ldots, e_n$ of V such that $\langle e_i, e_j\rangle \in \mathbb{F}$ and defining

$$u(\sum x_i e_i) = \sum \overline{x}_i e_i.$$

Any $u \in A$ is extended to W by the rule $u(v+\lambda e) = u(v) + \overline{\lambda} e$ and we define an action on $\mathrm{GL}(W)$ by $m \mapsto u m^{-1} u^{-1}$. The group G acts on $\mathrm{GL}(W)$ by conjugation.

Let $\widetilde{G}$ be the group of bijections of $\mathrm{GL}(W)$ onto itself generated by the actions of G and A. It is a semi direct product of G and S_2. We identify G with a subgroup of $\widetilde{G}$ and A with $\widetilde{G}\setminus G$. Note that $\widetilde{G}$ preserves M. When a confusion is possible we denote the product in $\widetilde{G}$ by $\times$.

We define a character χ of $\widetilde{G}$ by $\chi(g) = 1$ for $g \in G$ and $\chi(u) = -1$ for $u \in \widetilde{G} \setminus G$. Our overall goal is to prove that $\mathcal{S}'(M)^{\widetilde{G},\chi} = (0)$.

Let $\widetilde{G}$ act on $G \times V$ as follows:

$$g(x,v) = (gxg^{-1}, g(v)),\ u(x,v) = (ux^{-1}u^{-1}, -u(v)), \quad g \in G, u \in A, x \in G, v \in V$$

Our first step is to replace M by $G \times V$.

Proposition 5.1. *Suppose that for any V and any hermitian form $\mathcal{S}'(G \times V)^{\widetilde{G},\chi} = (0)$, then $\mathcal{S}'(M)^{\widetilde{G},\chi} = (0)$.*

Proof. We have in particular $\mathcal{S}'(M \times W)^{\widetilde{M},\chi} = (0)$. Let Y be the set of all (m, w) such that $\langle w, w\rangle = \langle e, e\rangle$; it is a closed subset, invariant under $\widetilde{M}$, hence $\mathcal{S}'(Y)^{\widetilde{M},\chi} = (0)$. By Witt's theorem M acts transitively on $\Gamma = \{w | \langle w, w\rangle = \langle e, e\rangle\}$. We can apply Frobenius descent (Theorem 2.2) to the map $(m, w) \mapsto w$ of Y onto Γ. The centralizer of e in $\widetilde{M}$ is isomorphic to $\widetilde{G}$ acting as before on the fiber $M \times \{e\}$. We have a linear bijection between $\mathcal{S}'(M)^{\widetilde{G}.\chi}$ and $\mathcal{S}'(Y)^{\widetilde{M},\chi}$; therefore $\mathcal{S}'(M)^{\widetilde{G}.\chi} = (0)$. □

The proof that $\mathcal{S}'(G \times V)^{\widetilde{G},\chi} = (0)$ is by induction on n. If $\mathfrak{g}$ is the Lie algebra of G we shall prove simultaneously that $\mathcal{S}'(\mathfrak{g} \times V)^{\widetilde{G},\chi} = (0)$. In this case G acts on its Lie algebra by the adjoint action and for $u \in \widetilde{G} \setminus G$ one puts, for $X \in \mathfrak{g}$, $u(X) = -uXu^{-1}$.

The case $n = 0$ is trivial so we may assume that $n \geq 1$. If $T \in \mathcal{S}'(G \times V)^{\widetilde{G},\chi}$ in this section we will prove that the support of T must be contained in the "singular set".

Let Z (resp. $\mathfrak{z}$) be the center of G (resp. $\mathfrak{g}$) and $\mathcal{U}$ (resp. $\mathcal{N}$) the (closed) set of all unipotent (resp. nilpotent) elements of G (resp. $\mathfrak{g}$).

Lemma 5.1. *If $T \in \mathcal{S}'(G \times V)^{\widetilde{G},\chi}$ (resp. $T \in \mathcal{S}'(\mathfrak{g} \times V)^{\widetilde{G},\chi}$) then the support of T is contained in $Z\mathcal{U} \times V$ (resp. $(\mathfrak{z} + \mathcal{N}) \times V$).*

Proof. This is Harish-Chandra's descent. We first review some facts about the centralizers of semi-simple elements, following [SS70].

Let $a \in G$, semi-simple; we want to describe its centralizer G_a (resp. $\widetilde{G}_a$) in G (resp. in $\widetilde{G}$) and to show that $\mathcal{S}'(G_a \times V)^{\widetilde{G}_a,\chi} = (0)$.

View a as a $\mathbb{D}$-linear endomorphism of V and call P its minimal polynomial. Then, as a is semi-simple, P decomposes into distinct irreducible factors $P = P_1 \ldots P_r$. Let $V_i = \operatorname{Ker} P_i(a)$ so that $V = \oplus V_i$. Any element x which commutes with a will satisfy $xV_i \subseteq V_i$ for each i. For

$$R(\xi) = d_0 + \cdots + d_m \xi^m, \quad d_0 d_m \neq 0$$

let

$$R^*(\xi) = \overline{d_0} \xi^m + \cdots + \overline{d_m}.$$

Then, from $aa^* = 1$ we obtain, if m is the degree of P

$$\langle P(a)v, v' \rangle = \langle v, a^{-m} P^*(a) v' \rangle$$

(note that the constant term of P can not be 0 because a is invertible). It follows that $P^*(a) = 0$ so that P^* is proportional to P. Now $P^* = P_1^* \ldots P_r^*$; hence there exists a bijection τ from $\{1, 2, \ldots, r\}$ onto itself such that P_i^* is proportional to $P_{\tau(i)}$. Let m_i be the degree of P_i. Then, for some non-zero constant c

$$0 = \langle P_i(a) v_i, v_j \rangle = \langle v_i, a^{-m_i} P_i^*(a) v_j \rangle = c \langle v_i, a^{-m_i} P_{\tau(i)}(a) v_j \rangle, \quad v_i \in V_i, \ v_j \in V_j.$$

We have two possibilities.

Case 1: $\tau(i) = i$. The space V_i is orthogonal to V_j for $j \neq i$; the restriction of the hermitian form to V_i is non-degenerate. Let $\mathbb{D}_i = \mathbb{D}[\xi]/(P_i)$ and consider V_i as a vector space over $\mathbb{D}_i$ through the action $(R(\xi), v) \mapsto R(a)v$. As $a_{|V_i}$ is invertible, ξ is invertible modulo (P_i); choose η such that $\xi\eta = 1$ modulo (P_i). Let σ_i be the semi-linear involution of $\mathbb{D}_i$, as an algebra over $\mathbb{D}$:

$$\sum d_j \xi^j \mapsto \sum \overline{d_j} \eta^j \pmod{P}_i$$

Let $\mathbb{F}_i$ be the subfield of fixed points for σ_i. It is a finite extension of $\mathbb{F}$, and $\mathbb{D}_i$ is either a quadratic extension of $\mathbb{F}_i$ or equal to $\mathbb{F}_i$. There exists a $\mathbb{D}$-linear form $\ell \neq 0$ on $\mathbb{D}_i$ such that $\ell(\sigma_i(d)) = \overline{\ell(d)}$ for all $d \in \mathbb{D}_i$. Then any $\mathbb{D}$-linear form L on $\mathbb{D}_i$ may be written as $d \mapsto \ell(\lambda d)$ for some unique $\lambda \in \mathbb{D}_i$.

If $v, v' \in V_i$ then $d \mapsto \langle d(a)v, v' \rangle$ is $\mathbb{D}$-linear map on $\mathbb{D}_i$; hence there exists $S(v, v') \in \mathbb{D}_i$ such that

$$\langle d(a)v, v' \rangle = \ell(dS(v, v')).$$

One checks that S is a non-degenerate hermitian form on V_i as a vector space over $\mathbb{D}_i$. Also a $\mathbb{D}$-linear map x_i from V_i into itself commutes with a_i if and only if it is $\mathbb{D}_i$-linear and it is unitary with respect to our original hermitian form if and only if it is unitary with respect to S. So in this case we call G_i the unitary group of S. It does not depend upon the choice of ℓ. As no confusion may arise, for $\lambda \in \mathbb{D}_i$ we define $\overline{\lambda} = \sigma_i(\lambda)$.

We choose an $\mathbb{F}_i$-linear map u_i from V_i onto itself, such that $u_i(\lambda v) = \overline{\lambda} u(v)$ and $S(u_i(v), u_i(v')) = \overline{S(v, v')}$. Then because of our original choice of ℓ we also have $\langle u_i(v), u_i(v') \rangle = \overline{\langle v, v' \rangle}$. Note that $u(a_{|V_i})^{-1} u^{-1} = a_{|V_i}$.

Case 2. Suppose now that $j = \tau(i) \neq i$. Then $V_i \oplus V_j$ is orthogonal to V_k for $k \neq i, j$ and the restriction of the hermitian form to $V_i \oplus V_j$ is non-degenerate, both V_i and V_j being totally isotropic subspaces. Choose an inverse η of ξ modulo P_j. Then for any $P \in \mathbb{D}[\xi]$

$$\langle P(a)v_i, v_j\rangle = \langle v_i, \overline{P}(\eta(a))v_j\rangle, \quad v_i \in V_i,\ v_j \in V_j$$

where $\overline{P}$ is the polynomial obtained from P by conjugating its coefficients. This defines a map, which we call σ_i from $\mathbb{D}_i$ onto $\mathbb{D}_j$. In a similar way we have a map σ_j which is the inverse of σ_i. Then, for $\lambda \in \mathbb{D}_i$ we have $\langle \lambda v_i, v_j\rangle = \langle v_i, \sigma_i(\lambda)v_j\rangle$.

View V_i as a vector space over $\mathbb{D}_i$. The action

$$(\lambda, v_j) \mapsto \sigma_i(\lambda)v_j$$

defines a structure of $\mathbb{D}_i$ vector space on V_j. However note that for $\lambda \in \mathbb{D}$ we have $\sigma_i(\lambda) = \overline{\lambda}$ so that $\sigma_i(\lambda)v_j$ may be different from λv_j. To avoid confusion we shall write, for $\lambda \in \mathbb{D}_i$

$$\lambda v_i = \lambda * v_i \quad \text{and} \quad \sigma_i(\lambda)v_j = \lambda * v_j.$$

As in the first case choose a non-zero $\mathbb{D}$-linear form ℓ on $\mathbb{D}_i$. For $v_i \in V_i$ and $v_j \in V_j$ the map $\lambda \mapsto \langle \lambda * v_i, v_j\rangle$ is a $\mathbb{D}$-linear form on $\mathbb{D}_i$; hence there exists a unique element $S(v_i, v_j) \in \mathbb{D}_i$ such that, for all λ

$$\langle \lambda * v_i, v_j\rangle = \ell(\lambda S(v_i, v_j)).$$

The form S is $\mathbb{D}_i$- bilinear and non-degenerate so that we can view V_j as the dual space over $\mathbb{D}_i$ of the $\mathbb{D}_i$ vector space V_i.

Let $(x_i, x_j) \in \mathrm{End}_{\mathbb{D}}(V_i) \times \mathrm{End}_{\mathbb{D}}(V_j)$. They commute with (a_i, a_j) if and only if they are $\mathbb{D}_i$-linear. The original hermitian form will be preserved, if and only if $S(x_i v_i, x_j v_j) = S(v_i, v_j)$ for all v_i, v_j. This means that x_j is the inverse of the transpose of x_i. In this situation we define G_i as the linear group of the $\mathbb{D}_i$-vector space V_i.

Let u_i be a $\mathbb{D}_i$-linear bijection of V_i onto V_j. Then $u_i(av_i) = a^{-1}u_i(v_i)$ and $u_i^{-1}(av_j) = a^{-1}u_i^{-1}(v_j)$.

Recall that G_a is the centralizer of a in G. Then (G_a, V) decomposes as a "product", each "factor" being either of type (G_i, V_i) with G_i a unitary group (case 1) or $(G_i, V_i \times V_j)$ with G_i a general linear group (case 2). Gluing together the u_i (case 1) and the (u_i, u_i^{-1}) (case 2) we get an element $u \in \widetilde{G} \backslash G$ such that $ua^{-1}u^{-1} = a$ which means that it belongs to the centralizer of a in $\widetilde{G}$. Finally if $\widetilde{G}_a$ is the centralizer of a in $\widetilde{G}$ then $(\widetilde{G}_a, V)$ is imbedded into a product each "factor" being either of type $(\widetilde{G}_i, V_i)$ with G_i a unitary group (case 1) or $(\widetilde{G}_i, V_i \times V_j)$ with G_i a general linear group (case 2).

If a is not central then for each i the dimension of V_i is strictly smaller than n and from the result for the general linear group and the inductive assumption in the orthogonal or unitary case we conclude that $\mathcal{S}'(G_a \times V)^{\widetilde{G}_a, \chi} = (0)$.

Proof of Lemma 5.1 in the group case. Consider the map $g \mapsto P_g$ where P_g is the characteristic polynomial of g. It is a continuous map from G into the set of polynomials of degree at most n. Each non-empty fiber $\mathcal{F}$ is stable under G but also under $\widetilde{G} \setminus G$. Bernstein's localization principle tells us that it is enough to prove that $\mathcal{S}'(\mathcal{F} \times V)^{\widetilde{G}, \chi} = (0)$.

Now it follows from [SS70, chapter IV] that $\mathcal{F}$ contains only a finite number of semi-simple orbits; in particular the set of semi-simple elements $\mathcal{F}_s$ in $\mathcal{F}$ is closed. Let us use the multiplicative Jordan decomposition into a product of a semi-simple and a unipotent element. Consider the map θ from $\mathcal{F} \times V$ onto $\mathcal{F}_s$ which associates to (g, v) the semi-simple part g_s of g. This map is continuous (see the corresponding proof for GL) and commutes with the action of $\widetilde{G}$. In $\mathcal{F}_s$ each orbit γ is both open and closed therefore $\theta^{-1}(\gamma)$ is open and closed and invariant under $\widetilde{G}$. It is enough to prove that for each such orbit $\mathcal{S}'(\theta^{-1}(\gamma))^{\widetilde{G},\chi} = (0)$. By Frobenius descent (Theorem 2.2), if $a \in \gamma$ and is not central, this follows from the above considerations on the centralizer of such an a and the fact that $\theta^{-1}(a)$ is a closed subset of the centralizer of a in $\widetilde{G}$, the product of the set of unipotent element commuting with a by V. Now g_s is central if and only if g belongs to $Z\mathcal{U}$, hence the Lemma. For the Lie algebra the proof is similar, using the additive Jordan decomposition. □

Going back to the group if a is central we see that it suffices to prove that $\mathcal{S}'(\mathcal{U} \times V)^{\widetilde{G},\chi} = (0)$ and similarly for the Lie algebra it is enough to prove that $\mathcal{S}'(\mathcal{N} \times V)^{\widetilde{G},\chi} = (0)$.

Now the exponential map (or the Cayley transform) is a homeomorphism of $\mathcal{N}$ onto $\mathcal{U}$ commuting with the action of $\widetilde{G}$. Therefore it is enough to consider the Lie algebra case.

We now turn our attention to V. Let

$$\Gamma = \{v \in V | \langle v, v\rangle = 0\}$$

Proposition 5.2. *If $T \in \mathcal{S}'(\mathcal{N} \times V)^{\widetilde{G},\chi}$ then the support of T is contained in $\mathcal{N} \times \Gamma$.*

Proof. Let

$$\Gamma_t = \{v \in V \mid \langle v, v\rangle = t\}$$

Each Γ_t is stable by $\widetilde{G}$, hence, by Bernstein's localization principle, to prove that the support of T is contained in $\mathcal{N} \times \Gamma_0$ it is enough to prove that, for $t \neq 0$, $\mathcal{S}'(\mathcal{N} \times \Gamma_t)^{\widetilde{G},\chi} = (0)$.

By Witt's theorem the group G acts transitively on Γ_t. We can apply Frobenius descent to the projection from $\mathcal{N} \times \Gamma_t$ onto Γ_t. Fix a point $v_0 \in \Gamma_t$. The fiber is $\mathcal{N} \times \{v_0\}$. Let $\widetilde{G}_1$ be the centralizer of v_0 in $\widetilde{G}$. We have to show that $\mathcal{S}'(\mathcal{N})^{\widetilde{G}_1,\chi} = (0)$ and it is enough to prove that $\mathcal{S}'(\mathfrak{g})^{\widetilde{G}_1,\chi} = (0)$.

The vector v_0 is not isotropic so we have an orthogonal decomposition

$$V = \mathbb{D}v_0 \oplus V_1$$

with V_1 orthogonal to v_0. The restriction of the hermitian form to V_1 is non-degenerate and G_1 is identified with the unitary group of this restriction, and $\widetilde{G}_1$ is the expected semi-direct product with S_2. As a $\widetilde{G}_1$-module the Lie algebra $\mathfrak{g}$ is isomorphic to a direct sum

$$\mathfrak{g} \approx \mathfrak{g}_1 \oplus V_1 \oplus W$$

where $\mathfrak{g}_1$ is the Lie algebra of G_1 and W a vector space over $\mathbb{F}$ of dimension 0 or 1 and on which the action of $\widetilde{G}_1$ is trivial. The action on $\mathfrak{g}_1 \oplus V_1$ is the usual one so that, by induction, we know that $\mathcal{S}'(\mathfrak{g}_1 \oplus V_1)^{\widetilde{G}_1,\chi} = (0)$. This readily implies that $\mathcal{S}'(\mathfrak{g})^{\widetilde{G}_1,\chi} = (0)$. □

Summarizing: it remains to prove that $\mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi} = (0)$.

6. End of the proof in the orthogonal and unitary cases

We keep our general notation. We have to show that a distribution on $\mathcal{N} \times \Gamma$ which is invariant under G is invariant under $\widetilde{G}$. To some extent the proof will be similar to the one we gave for the general linear group.

In particular we will use the fact that if T is such a distribution then its partial Fourier transform on V is also invariant under G. The Fourier transform on V is defined using the bilinear form

$$(v_1, v_2) \mapsto \langle v_1, v_2\rangle + \langle v_2, v_1\rangle$$

which is invariant under $\widetilde{G}$.

For $v \in V$ put

$$\varphi_v(x) = \langle x, v\rangle v, \quad x \in V.$$

It is a rank one endomorphism of V and $\langle \varphi_v(x), y\rangle = \langle x, \varphi_v(y)\rangle$.

Lemma 6.1.

(1) *In the unitary case, for $\lambda \in \mathbb{D}$ such that $\lambda = -\overline{\lambda}$ the map*

$$\nu_\lambda : \quad (X, v) \mapsto (X + \lambda\varphi_v, v)$$

is a homeomorphism of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ onto itself which commutes with $\widetilde{G}$.

(2) *In the orthogonal case, for $\lambda \in \mathbb{F}$ the map*

$$\mu_\lambda : \quad (X, v) \mapsto (X + \lambda X\varphi_v + \lambda\varphi_v X, v)$$

is a homeomorphism of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ onto itself which commutes with $\widetilde{G}$.

The proof is a trivial verification.

We now use the stratification of $\mathcal{N}$. Let us first check that a G-orbit is stable by $\widetilde{G}$. [1]

Choose a basis $e_1, \ldots, e_n$ of V such that $\langle e_i, e_j\rangle \in \mathbb{F}$; this gives a conjugation $u : v = \sum x_i e_i \mapsto \overline{v} = \sum \overline{x_i} e_i$ on V. If A is any endomorphism of V then $\overline{A}$ is the endomorphism $v \mapsto \overline{A(\overline{v})}$. The conjugation u is an element of $\widetilde{G} \setminus G$ and, as such, it acts on $\mathfrak{g} \times V$ by $(X, v) \mapsto (-uXu^{-1}, -u(v)) = (-\overline{X}, -\overline{v})$. In [MVW87, Chapter 4, Proposition 1-2] it is shown that for $X \in \mathfrak{g}$ there exists an $\mathbb{F}$-linear automorphism a of V such that $\langle a(x), a(y)\rangle = \overline{\langle x, y\rangle}$ (this implies that $a(\lambda x) = \overline{\lambda} x$) and such that $aXa^{-1} = -X$. Then $g = ua \in G$ and $gXg^{-1} = -\overline{X}$ so that $-\overline{X}$ belongs to the G-orbit of X. Note that $a \in \widetilde{G} \setminus G$ and as such acts as $a(X, v) = (X, -a(v))$; it is an element of the centralizer of X in $\widetilde{G} \setminus G$.

Let $\mathcal{N}_i$ be the union of all nilpotent orbits of dimension at most i. We shall prove, by descending induction on i, that the support of a distribution $T \in \mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ must be contained in $\mathcal{N}_i \times \Gamma$.

---

[1] In fact, we only need this for nilpotent orbits and this will be done later in an explicit way, using the canonical form of nilpotent matrices.

So now assume that $i \geq 0$ and that we already know that the support of any $T \in \mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ must be contained in $\mathcal{N}_i \times \Gamma$. Let $\mathcal{O}$ be a nilpotent orbit of dimension i; we have to show that the restriction of T to $\mathcal{O}$ is 0.

In the unitary case fix $\lambda \in \mathbb{D}$ such that $\lambda = -\overline{\lambda}$ and consider, for every $t \in \mathbb{F}$ the homeomorphism $\nu_{t\lambda}$; the image of T belongs to $\mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ so that the image of the support of T must be contained in $\mathcal{N}_i \times \Gamma$. If (X, v) belongs to this support this means that $X + t\lambda\varphi_v \in \mathcal{N}_i$.

If $i = 0$ so that $\mathcal{N}_i = \{0\}$ this implies that $v = 0$ so that T must be a multiple of the Dirac measure at the point $(0, 0)$ and hence is invariant under $\widetilde{G}$ so must be 0.

If $i > 0$ and $X \in \mathcal{O}$ then as $\mathcal{O}$ is open in $\mathcal{N}_i$, we get that, at least for $|t|$ small enough, $X + t\lambda\varphi_v \in \mathcal{O}$ and therefore $\lambda\varphi_v$ belongs to the tangent space Im ad(X) of $\mathcal{O}$ at the point X. Define

$$Q(X) = \{v \in V | \varphi_v \in \text{Im ad}(X)\}, \quad X \in \mathcal{N}, \text{ (unitary case).}$$

Then we know that the support of the restriction of T to $\mathcal{O}$ is contained in

$$\{(X, v) | X \in \mathcal{O}, v \in Q(X)\}$$

and the same is true for the partial Fourier transform of T on V.

In the orthogonal case for $i = 0$, the distribution T is the product of the Dirac measure at the origin of $\mathfrak{g}$ by a distribution T' on V. The distribution T' is invariant under G but the image of $\widetilde{G}$ in End(V) is the same as the image of G so that T' is invariant under $\widetilde{G}$ hence must be 0.

If $i > 0$ we proceed as in the unitary case, using μ_λ. We define

$$Q(X) = \{v \in V | X\varphi_v + \varphi_v X \in \text{Im ad}(X)\}, \quad X \in \mathcal{N}, \text{ (orthogonal case)}$$

and we have the same conclusion.

In both cases, for $i > 0$, fix $X \in \mathcal{O}$. We use Frobenius descent for the projection map $(Y, v) \mapsto Y$ of $\mathcal{O} \times V$ onto $\mathcal{O}$. Let C (resp. $\widetilde{C}$) be the stabilizer of X in G (resp. $\widetilde{G}$). We have a linear bijection of $\mathcal{S}'(\mathcal{O} \times \Gamma)^{\widetilde{G},\chi}$ onto $\mathcal{S}'(V)^{\widetilde{C},\chi}$.

Lemma 6.2. *Let $T \in \mathcal{S}'(V)^{\widetilde{C},\chi}$. If T and its Fourier transform are supported in $Q(X)$ then $T = 0$.*

Let us say that a nilpotent element X is nice if the above Lemma is true.

Suppose that we have a direct sum decomposition $V = V_1 \oplus V_2$ such that V_1 and V_2 are orthogonal. By restriction we get non-degenerate hermitian forms $\langle ., . \rangle_i$ on V_i. We call G_i the unitary group of $\langle ., . \rangle_i$, $\mathfrak{g}_i$ its Lie algebra and so on. Suppose that $X(V_i) \subseteq V_i$ so that $X_i = X_{|V_i}$ is a nilpotent element of $\mathfrak{g}_i$.

Lemma 6.3. *If X_1 and X_2 are nice so is X.*

Proof. We claim that $Q(X) \subseteq Q(X_1) \times Q(X_2)$. Indeed if

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \in \mathfrak{g}$$

then from

$$\langle A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle + \langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = 0$$

we get in particular

$$\langle A_{i,i} x_i, y_i \rangle + \langle x_i, A_{i,i} y_i \rangle = 0$$

so that $A_{i,i} \in \mathfrak{g}_i$. Note that

$$[X, A] = \begin{pmatrix} [X_1, A_{1,1}] & * \\ * & [X_2, A_{2,2}] \end{pmatrix}.$$

If $v_i \in V_i$ and $v_j \in V_j$ we define $\varphi_{v_i,v_j} : V_i \mapsto V_j$ by $\varphi_{v_i,v_j}(x_i) = \langle x_i, v_i \rangle v_j$. Then, for $v = v_1 + v_2$

$$\varphi_v = \begin{pmatrix} \varphi_{v_1,v_1} & \varphi_{v_2,v_1} \\ \varphi_{v_1,v_2} & \varphi_{v_1,v_2} \end{pmatrix}.$$

Therefore if, for $A \in \mathfrak{g}$ we have $\varphi_v = [X, A]$ then $\varphi_{v_i,v_i} = [X_i, A_{i,i}]$. This proves the assertion for the unitary case. The orthogonal case is similar.

The end of the proof is the same as the end of the proof of Lemma 4.2. □

Now in both orthogonal and unitary cases nilpotent elements have normal forms which are orthogonal direct sums of "simple" nilpotent matrices. This is precisely described in [SS70] IV 2-19 page 259. By the above Lemma it is enough to prove that each "simple" matrix is nice.

Unitary case. There is only one type to consider. There exists a basis $e_1, \dots, e_n$ of V such that $Xe_1 = 0$ and $Xe_i = e_{i-1}$, $i \geq 2$. The hermitian form is given by

$$\langle e_i, e_j \rangle = 0 \text{ if } i + j \neq n + 1, \quad \langle e_i, e_{n+1-i} \rangle = (-1)^{n-i} \alpha$$

with $\alpha \neq 0$. Note that $\overline{\alpha} = (-1)^{n-1}\alpha$. Suppose that $v \in Q(X)$; for some $A \in \mathfrak{g}$ we have $\lambda \varphi_v = XA - AX$. For any integer $p \geq 0$

$$\mathrm{Tr}(\lambda \varphi_v X^p) = \mathrm{Tr}(XAX^p - AX^{p+1}) = 0.$$

Now $\mathrm{Tr}(\varphi_v X^p) = \langle X^p v, v \rangle$ Let $v = \sum x_i e_i$. Hence

$$\langle X^p v, v \rangle = \sum_1^{n-p} x_{i+p} \langle e_i, v \rangle = \sum_1^{n-p} (-1)^{n-i} \alpha x_{i+p} \overline{x}_{n+1-i} = 0.$$

For $p = n - 1$ this gives $x_n \overline{x}_n = 0$. For $p = n - 2$ we get nothing new but for $p = n - 3$ we obtain $x_{n-1} = 0$. Going on, by an easy induction, we conclude that $x_i = 0$ if $i \geq (n+1)/2$.

If $n = 2p + 1$ is odd put $V_1 = \oplus_1^p \mathbb{D} e_i$, $V_0 = \mathbb{D} e_{p+1}$ and $V_2 = \oplus_{p+2}^{2p+1} \mathbb{D} e_i$. If $n = 2p$ is even put $V_1 = \oplus_1^p \mathbb{D} e_i$, $V_0 = (0)$ and $V_2 = \oplus_{p+1}^{2p} \mathbb{D} e_i$. In both cases we have $V = V_1 \oplus V_0 \oplus V_2$. We use the notation $v = v_2 + v_0 + v_1$

The distribution T is supported by V_1. Call δ_i the Dirac measure at 0 on V_i. Then we may write $T = U \otimes \delta_0 \otimes \delta_2$ with $U \in \mathcal{S}'(V_1)$. The same thing must be true of the Fourier transform of T. Note that $\widehat{U}$ is a distribution on V_2, that $\widehat{\delta_2}$ is a Haar measure dv_1 on V_1 and that, for n odd $\widehat{\delta_0}$ is a Haar measure dv_0 on V_0. So we have $\widehat{T} = dv_1 \otimes \widehat{U}$ if n is even and $\widehat{T} = dv_1 \otimes dv_0 \otimes \widehat{U}$ if n is odd. In the odd case this forces $T = 0$. In the even case, up to a scalar multiple the only possibility is $T = dv_1 \otimes \delta_2$.

Let

$$a : \sum x_i e_i \mapsto \sum (-1)^i \overline{x}_i e_i.$$

Then $a \in \widetilde{G} \setminus G$. It acts on $\mathfrak{g}$ by $Y \mapsto -aYa^{-1}$ and in particular $-aXa^{-1} = X$ so that $a \in \widetilde{C} \setminus C$. The action on V is given by $v \mapsto -a(v)$. It is an involution. The subspace V_1 is invariant and so dv_1 is invariant. This implies that T is invariant under $\widetilde{C}$ so it must be 0.

Orthogonal case. There are two different types of "simple" nilpotent matrices.

The first type is the same as the unitary case, with $\alpha = 1$ and thus n odd but now our condition is that $X\varphi_v + \varphi_v X = [X, A]$ for some $A \in \mathfrak{g}$. As before this implies that $\mathrm{Tr}(\varphi_v X^q) = 0$ but only for $q \geq 1$. Put $n = 2p+1$; we get $x_j = 0$ for $j > p+1$. Decompose V as before: $V = V_1 \oplus V_0 \oplus V_2$. Our distribution T is supported by the subspace $v_2 = 0$ so we write it $T = U \otimes \delta_2$ with $U \in \mathcal{S}'(V_1 \oplus V_0)$. This is also true for the distribution $\widehat{T}$ so we must have $U = dv_1 \otimes R$ with R a distribution on V_0. Finally $T = dv_1 \otimes R \otimes \delta_2$. Now $-\mathrm{Id} \in C$ and T is invariant under C so that R must be an even distribution. On the other end the endomorphism a of V defined by $a(e_i) = (-1)^{i-p-1} e_i$ belongs to C and $aXa^{-1} = -X$ and $u : (X, v) \mapsto (-X, -v)$ belongs to $\widetilde{G} \setminus G$. The product $a \times u$ of a and u in $\widetilde{G}$ belongs to $\widetilde{C} \setminus C$. Clearly T is invariant under $a \times u$ so that T is invariant under $\widetilde{C}$ so it must be 0.

The second type is as follows. We have $n = 2m$, an even integer and a decomposition $V = E \oplus F$ with both E and F of dimension m. We have a basis $e_1, \ldots, e_m$ of E and a basis $f_1, \ldots, f_m$ of F such that

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$$

and

$$\langle e_i, f_j \rangle = 0 \,\text{if}\, i + j \neq m+1 \quad \text{and} \quad \langle e_i, f_{m+1-i} \rangle = (-1)^{m-i}.$$

Finally X is such that $Xe_i = e_{i-1},\ Xf_i = f_{i-1}$.

Let ξ be the matrix of the restriction of X to E or to F. Write an element $A \in \mathfrak{g}$ as 2×2 matrix $A = (a_{i,j})$. Then

$$[X, A] = \begin{pmatrix} [\xi, a_{1,1}] & [\xi, a_{1,2}] \\ [\xi, a_{2,1}] & [\xi, a_{2,2}] \end{pmatrix}.$$

Suppose that $v \in Q(X)$ and let

$$v = e + f \quad \text{with} \quad e = \sum x_i e_i, \;\; f = \sum y_i f_i.$$

We get

$$X\varphi_v + \varphi_v X = \begin{pmatrix} \xi\varphi_{f,e} + \varphi_{f,e}\xi & \xi\varphi_{f,f} + \varphi_{f,f}\xi \\ \xi\varphi_{e,e} + \varphi_{e,e}\xi & \xi\varphi_{e,f} + \varphi_{e,f}\xi \end{pmatrix}$$

where, for example $\varphi_{e,e}$ is the map $f' \mapsto \langle f', e \rangle e$ from F into E. Thus, for some A,

$$\xi\varphi_{e,e} + \varphi_{e,e}\xi = \xi a_{2,1} - a_{2,1}\xi$$

In this formula, using the basis (e_i), (f_i) replace all the maps by their matrices.

Then, as before, we have $\mathrm{Tr}(\varphi_{e,e}\xi^q) = 0$ for $1 \leq q \leq m-1$. If $e' = \sum x_i f_i$ (the x_i are the coordinates of e), then $\mathrm{Tr}(\xi^q \varphi_{e,e})$ is $\langle \xi^q e, e' \rangle$. Thus, as in the other cases, we have $x_j = 0$ for $j > m/2$ if m is even and $j > (m+1)/2$ if m is odd. The same thing is true for the y_i.

If $m = 2p$ is even, let $V_1 = \oplus_{i \leq p}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$ and $V_2 = \oplus_{i>p}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$; write $v = v_1 + v_2$ the corresponding decomposition of an arbitrary element of V. Let δ_2 be the Dirac measure at the origin in V_2 and dv_1 a Haar measure on V_1. Then, as in the unitary case, using the Fourier transform, we see that the distribution T must be a multiple of $dv_1 \otimes \delta_2$.

The endomorphism a of V defined by $a(e_i) = (-1)^i e_i$ and $a(f_i) = (-1)^{i+1} f_i$ belongs to G and $aXa^{-1} = -X$. The map $u : (Y, v) \mapsto (-Y, -v)$ belongs to $\widetilde{G} \setminus G$ so that the product $a \times u$ in $\widetilde{G}$ belongs to $\widetilde{C} \setminus C$. It clearly leaves T invariant so that $T = 0$.

Finally if $m = 2p+1$ is odd we put $V_1 = \oplus_{i \leq p}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$, $V_0 = \mathbb{F}e_{p+1} \oplus \mathbb{F}f_{p+1}$, $V_2 = \oplus_{i \geq p+2}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$. As in the unitary case we find that $T = dv_1 \otimes R \otimes \delta_2$ with R a distribution on V_0. As $-\mathrm{Id} \in C$ we see that R must be even. Then again, define $a \in G$ by $a(e_i) = (-1)^i e_i$ and $a(f_i) = (-1)^i f_i$ and consider $a \times u$ with $u(Y, v) = (-Y, -v)$. As before $a \times u \in \widetilde{C} \setminus C$ and leaves T invariant so we have to take $T = 0$. □

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Multiplicity one theorem for $(\mathrm{GL}_{n+1}(\mathbb{R}), \mathrm{GL}_n(\mathbb{R}))$

Avraham Aizenbud and Dmitry Gourevitch

Abstract. Let F be either $\mathbb{R}$ or $\mathbb{C}$. Consider the standard embedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$ and the action of $\mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ by conjugation.

In this paper we show that any $\mathrm{GL}_n(F)$-invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.

We show that this implies that for any irreducible admissible smooth Fréchet representations π of $\mathrm{GL}_{n+1}(F)$ and τ of $\mathrm{GL}_n(F)$,

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1.$$

For p-adic fields those results were proven in [AGRS].

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Contents

1. Introduction

Let F be an archimedean local field, i.e. $F = \mathbb{R}$ or $F = \mathbb{C}$. Consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the action of $\mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ by conjugation. In this paper we prove the following theorem:

Theorem A. *Any $\mathrm{GL}_n(F)$ - invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.*

It has the following corollary in representation theory.

Theorem B. *Let π be an irreducible admissible smooth Fréchet representation of $\mathrm{GL}_{n+1}(F)$ and τ be an irreducible admissible smooth Fréchet representation of $\mathrm{GL}_n(F)$. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1. \tag{1}$$

We deduce Theorem B from Theorem A using an argument due to Gelfand and Kazhdan adapted to the archimedean case in [AGS].

Property (1) is sometimes called *strong Gelfand property* of the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$. It is equivalent to the fact that the pair $(\mathrm{GL}_{n+1}(F) \times \mathrm{GL}_n(F), \Delta\mathrm{GL}_n(F))$ is a *Gelfand pair*.

Remark C. *Using the tools developed here, combined with* [AGRS]*, one can easily show that Theorem A implies an analogous theorem for the unitary groups.*

Remark. *After the completion of this work we found out that Chen-Bo Zhu and Sun Binyong have obtained the same results simultaneously, independently and in a different way, see* [SZ].

They also proved an analogous theorem for the orthogonal groups.

1.1. Some related results

For non-archimedean local fields of characteristic zero Theorems A and B were proven in [AGRS]. The proof in [AGRS] does not work in the archimedean case because of the presence of transversal derivatives. For this reason we need to use a new ingredient - the theory of D-modules and in particular the Integrability Theorem (see Theorem 2.3.6 below).

We hope that this method will be very useful in the future. It already has been used in subsequent works [AS08, Say09, Aiz08].

The proof given here cannot be literally repeated to get a new proof in the non-Archimedean case since the theory of D-modules is not available there. However one can develop a non-Archimedean analog of the tools that we gain from the theory of D-modules and obtain a proof that works uniformly in both cases. This is done in the subsequent work [Aiz08].

In [AGS], a special case of Theorem B was proven for all local fields; namely the case when τ is one-dimensional.

Theorem A easily implies the following corollary.

Corollary D. *Let* $P_n \subset \mathrm{GL}_n$ *be the subgroup consisting of all matrices whose last row is* $(0, ..., 0, 1)$. *Let* GL_n *act on itself by conjugation. Then every* $P_n(F)$ *- invariant distribution on* $\mathrm{GL}_n(F)$ *is* $\mathrm{GL}_n(F)$ *- invariant.*

This theorem has been proven in [Bar] for eigendistributions with respect to the center of $U_{\mathbb{C}}(\mathrm{gl}_n)$. In [Bar] it is also shown that this implies Kirillov's conjecture.

1.2. Structure of the proof

We will now briefly sketch the main ingredients of our proof of Theorem A.

First we show that we can switch to the following problem. The group $\mathrm{GL}_n(F)$ acts on a certain linear space X_n and σ is an involution of X_n. We have to prove that every $\mathrm{GL}_n(F)$-invariant distribution on X_n is also σ-invariant. We do that by induction on n. Using the Harish-Chandra descent method we show that the induction hypothesis implies that this holds for distributions on the complement to a certain small closed subset $S \subset X_n$. We call this set *the singular set.* This is done in section 3.

Next we assume the contrary: there exists a non-zero $\mathrm{GL}_n(F)$-invariant distribution ξ on X which is anti-invariant with respect to σ.

We use the notion of singular support of a distribution from the theory of D-modules. Let $T \subset T^*X$ denote the singular support of ξ. Using Fourier transform and the fact any such distribution is supported in S we obtain that T is contained in $\check{S}$ where $\check{S}$ is a certain small subset in T^*X. This is done in section 4.

Then we use a deep result from the theory of D-modules which states that the singular support of a distribution is a coisotropic variety in the cotangent bundle.

This enables us to show, using a complicated but purely geometric argument, that the support of ξ is contained in a much smaller subset of S. This is done in section 5.

Finally it remains to prove that any $\mathrm{GL}_n(F)$-invariant distribution that is supported on this subset together with its Fourier transform is zero. This is proven in subsection 4.1 using Homogeneity Theorem (Theorem 2.2.13) which in turn uses Weil representation.

1.3. Content of the paper

In section 2 we give the necessary preliminaries for the paper.

In subsection 2.1 we fix the general notation that we will use.

In subsection 2.2 we discuss invariant distributions and introduce some tools to work with them. The most advanced are

- The Homogeneity theorem and Fourier transform.
- The Harish-Chandra descent method.

In subsection 2.3 we discuss the notion of singular support of a distribution. The most important for us property of this singular support is being coisotropic. This fact is a crucial tool of this paper.

In subsection 2.4 we introduce notation that we will use in our proof.

In section 3 we use the Harish-Chandra descent method.

In subsection 3.1 we linearize the problem to a problem on the linear space $X = \mathrm{sl}(V) \times V \times V^*$, where $V = F^n$.

In subsection 3.2 we perform the Harish-Chandra descent on the $\mathrm{sl}(V)$-coordinate and $V \times V^*$ coordinate separately and then use automorphisms ν_λ of X to descend further to the singular set S.

In section 4 we reduce Theorem A to the following geometric statement: any coisotropic subvariety of $\check{S}$ is contained in a certain set $\check{C}_{X \times X}$. The reduction is done using the fact that the singular support of a distribution has to be coisotropic, and the following proposition: any $\mathrm{GL}(V)$-invariant distribution on X such that it and its Fourier transform are supported on $\mathrm{sl}(V) \times (V \times 0 \cup 0 \times V^*)$ is zero.

In subsection 4.1 we prove this proposition using Homogeneity theorem.

In section 5 we prove the geometric statement. Technically this is the most complicated part of the paper. However we would like to note that it is purely algebro-geometric statement that involves no analysis.

In subsection 5.1 we give preliminaries on coisotropic subvarieties. In particular, we give a geometric partial analog of Frobenius reciprocity for coisotropic subvarieties (Corollaries 5.1.7 and 5.1.8).

In subsection 5.2 we stratify the set $\check{S}$ and use an inductive argument on the strata. This reduces the geometric statement to a proposition on one stratum that we call *the Key Proposition*.

In subsection 5.3 we analyze a stratum of $\check{S}$ and then use the geometric analog of Frobenius reciprocity to reduce the Key Proposition to a lemma on $V \times V^* \times V \times V^*$ that we call *the Key Lemma*.

In subsection 5.4 we prove the Key Lemma.

In Appendix A we prove that Theorem A implies Theorem B using an archimedean analog of Gelfand-Kazhdan technique.

In Appendix B we give more details on the facts concerning the theory of D-modules listed in subsection 2.3.

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2. Preliminaries

2.1. General notation

- In this paper all the algebraic varieties are defined over F.
- For an algebraic variety X we denote by $X(F)$ the topological space or smooth manifold of F points of X.
- We consider linear spaces as algebraic varieties and treat them in the same way.
- For an algebraic variety X defined over $\mathbb{R}$ we denote by $X_{\mathbb{C}}$ the natural algebraic variety defined over $\mathbb{R}$ such that $X_{\mathbb{C}}(\mathbb{R}) = X(\mathbb{C})$. Note that over $\mathbb{C}$, $X_{\mathbb{C}}$ is isomorphic to $X \times X$.
- For a group G acting on a set X and a point $x \in X$ we denote by Gx or by $G(x)$ the orbit of x and by G_x the stabilizer of x.
- An action of a Lie algebra $\mathfrak{g}$ on a (smooth, algebraic, etc) manifold M is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Note that an action of a (Lie, algebraic, etc) group on M defines an action of its Lie algebra on M.
- For a Lie algebra $\mathfrak{g}$ acting on M, an element $\alpha \in \mathfrak{g}$ and a point $x \in M$ we denote by $\alpha(x) \in T_xM$ the value at point x of the vector field corresponding to α. We denote by $\mathfrak{g}x \subset T_xM$ or by $\mathfrak{g}(x)$ the image of the map $\alpha \mapsto \alpha(x)$ and by $\mathfrak{g}_x \subset \mathfrak{g}$ its kernel.
- For manifolds $L \subset M$ we denote by $N_L^M := (T_M|_L)/T_L$ the normal bundle to L in M.
- Denote by $CN_L^M := (N_L^M)^*$ the conormal bundle.

- For a point $y \in L$ we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space.

2.2. Invariant distributions

2.2.1. Distributions on smooth manifolds.

Notation 2.2.1. *Let X be a smooth manifold. Denote by $C_c^\infty(X)$ the space of test functions on X, that is smooth compactly supported functions, with the standard topology, i.e. the topology of inductive limit of Fréchet spaces.*

Denote $\mathcal{D}(X) := C_c^\infty(X)^*$ to be the dual space to $C_c^\infty(X)$.

For any vector bundle E over X we denote by $C_c^\infty(X, E)$ the space of smooth compactly supported sections of E and by $\mathcal{D}(X, E)$ its dual space. Also, for any finite dimensional real vector space V we denote by $C_c^\infty(X, V)$ the space of smooth compactly supported sections of the trivial bundle with fiber V and by $\mathcal{D}(X, V)$ its dual space.

2.2.2. Schwartz distributions on Nash manifolds.

Our proof of Theorem A widely uses Fourier transform which cannot be applied to general distributions. For this we require a theory of Schwartz functions and distributions as developed in [AG1].

This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic.*

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

Notation 2.2.2. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.*

Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the space of Schwartz distributions on X.

For any Nash vector bundle E over X we denote by $\mathcal{S}(X, E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X, E)$ its dual space.

Notation 2.2.3. *Let X be a smooth manifold and let $Z \subset X$ be a closed subset. We denote $\mathcal{S}_X^*(Z) := \{\xi \in \mathcal{S}^*(X) | \mathrm{Supp}(\xi) \subset Z\}$.*

For a locally closed subset $Y \subset X$ we denote $\mathcal{S}_X^*(Y) := \mathcal{S}_{X \setminus (\overline{Y} \setminus Y)}^*(Y)$. In the same way, for any bundle E on X we define $\mathcal{S}_X^*(Y, E)$.

Remark 2.2.4. *Schwartz distributions have the following two advantages over general distributions:*
(i) For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence

$$0 \to \mathcal{S}_X^*(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

(ii) Fourier transform defines an isomorphism $\mathcal{F} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.

The following theorem allows us to switch between general distributions and Schwartz distributions.

Theorem 2.2.5. *Let a reductive group G act on a smooth affine variety X. Let V be a finite dimensional continuous representation of $G(F)$ over $\mathbb{R}$. Suppose that $\mathcal{S}^*(X(F), V)^{G(F)} = 0$. Then $\mathcal{D}(X(F), V)^{G(F)} = 0$.*

For proof see [AG2], Theorem 4.0.2.

2.2.3. Basic tools.
We present here some basic tools on equivariant distributions that we will use in this paper.

Proposition 2.2.6. *Let a Nash group G act on a Nash manifold X. Let $Z \subset X$ be a closed subset.*

Let $Z = \bigcup_{i=0}^{l} Z_i$ be a Nash G-invariant stratification of Z. Let χ be a character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i, Sym^k(CN_{Z_i}^X))^{G,\chi} = 0$. Then $\mathcal{S}_X^*(Z)^{G,\chi} = 0$.

For proof see section B.2 in [AGS].

Proposition 2.2.7. *Let G_i be Nash groups acting on Nash manifolds X_i for $i = 1 \ldots n$. Let $E_i \to X_i$ be G_i-equivariant Nash vector bundles.*
(i) Suppose that $\mathcal{S}^*(X_j, E_j)^{G_j} = 0$ for some $1 \leq j \leq n$. Then

$$\mathcal{S}^*(\prod_{i=1}^{n} X_i, \boxtimes E_i)^{\prod G_i} = 0,$$

where $\boxtimes$ denotes the external product of vector bundles.
(ii) Let $H_i < G_i$ be Nash subgroups. Suppose that $\mathcal{S}^*(X_i, E_i)^{H_i} = \mathcal{S}^*(X_i, E_i)^{G_i}$ for all i. Then

$$\mathcal{S}^*(\prod X_i, \boxtimes E_i)^{\prod H_i} = \mathcal{S}^*(\prod X_i, \boxtimes E_i)^{\prod G_i},$$

The proof is trivial and the same as the proof of Proposition 3.1.5 in [AGS].

Theorem 2.2.8 (Frobenius reciprocity). *Let a unimodular Nash group G act transitively on a Nash manifold Z. Let $\varphi : X \to Z$ be a G-equivariant Nash map. Let $z \in Z$. Suppose that its stabilizer G_z is unimodular. Note that this implies that there exists a G-invariant measure on Z. Fix such a measure. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z,\chi}$.*

Moreover, for any G-equivariant bundle E on X, the space $\mathcal{S}^*(X, E)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z, E|_{X_z})^{G_z,\chi}$.

For proof see [AG2], Theorem 2.5.7.

2.2.4. Fourier transform.
From now till the end of the paper we fix an additive character κ of F given by $\kappa(x) := e^{2\pi i \operatorname{Re}(x)}$.

Notation 2.2.9. *Let V be a vector space over F. Let B be a non-degenerate bilinear form on V. Then B defines Fourier transform with respect to the self-dual Haar measure on V. We denote it by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$.*

For any Nash manifold M we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the fiberwise Fourier transform.

If there is no ambiguity, we will write $\mathcal{F}_V$ instead $\mathcal{F}_B$.

We will use the following trivial observation.

Lemma 2.2.10. *Let V be a finite dimensional vector space over F. Let a Nash group G act linearly on V. Let B be a G-invariant non-degenerate symmetric bilinear form on V. Let M be a Nash manifold with an action of G. Let $\xi \in \mathcal{S}^*(V(F) \times M)$ be a G-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also G-invariant.*

2.2.5. Homogeneity Theorem.

Notation 2.2.11. *Let V be a vector space over F. Consider the homothety action of $F^\times$ on V by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$.*

Also, for any $\lambda \in F$ we denote $|\lambda|_F := |\lambda|^{dim_{\mathbb{R}} F}$.

Notation 2.2.12. *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. We denote*

$$Z(B) := \{x \in V(F) | B(x,x) = 0\}.$$

Theorem 2.2.13 (Homogeneity Theorem). *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a Nash manifold. Let $L \subset \mathcal{S}^*_{V(F)\times M}(Z(B) \times M)$ be a non-zero subspace such that $\forall \xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).*

Then there exist a non-zero distribution $\xi \in L$ and a unitary character u of $F^\times$ such that either $\rho(\lambda)\xi = |\lambda|_F^{\frac{dim V}{2}} u(\lambda)\xi$ for any $\lambda \in F^\times$ or $\rho(\lambda)\xi = |\lambda|_F^{\frac{dim V}{2}+1} u(\lambda)\xi$ for any $\lambda \in F^\times$.

For proof see [AG2], Theorem 5.1.7.

2.2.6. Harish-Chandra descent.

Definition 2.2.14. *Let an algebraic group G act on an algebraic variety X. We say that an element $x \in X(F)$ is G-**semisimple** if its orbit $G(F)x$ is closed.*

Theorem 2.2.15 (Generalized Harish-Chandra descent). *Let a reductive group G act on smooth affine varieties X and Y. Let χ be a character of $G(F)$. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{S}^*((N^X_{Gx,x} \times Y)(F))^{G(F)_x,\chi} = 0.$$

Then $\mathcal{S}^*(X(F) \times Y(F))^{G(F)_x,\chi} = 0$.

For proof see [AG2], Theorem 3.1.6.

2.3. D-modules and singular support

In this paper we will use the algebraic theory of D-modules. We will now summarize the facts that we need and give more details in Appendix B. For a good introduction to the algebraic theory of D-modules we refer the reader to [Ber] and [Bor].

More specifically, we will use the notion of singular support of a distribution. For those who are not familiar with the theory of D-modules, Corollary 2.3.7 and the facts that are listed after it are the only properties of singular support that we use.

In this subsection $F = \mathbb{R}$.

Definition 2.3.1. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Consider the D_X-submodule $\mathcal{M}_\xi$ of $\mathcal{S}^*(X(\mathbb{R}))$ generated by ξ. We define the* **singular support** *of ξ to be the singular support of $\mathcal{M}_\xi$. We denote it by $SS(\xi)$.*

Remark 2.3.2. *A similar, but not equivalent notion is sometimes called in the literature a 'wave front of ξ' .*

Notation 2.3.3. *Let (V, B) be a quadratic space. Let X be a smooth algebraic variety. Consider B as a map $B : V \to V^*$. Identify $T^*(X \times V)$ with $T^*X \times V \times V^*$. We define $F_V : T^*(X \times V) \to T^*(X \times V)$ by $F_V(\alpha, v, \phi) := (\alpha, -B^{-1}\phi, Bv)$.*

Definition 2.3.4. *Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M-***coisotropic** *if one of the following equivalent conditions holds.*
(i) The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.
(ii) At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$. Here, $(T_zZ)^\perp$ denotes the orthogonal complement to (T_zZ) in (T_zM) with respect to ω.
(iii) For a generic smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$.

If there is no ambiguity, we will call Z a coisotropic variety.

Note that every non-empty M-coisotropic variety is of dimension at least $\frac{1}{2} dim M$.

Notation 2.3.5. *For a smooth algebraic variety X we always consider the standard symplectic form on T^*X. Also, we denote by $p_X : T^*X \to X$ the standard projection.*

The following theorem is crucial in this paper.

Theorem 2.3.6. *[Integrability Theorem] Let X be a smooth algebraic variety. Let $\mathcal{M}$ be a finitely generated D_X-module. Then $SS(\mathcal{M})$ is a T^*X-coisotropic variety.*

This is a special case of Theorem I in [Gab]. For similar versions see also [KKS, Mal].

Corollary 2.3.7. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $SS(\xi)$ is coisotropic.*

We will also use the following well-known facts from the theory of D-modules. Let X be a smooth algebraic variety.

Fact 2.3.8. *Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))(\mathbb{R})$, where $\overline{\mathrm{Supp}(\xi)}_{Zar}$ denotes the Zariski closure of* Supp(ξ).

Fact 2.3.9.
Let an algebraic group G act on X. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))^{G(\mathbb{R})}$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g}\, \phi(\alpha(x)) = 0\}.$$

Fact 2.3.10. *Let (V, B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that* Supp(ξ) $\subset Z(\mathbb{R})$. *Then*

$$SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X\times V}^{-1}(Z)).$$

For proofs of those facts see Appendix B.

2.4. Specific notation

The following notations will be used in the body of the paper.

- Let $V := V_n$ be the standard n-dimensional linear space defined over F.
- Let $\mathrm{sl}(V)$ denote the Lie algebra of operators with zero trace.
- Denote $X := X_n := \mathrm{sl}(V_n) \times V_n \times V_n^*$
- $G := G_n := \mathrm{GL}(V_n)$
- $\mathfrak{g} := \mathfrak{g}_n := \mathrm{Lie}(G_n) = \mathrm{gl}(V_n)$
- $\widetilde{G} := \widetilde{G}_n := G_n \rtimes \{1, \sigma\}$, where the action of the 2-element group $\{1, \sigma\}$ on G is given by the involution $g \mapsto {g^t}^{-1}$.
- We define a character χ of $\widetilde{G}$ by $\chi(G) = \{1\}$ and $\chi(\widetilde{G} - G) = \{-1\}$.
- Let G_n act on G_{n+1}, $\mathfrak{g}_{n+1}$ and on $\mathrm{sl}(V_n)$ by $g(A) := gAg^{-1}$.
- Let G act on $V \times V^*$ by $g(v, \phi) := (gv, (g^*)^{-1}\phi)$. This gives rise to an action of G on X.
- Extend the actions of G to actions of $\widetilde{G}$ by $\sigma(A) := A^t$ and $\sigma(v, \phi) := (\phi^t, v^t)$.
- We consider the standard scalar products on $\mathrm{sl}(V)$ and $V \times V^*$. They give rise to a scalar product on X.
- We identify the cotangent bundle T^*X with $X \times X$ using the above scalar product.
- Let $\mathcal{N} := \mathcal{N}_n \subset \mathrm{sl}(V_n)$ denote the cone of nilpotent operators.
- $C := (V \times 0) \cup (0 \times V^*) \subset V \times V^*$.
- $\check{C} := (V \times 0 \times V \times 0) \cup (0 \times V^* \times 0 \times V^*) \subset V \times V^* \times V \times V^*$.
- $\check{C}_{X\times X} := (\mathrm{sl}(V) \times V \times 0 \times \mathrm{sl}(V) \times V \times 0) \cup (\mathrm{sl}(V) \times 0 \times V^* \times \mathrm{sl}(V) \times 0 \times V^*) \subset X \times X$.
- $S := \{(A, v, \phi) \in X_n | A^n = 0 \text{ and } \phi(A^i v) = 0 \text{ for any } 0 \leq i \leq n\}$.

- $$\check{S} := \{((A_1,v_1,\phi_1),(A_2,v_2,\phi_2)) \in X \times X \,|\, \forall i,j \in \{1,2\} \quad (A_i,v_j,\phi_j) \in S \text{ and } \forall \alpha \in \mathrm{gl}(V), \alpha(A_1,v_1,\phi_1) \perp (A_2,v_2,\phi_2)\}$$
.
- Note that
$$\check{S} = \{((A_1,v_1,\phi_1),(A_2,v_2,\phi_2)) \in X \times X \,|\, \forall i,j \in \{1,2\} \quad (A_i,v_j,\phi_j) \in S \text{ and } [A_1,A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$
- $\check{S}' := \check{S} - \check{C}_{X \times X}$.
- $\Gamma := \{(v,\phi) \in V \times V^* \,|\, \phi(V) = 0\}$.
- For any $\lambda \in F$ we define $\nu_\lambda : X \to X$ by $\nu_\lambda(A,v,\phi) := (A + \lambda v \otimes \phi - \lambda \frac{\langle \phi, v \rangle}{n} \mathrm{Id}, v, \phi)$.
- It defines $\check{\nu}_\lambda : X \times X \to X \times X$. It is given by
$$\check{\nu}_\lambda((A_1,v_1,\phi_1),(A_2,v_2,\phi_2)) = = ((A_1 + \lambda v_1 \otimes \phi_1 - \lambda \frac{\langle \phi_1, v_1 \rangle}{n} \mathrm{Id}, v_1, \phi_1), (A_2, v_2 - \lambda A_2 v_1, \phi_2 - \lambda A_2^* \phi_1)).$$

3. Harish-Chandra descent

3.1. Linearization

In this subsection we reduce Theorem A to the following one

Theorem 3.1.1. $\mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi} = 0$.

We will divide this reduction to several propositions.

Proposition 3.1.2. *If* $\mathcal{D}(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then Theorem A holds.*

The proof is straightforward.

Proposition 3.1.3. *If* $\mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{D}(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Follows from Theorem 2.2.5.

Proposition 3.1.4. *If* $\mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Proof. Let $\xi \in \mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \mathrm{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in \mathcal{S}(F)$ be such that f vanishes in a neighborhood of 0 and $f(t) \neq 0$. Consider the determinant map $\det : G_{n+1}(F) \to F$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi}$ and $p \in \mathrm{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F),\chi}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. ☐

Proposition 3.1.5. *If* $\mathcal{S}^*(X_n(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Proof. The $\widetilde{G}_n(F)$-space $\mathrm{gl}_{n+1}(F)$ is isomorphic to $X_n(F)\times F\times F$ with trivial action on $F \times F$. This isomorphism is given by

$$\begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & \lambda \end{pmatrix} \mapsto ((A - \frac{\mathrm{Tr}\, A}{n}\mathrm{Id}, v, \phi), \lambda, \mathrm{Tr}\, A).$$

□

3.2. Harish-Chandra descent

Now we start to prove Theorem 3.1.1. The proof is by induction on n. Till the end of the paper we will assume that Theorem 3.1.1 holds for all $k < n$ for both archimedean local fields.

The theorem obviously holds for $n = 0$. Thus from now on we assume $n \geq 1$. The goal of this subsection is to prove the following theorem.

Proposition 3.2.1. $\mathcal{S}^*(X(F) - S(F))^{\widetilde{G}(F),\chi} = 0$.

In fact, one can prove this theorem directly using Theorem 2.2.15. However, this will require long computations. Thus, we will divide the proof to several steps and use some tricks to avoid part of those computations.

Proposition 3.2.2. $\mathcal{S}^*(X(F) - (\mathcal{N} \times V \times V^*)(F))^{\widetilde{G}(F),\chi} = 0$.

Proof. By Theorem 2.2.15 it is enough to prove that for any semi-simple $A \in \mathrm{sl}(V)$ we have

$$\mathcal{S}^*((N^{\mathrm{sl}(V)}_{GA,A} \times (V \times V^*))(F))^{\widetilde{G}(F)_A,\chi} = 0.$$

Now note that $\widetilde{G}(F)_A \cong \prod \widetilde{G}_{n_i}(F_i)$ where $n_i < n$ and F_i are some field extensions of F. Note also that

$$(N^{\mathrm{sl}(V)}_{GA,A} \times V \times V^*)(F) \cong \mathrm{sl}(V)_A \times (V \times V^*)(F) \cong \prod X_{n_i}(F_i) \times \mathcal{Z}(\mathrm{sl}(V)_A)(F),$$

where $\mathcal{Z}(\mathrm{sl}(V)_A)$ is the center of $\mathrm{sl}(V)_A$. Clearly, $\widetilde{G}_A$ acts trivially on $\mathcal{Z}(\mathrm{sl}(V)_A)$.

Now by Proposition 2.2.7 the induction hypothesis implies that

$$\mathcal{S}^*(\prod X_{n_i}(F_i) \times \mathcal{Z}(\mathrm{sl}(V)_A)(F))^{\prod \widetilde{G}_{n_i}(F_i),\chi} = 0.$$

□

In the same way we obtain the following proposition.

Proposition 3.2.3. $\mathcal{S}^*(X(F) - (\mathrm{sl}(V) \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$.

Corollary 3.2.4. $\mathcal{S}^*(X(F) - (\mathcal{N} \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$.

Lemma 3.2.5. *Let* $A \in \mathrm{sl}(V)$, $v \in V$ *and* $\phi \in V^*$. *Suppose* $A + \lambda v \otimes \phi$ *is nilpotent for all* $\lambda \in F$. *Then* $\phi(A^i v) = 0$ *for any* $i \geq 0$.

Proof. Since $A + \lambda v \otimes \phi$ is nilpotent, we have $tr(A + \lambda v \otimes \phi)^k = 0$ for any $k \geq 0$ and $\lambda \in F$. By induction on i this implies that $\phi(A^i v) = 0$. □

Proof of Theorem 3.2.1. By the previous lemma, $\bigcap_{\lambda \in F} \nu_\lambda(\mathcal{N} \times \Gamma) \subset S$. Hence $\bigcup_{\lambda \in F} \nu_\lambda(X - \mathcal{N} \times \Gamma) \supset X - S$.

By Corollary 3.2.4 $\mathcal{S}^*(X(F) - (\mathcal{N} \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$. Note that ν_λ commutes with the action of $\widetilde{G}(F)$. Thus $\mathcal{S}^*(\nu_\lambda(X(F) - (\mathcal{N} \times \Gamma)(F)))^{\widetilde{G}(F),\chi} = 0$ and hence $\mathcal{S}^*(X(F) - S(F))^{\widetilde{G}(F),\chi} = 0$. □

4. Reduction to the geometric statement

In this section coisotropic variety means $X \times X$-coisotropic variety.

The goal of this section is to reduce Theorem 3.1.1 to the following statement, which is purely geometric and involves no distributions.

Theorem 4.0.1 (geometric statement). *For any coisotropic subvariety of $T \subset \check{S}$ we have $T \subset \check{C}_{X \times X}$.*

Till the end of this section we will assume the geometric statement.

Proposition 4.0.2. *Let $\xi \in \mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi} = 0$. Then* $\mathrm{Supp}(\xi) \subset (\mathrm{sl}(V) \times C)(F)$.

Proof for the case $F = \mathbb{R}$.

Step 1. $SS(\xi) \subset \check{S}$.

We know that

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}^{-1}_{\mathrm{sl}(V)}\xi), \mathrm{Supp}(\mathcal{F}^{-1}_{V \times V^*}(\xi)), \mathrm{Supp}(\mathcal{F}^{-1}_X(\xi)) \subset S(F).$$

By Fact 2.3.10 this implies that

$$SS(\xi) \subset (S \times X) \cap F_{\mathrm{sl}(V)}(S \times X) \cap F_{V \times V^*}(S \times X) \cap F_X(S \times X).$$

On the other hand, ξ is $G(F)$-invariant and hence by Fact 2.3.9

$$SS(\xi) \subset \{((x_1, x_2) \in X \times X \,|\, \forall g \in \mathfrak{g}, g(x_1) \perp x_2\}.$$

Thus $SS(\xi) \subset \check{S}$.

Step 2. $SS(\xi) \subset \check{C}_{X \times X}$.

By Corollary 2.3.6, $SS(\xi)$ is $X \times X$-coisotropic and hence by the geometric statement $SS(\xi) \subset \check{C}_{X \times X}$.

Step 3. $\mathrm{Supp}(\xi) \subset (\mathrm{sl}(V) \times C)(F)$.

Follows from the previous step by Fact 2.3.8. □

The case $F = \mathbb{C}$ is proven in the same way using the following corollary of the geometric statement.

Proposition 4.0.3. *Any $(X \times X)_{\mathbb{C}}$-coisotropic subvariety of $\check{S}_{\mathbb{C}}$ is contained in $(\check{C}_{X \times X})_{\mathbb{C}}$.*

Now it is left to prove the following proposition.

Proposition 4.0.4. *Let $\xi \in \mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi}$ be such that*

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}_{V\times V^*}(\xi)) \subset (\mathrm{sl}(V)\times C)(F).$$

Then $\xi = 0$.

4.1. Proof of proposition 4.0.4

Proposition 4.0.4 follows from the following lemma.

Lemma 4.1.1. *Let $F^\times$ act on $V\times V^*$ by $\lambda(v,\phi) := (\lambda v, \frac{\phi}{\lambda})$. Let $\xi \in \mathcal{S}^*((V\times V^*)(F))^{F^\times}$ be such that*

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}_{V\times V^*}(\xi)) \subset C(F).$$

Then $\xi = 0$.

By Homogeneity Theorem (Theorem 2.2.13) it is enough to prove the following lemma.

Lemma 4.1.2. *Let μ be a character of $F^\times$ given by $|\cdot|_F^n u$ or $|\cdot|_F^{n+1}u$ where u is some unitary character. Let $F^\times\times F^\times$ act on $V\times V^*$ by $(x,y)(v,\phi) = (\frac{y}{x}v, \frac{1}{xy}\phi)$. Then $\mathcal{S}^*_{(V\times V^*)(F)}(C(F))^{F^\times\times F^\times,\mu\times 1} = 0$.*

By Proposition 2.2.6 this lemma follows from the following one.

Lemma 4.1.3. *For any $k \geq 0$ we have*
(i) $\mathcal{S}^*(((V-0)\times 0)(F), Sym^k(CN^{V\times V^*}_{(V-0)\times 0}(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.
(ii) $\mathcal{S}^*((0\times(V^*-0))(F), Sym^k(CN^{V\times V^*}_{0\times(V^*-0)}(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.
(iii) $\mathcal{S}^*(0, Sym^k(CN_0^{V\times V^*}(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.

Proof.
(i) Cover $V-0$ by standard affine open sets $V_i := \{x_i \neq 0\}$. It is enough to show that $\mathcal{S}^*((V_i\times 0)(F), Sym^k(CN^{V\times V^*}_{(V_i\times 0)(F)}(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.

Note that V_i is isomorphic as an $F^\times\times F^\times$-manifold to $F^{n-1}\times F^\times$ with the action given by $(x,y)(v,\alpha) = (v, \frac{y}{x}\alpha)$. Note also that the bundle $Sym^k(CN^{V\times V^*}_{(V_i\times 0)(F)}(F))$ is a constant bundle with fiber $Sym^k(V)$.

Hence by Proposition 2.2.7 it is enough to show that $\mathcal{S}^*(F^\times, Sym^k(V))^{F^\times\times F^\times,\mu\times 1} = 0$. Let $H := (F^\times\times F^\times)_1 = \{(t,t)\in F^\times\times F^\times\}$. Now by Frobenius reciprocity (Theorem 2.2.8) it is enough to show that $(Sym^k(V^*(F))\otimes_{\mathbb{R}}\mathbb{C})^{H,\mu\times 1|_H} = 0$. This is clear since (t,t) acts on $(Sym^k(V^*(F))$ by multiplication by t^{-2k}.
(ii) is proven in the same way.
(iii) is equivalent to the statement $((Sym^k(V\times V^*)(F))\otimes_{\mathbb{R}}\mathbb{C})^{F^\times\times F^\times,\mu\times 1} = 0$. This is clear since $(t,1)$ acts on $Sym^k(V\times V^*)(F)$ by multiplication by t^{-k}. □

5. Proof of the geometric statement

5.1. Preliminaries on coisotropic subvarieties

Proposition 5.1.1. *Let M be a smooth algebraic variety with a symplectic form on it. Let $R \subset M$ be an algebraic subvariety. Then there exists a maximal M-coisotropic subvariety of R i.e. an M-coisotropic subvariety $T \subset M$ that includes all M-coisotropic subvarieties of R.*

Proof. Let T' be the union of all smooth M-coisotropic subvarieties of R. Let T be the Zariski closure of T' in R. Clearly, T includes all M-coisotropic subvarieties of R. Let U denote the set of regular points of T. Clearly $U \cap T'$ is dense in U. On the other hand, for any $x \in U \cap T'$, the tangent space to T at x is coisotropic. Hence T is coisotropic. □

Remark 5.1.2. *Suppose M is affine. Then T can be computed explicitly in the following way. Let I be the ideal of regular functions that vanish on $\overline{R}$. We can iteratively close it with respect to Poisson brackets and taking radical. Since $\mathcal{O}(M)$ is Noetherian, this process will stabilize. Let J denote the obtained closure and $Z(J)$ denote the zero set of J. Then $T = Z(J) \cap R$.*

The following lemma is trivial.

Lemma 5.1.3. *Let M be a smooth algebraic variety and ω be a symplectic form on it. Let a group G act on M preserving ω. Let S be a G -invariant subvariety. Then the maximal M-coisotropic subvariety of S is also G-invariant.*

Definition 5.1.4. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be any subvariety. We define* **the restriction** $R|_Z \subset T^*Z$ **of** R **to** Z *in the following way. Let $R' = p_Y^{-1}(Z) \cap R$. Let $q : p_Y^{-1}(Z) \to T^*Z$ be the projection. We define $R|_Z := q(R')$.*

Lemma 5.1.5. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be a coisotropic subvariety. Assume that any smooth point $z \in p_Y^{-1}(Z) \cap R$ is also a smooth point of R and we have $T_z(p_Y^{-1}(Z) \cap R) = T_z(p_Y^{-1}(Z)) \cap T_z R$.*

Then $R|_Z$ is T^*Z coisotropic.

In the proof we will use the following straightforward lemma.

Lemma 5.1.6. *Let W be a linear space. Let $L \subset W$ be a linear subspace and $R \subset W \oplus W^*$ be a coisotropic subspace. Then $R|_L$ is $L \oplus L^*$ coisotropic.*

Proof of lemma 5.1.5. Without loss of generality we assume that R is irreducible. Let $R' = p_Y^{-1}(Z) \cap R$. Without loss of generality we assume that R' is irreducible. Let R'' be the set of smooth points of R'. Let $q : p_Y^{-1}(Z) \to T^*Z$ be the projection. Let R''' be the set of smooth points in $q(R'')$. Clearly R''' is dense in $R|_Z$. Hence it is enough to prove that for any $x \in R'''$ the space $T_x(R|_Z)$ is coisotropic. Let $y \in R''$ s.t. $x = q(y)$. Denote $W := T_{p_Y(y)}Y$. Let

$Q := T_yR \subset W \oplus W^*$. Let $L := T_{p_Y(y)}Z$. By the assumption $T_x(R|_Z) \supset Q|_L$. By the lemma, $Q|_L$ is coisotropic and hence $T_x(R|_Z)$ is also coisotropic. $\Box$

Corollary 5.1.7. *Let Y be a smooth algebraic variety. Let an algebraic group H act on Y. Let $q : Y \to B$ be an H-equivariant morphism. Let $O \subset B$ be an orbit. Consider the natural action of G on T^*Y and let $R \subset T^*Y$ be an H-invariant subvariety. Suppose that $p_Y(R) \subset q^{-1}(O)$. Let $x \in O$. Denote $Y_x := q^{-1}(x)$. Then*

– *if R is T^*Y-coisotropic then $R|_{Y_x}$ is $T^*(Y_x)$-coisotropic.*

Corollary 5.1.8. *In the notation of the previous corollary, if $R|_{Y_x}$ has no (non-empty) $T^*(Y_x)$-coisotropic subvarieties then R has no (non-empty) $T^*(Y)$-coisotropic subvarieties.*

Note that the converse statement does not hold in general.

5.2. Reduction to the Key Proposition

In this subsection coisotropic variety means $X \times X$-coisotropic variety.

We will use the following notation.

Notation 5.2.1.
(i) For any nilpotent operator $A \in \mathrm{sl}(V)$ we denote

$$Q_A := \{(v, \phi) \in V \times V^* \,|\, v \otimes \phi \in [A, \mathfrak{g}]\} = \{(v, \phi) \in V \times V^* \,|\, (v \otimes \phi) \bot \mathfrak{g}_A\}.$$

(ii) Denote by T the maximal coisotropic subvariety of $\check{S}'$.
(iii) For any two nilpotent orbits $O_1, O_2 \subset N$ denote

$$U(O_1, O_2) := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in X \times X | \forall i, j \in \{1, 2\}$$
$$A_i \in O_i, (v_j, \phi_j) \in Q_{A_i},\ [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0, (v_1, \phi_1, v_2, \phi_2) \notin \check{C}\}.$$

The geometric statement is equivalent to the following theorem

Theorem 5.2.2. $T = \emptyset$.

The goal of this subsection is to reduce the geometric statement to the following Key Proposition.

Proposition 5.2.3 (Key Proposition). *For any two nilpotent orbits O_1, O_2 there are no (non-empty) coisotropic subvarieties in $U(O_1, O_2)$.*

The reduction will be in the spirit of the beginning of section 3 in [AGRS].

Notation 5.2.4. *Let*

$$\mathcal{N}^i = \{(A_1, A_2) \in \mathcal{N} \times \mathcal{N} | \dim G(A_1) + \dim G(A_2) \leq i\}.$$

$$\widehat{\mathcal{N}^i} := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in \check{S}' | (A_1, A_2) \in \mathcal{N}^i\}.$$

We will prove by descending induction that $T \subset \widehat{\mathcal{N}^i}$. From now on we fix i, suppose that this holds for i and prove that holds for $i-1$. Let $\mathfrak{S}$ denote the subgroup of automorphisms of $X \times X$ generated by $\check{\nu}_\lambda$, $F_{\mathrm{sl}(V)}$ and $F_{V\times V^*}$.

Denote $\widetilde{\mathcal{N}^i} := \bigcap_{\nu\in\mathfrak{S}} \nu(\widehat{\mathcal{N}^i})$. We know that $T \subset \widehat{\mathcal{N}^i}$, and hence $T \subset \widetilde{\mathcal{N}^i}$. Let $U^i := \widetilde{\mathcal{N}^i} - \widehat{\mathcal{N}^{i-1}}$. It is enough to show that U^i does not have (non-empty) coisotropic subvarieties.

Notation 5.2.5. *Let O_1, O_2 be nilpotent orbits such that* $\dim O_1 + \dim O_2 = i$. *Denote* $U'(O_1, O_2) := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in U^i | A_1 \in O_1,\ A_2 \in O_2\}$.

Since the sets $U'(O_1, O_2)$ form an open cover of U^i, it is enough to show that each $U'(O_1, O_2)$ does not have (non-empty) coisotropic subvarieties. This fact clearly follows from the Key Proposition using the following easy lemma.

Lemma 5.2.6. $U'(O_1, O_2) \subset U(O_1, O_2)$.

5.3. Reduction to the Key Lemma

We will use the following notation

Notation 5.3.1.

$$R_A := \{(v_1, \phi_1, v_2, \phi_2) \in Q_A \times Q_A - \check{C} \mid \exists B \in [A, \mathfrak{g}] \cap \mathcal{N} \text{ such that } [A, B] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

The goal of this subsection is to reduce the Key Proposition to the following Key Lemma.

Lemma 5.3.2 (Key Lemma). *R_A does not have (non-empty) $V \times V^* \times V \times V^*$-coisotropic subvarieties.*

Notation 5.3.3. *Denote*

$$U''(O_1, O_2) := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in U(O_1, O_2) | \mathfrak{g}_{A_1} \perp \mathfrak{g}_{A_2}\}.$$

Lemma 5.3.4. *Any $X \times X$-coisotropic subvariety of $U(O_1, O_2)$ lies in $U''(O_1, O_2)$.*

Proof. Denote $M := O_1 \times V \times V^* \times O_2 \times V \times V^*$. Note that $U(O_1, O_2) \subset M$. Note that any coisotropic subvariety of M is contained in $M' := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in M \mid \mathfrak{g}_{A_1} \perp \mathfrak{g}_{A_2}\}$. Hence any coisotropic subvariety of $U(O_1, O_2)$ is contained in $U(O_1, O_2) \cap M'$. ☐

The following straightforward lemma together with Corollary 5.1.8 finish the reduction.

Lemma 5.3.5. $U''(O_1, O_2)|_{A \times V \times V^*} \subset R_A$.

5.4. Proof of the Key Lemma

We will first give a short description of the proof for the case when A is one Jordan block. Then we will present the proof in the general case.

During the whole subsection coisotropic variety means $V \times V^* \times V \times V^*$-coisotropic variety.

5.4.1. Proof in the case when A is one Jordan block.
In this case $Q_A = \bigcup_{i=0}^{n}(KerA^i) \times (Ker(A^*)^{n-i})$. Hence

$$Q_A \times Q_A = \bigcup_{i,j=0}^{n} (KerA^i) \times (Ker(A^*)^{n-i}) \times (KerA^j) \times (Ker(A^*)^{n-j}).$$

Denote $L_{ij} := (KerA^i) \times (Ker(A^*)^{n-i}) \times (KerA^j) \times (Ker(A^*)^{n-j})$.

It is easy to see that any coisotropic subvariety of $Q_A \times Q_A$ is contained in $\bigcup_{i=0}^{n} L_{ii}$. Hence it is enough to show that for any i, $\dim R_A \cap L_{ii} < 2n$. For $i = 0, n$ it is clear since $R_A \cap L_{ii}$ is empty. So we will assume $0 < i < n$.

Let $f \in \mathcal{O}(L_{ii})$ be the polynomial defined by $f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1}$, where $(\cdot)_i$ means the i-th coordinate. It is enough to show that $f(R_A \cap L_{ii}) = \{0\}$.

Let $(v_1, \phi_1, v_2, \phi_2) \in L_{ii}$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$. Clearly, M is of the form

$$M = \begin{pmatrix} 0_{i\times i} & * \\ 0_{(n-i)\times i} & 0_{(n-i)\times(n-i)} \end{pmatrix}.$$

Note also that $M_{i,i+1} = f(v_1, \phi_1, v_2, \phi_2)$.

It is easy to see that any B satisfying $[A, B] = M$ is upper triangular. On the other hand, we know that there exists a nilpotent B satisfying $[A, B] = M$. Hence this B is upper nilpotent, which implies $M_{i,i+1} = 0$ and hence $f(v_1, \phi_1, v_2, \phi_2) = 0$.

5.4.2. Notation on filtrations.

Notation 5.4.1.
(i) Let L be a vector space with a gradation G^iL. It defines a filtration $G^{\geq i}L$ by $G^{\geq i}L := \bigoplus_{j\geq i} G^jL$.
(ii) Let L be a vector space with a descending filtration $F^{\geq i}$. We define $F^{>i}L := \bigcup_{j>i} F^{\geq j}L$.

Notation 5.4.2. *Let L and M be vector spaces with descending filtrations $F^{\geq i}L$ and $F^{\geq i}M$.*

Define filtrations $F^{\geq i}(L\otimes M) := \sum_{k+l=i} F^{\geq k}L\otimes F^{\geq l}M$ and $F^{\geq i}(L^*) := (F^{>-i}L)^{\perp}$.

Similarly for gradations G^iL and G^iM we define gradations $G^i(L \oplus M) := \bigoplus_{k+l=i} G^kL \otimes G^lM$ and $G^i(L^*) := (\bigoplus_{j\neq -i} G^jL)^{\perp}$.

We fix a standard basis $\{E, H, F\}$ of sl_2.

Notation 5.4.3. *Let L be a representation of sl_2. We define*

- *A gradation $W^\alpha(L) := Ker(H - \alpha Id)$ and*
- *An ascending filtration $K_i(L) := Ker(E^i)$.*

Note that if L is an irreducible representation then $K_i(L) = W^{\geq dimL+1-2i}(L)$.

5.4.3. Proof of the Key Lemma.
We will cover R_A by linear spaces and show that every one of them does not include coisotropic subvarieties of R_A.

Fix a morphism of Lie algebras $\psi : \mathrm{sl}_2 \to \mathrm{sl}(V)$ such that $\psi(E) = A$. Decompose V to irreducible representations of sl_2: $V = \bigoplus_{i=1}^k V_i$ such that $dim V_i \geq \dim V_{i+1}$.

Notation 5.4.4. *Denote* $D_i := \dim V_i$. *Let* D *denote the multiindex* $D := (D_1, ..., D_k)$.

For any multiindex $I = (I_1, ..., I_k)$ *such that* $0 \leq I_l \leq D_l$, $I \neq 0$ *and* $I \neq D$ *we define*

- $L_I := W^{\geq D_1+1-2I_1}(V_1) \oplus ... \oplus W^{\geq D_k+1-2I_k}(V_k) = K_{I_1}(V_1) \oplus ... \oplus K_{I_k}(V_k)$
- $L'_I := W^{\geq D_1+1-2I_1}(V_1^*) \oplus ... \oplus W^{\geq D_k+1-2I_k}(V_k^*) = K_{I_1}(V_1^*) \oplus ... \oplus K_{I_k}(V_k^*)$
- $L_{IJ} := L_I \times L'_{D-I} \times L_J \times L'_{D-J}$

The following two lemmas are straightforward

Lemma 5.4.5.
$$R_A \subset \bigcup_{I,J} L_{IJ}$$

Lemma 5.4.6. L_{IJ} *is not coisotropic if* $I \neq J$.

Hence it is enough to prove the following proposition.

Proposition 5.4.7. $\dim L_{II} \cap R_A < 2n$.

From here on we fix I and suppose that the proposition does not hold for this I. Our aim now is to get a contradiction. Note that if Proposition 5.4.7 holds for I then it holds for $D - I$. Hence without loss of generality we can (and will) assume $I_k < D_k$.

Lemma 5.4.8. *For any* $m < l$ *we have* $D_m - D_l \geq I_m - I_l \geq 0$.

Before we prove this lemma we introduce some notation.

We fix a Jordan basis of A in each V_i.

Notation 5.4.9. *For any* $v \in V, \phi \in V^*, X \in V \otimes V^*$ *we define* v^l *to be the* l*-th component of* v *with respect to the decomposition* $V = \oplus V_l$ *and* v^l_α *to be its* α *coordinate.*

Similarly we define $\phi^l, \phi^l_\alpha, X^{lm}, X^{lm}_{\alpha,\beta}$

Proof of lemma 5.4.8. It is enough to prove that for any l, m we have $I_l + (D_m - I_m) \leq max(D_l, D_m)$. Assume that the contrary holds for some l, m. It is enough to show that in this case $\dim(Q_A \cap (L_I \times L'_{D-I})) < n$. Consider the function $g \in \mathcal{O}(L_I \times L'_{D-I})$ defined by $g(v, \phi) = \phi^m_{I_m+1} \cdot v^l_{I_l}$. It is enough to show that $g(Q_A \cap (L_I \times L_{D-I})) = \{0\}$.

Let $B \in V_m \otimes V_l^*$ be defined by $B_{\alpha,\beta} = \delta_{\alpha-\beta, I_m-I_l+1}$. Consider B as an element of $\mathfrak{g}$. Note that $B \in \mathfrak{g}_A$ and $\langle B, v \otimes \phi \rangle = g(v, \phi)$ for any $(v, \phi) \in L_I \times L'_{D-I}$. Hence $g(Q_A \cap (L_I \times L_{D-I})) = \{0\}$. ☐

Corollary 5.4.10.
(i) If $I_m = 0$ then $I_l = 0$ for any $l > m$.
(ii) If $I_m = D_m$ then $I_l = D_l$ for any $l > m$.

Corollary 5.4.11. $D_1 > I_1 > 0$.

Notation 5.4.12. *Let k' be the maximal index such that $D_{k'} > I_{k'} > 0$.*

Notation 5.4.13. *Define $f_l \in \mathcal{O}(V \times V^* \times V \times V^*)$ by*

$$f_l(v_1, \phi_1, v_2, \phi_2) := (v_1)^l_{I_l}(\phi_2)^l_{I_l+1} - (v_2)^l_{I_l}(\phi_1)^l_{I_l+1}.$$

Define also $f := \sum_{l=1}^{k'} \frac{D_l - I_l}{D_l} f_l$.

Now it is enough to prove the following proposition.

Proposition 5.4.14.

$$f(R_A \cap L_{II}) = \{0\}.$$

We will need several notations and straightforward lemmas.

Lemma 5.4.15. *For any $\alpha \leq |D_m - D_l|$ we have $W^{\geq \alpha}(V_l \otimes V_m^*) = \{X \in V_l \otimes V_m^* | E(X) \in W^{\geq \alpha+2}(V_l \otimes V_m^*)\}$.*

Definition 5.4.16. *Define gradation W_I^i on V_l by $W_I^i(V_l) = W^{i+(D_l+l-2I_l)}(V_l)$. It gives rise to gradations W_I^i on $V_l^*, V_m \otimes V_l^*, V, V^*$.*

Lemma 5.4.17.
(i) If i is odd then $W_I^i = 0$.
(ii) $W_I^{\geq 0}(V) = L_I$.
(iii) $W_I^{\geq 2}(V^*) = L'_{D-I}$.

Definition 5.4.18. *Let $\mathcal{A}$ be the algebra $W_I^{\geq 0}(V \otimes V^*)$ and $\mathcal{I}$ be its ideal $W_I^{>0}(V \otimes V^*) = W_I^{\geq 2}(V \otimes V^*)$. Clearly $\mathcal{A}/\mathcal{I} \cong \prod End(W_I^i(V))$. This gives rise to a homomorphism $\varepsilon : \mathcal{A} \to End(W_I^0(V))$.*

Lemma 5.4.19.
(i) $\mathcal{A} = \bigoplus_{1 \leq l,m \leq k} W^{\geq D_l - D_m - 2(I_l - I_m)}(V_l \otimes V_m^*)$.
(ii) $\mathcal{I} := \bigoplus_{1 \leq l,m \leq k} W^{\geq D_l - D_m - 2(I_l - I_m)+2}(V_l \otimes V_m^*)$.
(iii) $dim(W_I^0(V)) = k'$
(iv) Consider the basis on $W_I^0(V)$ corresponding to the one on V and identify $End(W_I^0(V))$ with $\mathrm{gl}(k')$. Then

$$\varepsilon(X)_{lm} := X^{lm}_{I_l, I_m}.$$

Corollary 5.4.20. $\mathcal{A} = \{X \in End(V) | [A, X] \in \mathcal{I}\}$.

Proof. Follows from the previous lemma using Lemma 5.4.15. □

Proof of Proposition 5.4.14. Let $(v_1, \phi_1, v_2, \phi_2) \in L_{II} \cap R_A$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$. We know that there exists a nilpotent matrix $B \in [A, \mathrm{gl}(V)]$ such that $[A, B] = M$. By Corollary 5.4.20 $B \in \mathcal{A}$. Denote $\Delta := \varepsilon(B)$. Fix $1 \leq l \leq k'$. Denote $a_l := M^{ll}_{I_l, I_l+1}$. Note that $[A_l, B^{ll}] = M^{ll}$. Hence $B^{11}_{ll} = ... = B^{I_l, I_l}_{ll} = \Delta_{ll} = B^{I_l+1, I_l+1}_{ll} - a_l = ... = B^{D_l, D_l}_{ll} - a_l$. Since $B \in [A, End(V)]$ we have $tr(B_{ll}) = 0$. Thus $\Delta_{ll} = \frac{D_l - I_l}{D_l} a_l$.

Since B is nilpotent Δ is nilpotent. Hence $tr(\Delta) = 0$ and thus $\sum_{l=1}^{k'} \frac{D_l - I_l}{D_l} a_l = 0$ which means $f(v_1, \phi_1, v_2, \phi_2) = 0$. □

Appendix A. Theorem A implies Theorem B

This appendix is analogous to section 1 in [AGRS]. There, the classical theory of Gelfand and Kazhdan (see [GK]) is used. Here we use an archimedean analog of this theory which is described in [AGS], section 2. We work in the notations of [AGS]. In particular, what we call a smooth Fréchet representation is sometimes called in the literature smooth Fréchet representation of moderate growth (see e.g. [Wal]).

We will also use the theory of nuclear Fréchet spaces. For a good brief survey on this theory we refer the reader to [CHM], Appendix A.

Notation A.0.1.
(i) For a smooth Fréchet representation π of a real reductive group we denote by $\widetilde{\pi}$ the smooth dual of π.
(ii) For a representation π of $\mathrm{GL}_n(F)$ we let $\widehat{\pi}$ be the representation of $\mathrm{GL}_n(F)$ defined by $\widehat{\pi} = \pi \circ \theta$, where θ is the (Cartan) involution $\theta(g) = {g^{-1}}^t$.

We will use the following theorem.

Theorem A.0.2 (Casselman - Wallach globalization). *Let G be a real reductive group. There is a canonical equivalence of categories between the category of admissible smooth Fréchet representations of G and the category of admissible $(\mathfrak{g}, K)$- modules.*

See e.g. [Wal], chapter 11.
We will also use the embedding theorem of Casselman.

Theorem A.0.3. *Any irreducible $(\mathfrak{g}, K)$-module can be imbedded into a $(\mathfrak{g}, K)$-module of principal series.*

Those two theorems have the following corollary.

Corollary A.0.4. *The underlying topological vector space of any admissible smooth Fréchet representation is a nuclear Fréchet space.*

Definition A.0.5. *Let G and H be real reductive groups. Let (π, E) and (τ, W) be admissible smooth Fréchet representations of G and H respectively. We define $\pi \otimes \tau$ to be the natural representation of $G \times H$ on the space $E \widehat{\otimes} W$.*

Proposition A.0.6. *Let G and H be real reductive groups. Let π and τ be irreducible admissible Harish-Chandra modules of G and H respectively. Then $\pi \otimes \tau$ is irreducible Harish-Chandra module of $G \times H$.*

This proposition is well known. For the benefit of the reader we include its proof in subsection A.1. An analogous proposition in the non-Archimedean case appears in [BZ, subsection 2.16], and the proof we give here is along the same lines.

Corollary A.0.7. *Let G and H be real reductive groups. Let π and τ be irreducible admissible smooth Fréchet representations of G and H respectively. Then $\pi \otimes \tau$ is an irreducible representation of $G \times H$.*

Lemma A.0.8. *Let G and H be real reductive groups. Let (π, E) and (τ, W) be admissible smooth Fréchet representations of G and H respectively. Then $Hom_{\mathbb{C}}(\pi, \tau)$ is canonically embedded to $Hom_{\mathbb{C}}(\pi \otimes \widetilde{\tau}, \mathbb{C})$.*

Proof. For a nuclear Fréchet space V we denote by V' its dual space equipped with the strong topology. Let $\widetilde{W}$ denote the underlying space of $\widetilde{\tau}$. By the theory of nuclear Fréchet spaces, we know $Hom_{\mathbb{C}}(E, W) \cong E' \widehat{\otimes} W$ and $Hom_{\mathbb{C}}(E \widehat{\otimes} \widetilde{W}, \mathbb{C}) \cong E' \widehat{\otimes} \widetilde{W}'$. The lemma follows now from the fact that W is canonically embedded to $\widetilde{W}'$. □

We will use the following two archimedean analogs of theorems of Gelfand and Kazhdan.

Theorem A.0.9. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Then $\widehat{\pi}$ is isomorphic to $\widetilde{\pi}$.*

For proof see [AGS], Theorem 2.4.4.

Theorem A.0.10. *Let $H \subset G$ be real reductive groups and let σ be an involutive anti-automorphism of G and assume that $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all H-bi-invariant Schwartz distributions ξ on G. Let π be an irreducible admissible smooth Fréchet representation of G. Then*

$$\dim \mathrm{Hom}_H(\pi, \mathbb{C}) \cdot \dim \mathrm{Hom}_H(\widetilde{\pi}, \mathbb{C}) \leq 1.$$

For proof see [AGS], Theorem 2.3.2.

Corollary A.0.11. *Let $H \subset G$ be real reductive groups and let σ be an involutive anti-automorphism of G such that $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all Schwartz distributions ξ on G which are invariant with respect to conjugation by H.*

Let π be an irreducible admissible smooth Fréchet representation of G and τ be an irreducible admissible smooth Fréchet representation of H. Then

$$dim\, \mathrm{Hom}_H(\pi, \tau) \cdot \dim \mathrm{Hom}_H(\widetilde{\pi}, \widetilde{\tau}) \leq 1.$$

Proof. Define $\sigma' : G \times H \to G \times H$ by $\sigma'(g,h) := (\sigma(g), \sigma(h))$. Let $\Delta H < G \times H$ be the diagonal. Consider the projection $G \times H \to H$. By Frobenius reciprocity (Theorem 2.2.8), the assumption implies that any ΔH-bi-invariant distribution on $G \times H$ is invariant with respect to σ'.

Hence by the previous theorem, for any irreducible admissible smooth Fréchet representation π' of $G \times H$ we have $\dim \mathrm{Hom}_{\Delta H}(\pi', \mathbb{C}) \cdot \dim \mathrm{Hom}_{\Delta H}(\widetilde{\pi'}, \mathbb{C}) \leq 1$.

Taking $\pi' := \pi \otimes \widetilde{\tau}$ we obtain the required inequality. □

Corollary A.0.12. *Theorem A implies Theorem B.*

Proof. By Theorem A.0.9, $\dim \mathrm{Hom}_H(\widetilde{\pi}, \widetilde{\tau}) = \dim \mathrm{Hom}_H(\widehat{\pi}, \widehat{\tau}) = \dim \mathrm{Hom}_H(\pi, \tau)$. □

A.1. Proof of proposition A.0.6

Notation A.1.1. *Let G be a reductive group, $\mathfrak{g}$ be its Lie algebra and K be its maximal compact subgroup. Let π be an admissible $(\mathfrak{g}, K)$-module.*
Let ρ be an irreducible representation of K.
(i) We denote by $e_\rho : \pi \to \pi$ the projection to the K-type ρ.
(ii) We denote by G^π_ρ the subalgebra of $End(e_\rho(\pi))$ generated by the actions of K and $e_\rho U(\mathfrak{g}) e_\rho$.

The following lemma is well-known

Lemma A.1.2. *Let π be an irreducible admissible $(\mathfrak{g}, K)$-module. Let ρ be an irreducible representation of K. Suppose that $e_\rho(\pi) \neq 0$. Then $e_\rho(\pi)$ is an irreducible representation of G^π_ρ.*

We will also use Bernside theorem.

Theorem A.1.3. *Let V be a finite dimensional complex vector space. Let $A \subset End(V)$ be a subalgebra such that V does not have any non-trivial A-invariant subspaces. Then $A = End(V)$.*

Now we are ready to prove proposition A.0.6.

Proof of proposition A.0.6. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. Let K and L be maximal compact subgroups of G and H. Let $\omega \subset \pi \otimes \tau$ be a nonzero $(\mathfrak{g} \times \mathfrak{h}, K \times L)$-submodule. Then ω intersects non-trivially some $K \times L$ type. Denote this type by $\rho \otimes \sigma$. By Lemma A.1.2, $e_\rho(\pi)$ is an irreducible representation of G^π_ρ and $e_\sigma(\tau)$ is an irreducible representation of H^τ_σ. Hence by Bernside theorem $G^\pi_\rho = End(e_\rho(\pi))$ and $H^\tau_\sigma = End(e_\sigma(\tau))$. Hence $(G \times H)^{\pi \otimes \tau}_{\rho \otimes \sigma} = End(e_\rho(\pi) \otimes e_\sigma(\tau))$. Thus $\omega \cap e_{\rho \otimes \sigma}(\pi \otimes \tau) = e_{\rho \otimes \sigma}(\pi \otimes \tau)$.

This means that ω contains an element of the form $v \otimes w$, which implies that $\omega = \pi \otimes \tau$. □

Appendix B. D-modules

In this appendix X denotes a smooth affine variety defined over $\mathbb{R}$. All the statements of this section extend automatically to general smooth algebraic varieties defined over $\mathbb{R}$. In this paper we use only the case when X is an affine space.

Definition B.0.1. *Let $D(X)$ denote the algebra of polynomial differential operators on X. We consider the filtration $F^{\leq i}D(X)$ on $D(X)$ given by the order of differential operator.*

Definition B.0.2. *We denote by $\operatorname{Gr} D(X)$ the associated graded algebra of $D(X)$.*

Define the symbol map $\sigma : D(X) \to \operatorname{Gr} D(X)$ in the following way. Let $d \in D(X)$. Let i be the minimal index such that $d \in F^{\leq i}$. We define $\sigma(d)$ to be the image of d in $(F^{\leq i}D(X))/(F^{\leq i-1}D(X))$

Proposition B.0.3. $\operatorname{Gr} D(X) \cong \mathcal{O}(T^*X)$.

For proof see e.g. [Bor].

Notation B.0.4. *Let (V, B) be a quadratic space.*
(i) We define a morphism of algebras $\Phi_V^D : D(X \times V) \to D(X \times V)$ in the following way.

Consider B as a map $B : V \to V^*$. For any $f \in V^*$ we set $\Phi_V^D(f) := \partial_{B^{-1}(f)}$. For any $v \in V$ we set $\Phi_V^D(\partial_v) := -B(v)$ and for any $d \in D(X)$ we set $\Phi_V^D(d) := d$.
(ii) It defines a morphism of algebras $\Phi_V^O : \mathcal{O}(T^*X) \to \mathcal{O}(T^*X)$.

The following lemma is straightforward.

Lemma B.0.5. *Let f be a homogeneous polynomial. Consider it as a differential operator. Then $\sigma(\Phi_V^D(f)) = \Phi_V^O(\sigma(f))$.*

The D-modules we use in the paper are right D-modules. The difference between right and left D-modules is not essential (see e.g. section VI.3 in [Bor]). We will use the notion of good filtration on a D-module, see e.g. section II.4 in [Bor]. Let us now remind the definition of singular support of a module and a distribution.

Notation B.0.6. *Let M be a $D(X)$-module. Let $\alpha \in M$ be an element. Then we denote by $Ann_{D(X)}$ the annihilator of α.*

Definition B.0.7. *Let M be a $D(X)$-module. Choose a good filtration on M. Consider grM as a module over $\operatorname{Gr} D(X) \cong \mathcal{O}(T^*X)$. We define*

$$SS(M) := \operatorname{Supp}(\operatorname{Gr} M) \subset T^*X.$$

This does not depend on the choice of the good filtration on M (see e.g. [Bor], *section II.4).*

For a distribution $\xi \in S^*(X(\mathbb{R}))$ we define $SS(\xi)$ to be the singular support of the module of distributions generated by ξ.

The following proposition is trivial.

Proposition B.0.8. *Let $I < D(X)$ be a right ideal. Consider the induced filtrations on I and $D(X)/I$. Then $\mathrm{Gr}(D(X)/I) \cong \mathrm{Gr}(D(X))/\mathrm{Gr}(I)$.*

Corollary B.0.9. *Let $\xi \in S^*(X)$. Then $SS(\xi)$ is the zero set of $\mathrm{Gr}(Ann_{D(X)}\xi)$.*

Corollary B.0.10. *Let $I < \mathcal{O}(T^*X)$ be the ideal generated by $\{\sigma(d) \mid d \in Ann_{D(X)}(\xi)\}$. Then $SS(\xi)$ is the zero set of I.*

Corollary B.0.11. *Fact 2.3.9 holds.*

Lemma B.0.12. *Let $\xi \in S^*(X)$. Let $Z \subset X$ be a closed subvariety such that $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$. Let $f \in \mathcal{O}(X)$ be a polynomial that vanishes on Z. Then there exists $k \in \mathbb{N}$ such that $f^k\xi = 0$.*

Proof.

Step 1. Proof for the case when X is affine space and f is a coordinate function.

This follows from the proof of Corollary 5.5.4 in [AG1].

Step 2. Proof for the general case.

Embed X into an affine space A^N such that f will be a coordinate function and consider ξ as distribution on A^N supported in X. By Step 1, $f^k\xi = 0$ for some k. □

Corollary B.0.13. *Fact 2.3.8 holds.*

Proposition B.0.14. *Fact 2.3.10 holds. Namely:*

Let (V, B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X\times V}^{-1}(Z))$.

Proof. Let $f \in \mathcal{O}(X \times V)$ be homogeneous with respect to homotheties in V. Suppose that f vanishes on Z. Then $\Phi_V^D(f^k) \in Ann_{D(X)}(\mathcal{F}_V(\xi))$. Therefore $\sigma(\Phi_V^D(f^k))$ vanishes on $SS(\mathcal{F}_V(\xi))$. On the other hand, $\sigma(\Phi_V^D(f^k)) = \Phi_V^O(\sigma(f^k)) = (\Phi_V^O(\sigma(f)))^k$. Hence $SS(\mathcal{F}_V(\xi))$ is included in the zero set of $\Phi_V^O(\sigma(f))$. Intersecting over all such f we obtain the required inclusion. □

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SPHERICAL PAIRS OVER CLOSE LOCAL FIELDS

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Abstract. Extending results of [Kaz86] to the relative case, we relate harmonic analysis over some spherical spaces $G(F)/H(F)$, where F is a field of positive characteristic, to harmonic analysis over the spherical spaces $G(E)/H(E)$, where E is a suitably chosen field of characteristic 0.

We apply our results to show that the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a strong Gelfand pair for all local fields of arbitrary characteristic, and that the pair $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ is a Gelfand pair for local fields of any characteristic different from 2. We also give a criterion for finite generation of the space of K-invariant compactly supported functions on $G(E)/H(E)$ as a module over the Hecke algebra.

Contents

0. Introduction

Local fields of positive characteristic can be approximated by local fields of characteristic zero. If F and E are local fields, we say that they are m-close if $O_F/\mathcal{P}_F^m \cong O_E/\mathcal{P}_E^m$, where O_F, O_E are the rings of integers of F and E, and $\mathcal{P}_F, \mathcal{P}_E$ are their maximal ideals. For example, $F_p((t))$ is m-close to $\mathbb{Q}_p(\sqrt[m]{p})$. More generally, for any local field F of positive characteristic p and any m there exists a (sufficiently ramified) extension of $\mathbb{Q}_p$ that is m-close to F.

Let G be a reductive group defined over $\mathbb{Z}$. For any local field F and conductor $\ell \in \mathbb{Z}_{\geq 0}$, the Hecke algebra $\mathcal{H}_\ell(G(F))$ is finitely generated and finitely presented. Based on this fact, Kazhdan showed in

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[Kaz86] that for any ℓ there exists $m \geq \ell$ such that the algebras $\mathcal{H}_\ell(G(F))$ and $\mathcal{H}_\ell(G(E))$ are isomorphic for any m-close fields F and E. This allows one to transfer certain results in representation theory of reductive groups from local fields of zero characteristic to local fields of positive characteristic.

In this paper we investigate a relative version of this technique. Let G be a reductive group and H be a spherical subgroup. Suppose for simplicity that both are defined over $\mathbb{Z}$.

In the first part of the paper we consider the space $\mathcal{S}(G(F)/H(F))^K$ of compactly supported functions on $G(F)/H(F)$ which are invariant with respect to a compact open subgroup K. We prove under certain assumption on the pair (G, H) that this space is finitely generated (and hence finitely presented) over the Hecke algebra $\mathcal{H}_K(G(F))$.

Theorem A (see Theorem 2.3.1)**.** *Let F be a (non-Archimedean) local field. Let G be a reductive group and $H < G$ be an algebraic subgroup both defined over F. Suppose that for any parabolic subgroup $P \subset G$, there is a finite number of double cosets $P(F) \setminus G(F)/H(F)$. Suppose also that for any irreducible smooth representation ρ of $G(F)$ we have*

$$\dim \operatorname{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) < \infty. \tag{1}$$

Then for any compact open subgroup $K < G(F)$, the space $\mathcal{S}(G(F)/H(F))^K$ is a finitely generated module over the Hecke algebra $\mathcal{H}_K(G(F))$.

Assumption (1) is rather weak in light of the results of [Del, SV]. In particular, it holds for all symmetric pairs over fields of characteristic different from 2. One can easily show that the converse is also true. Namely, that if $\mathcal{S}(G(F)/H(F))^K$ is a finitely generated module over the Hecke algebra $\mathcal{H}_K(G(F))$ for any compact open subgroup $K < G(F)$, then (1) holds.

Remark. *Theorem A implies that, if $\dim \operatorname{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C})$ is finite, then it is bounded on every Bernstein component.*

In the second part of the paper we introduce the notion of a uniform spherical pair and prove for them the following analog of Kazhdan's theorem.

Theorem B. *[See Theorem 3.2.3] Let $H < G$ be reductive groups defined over $\mathbb{Z}$. Suppose that the pair (G, H) is uniform spherical.*

Then for any l there exists n such that for any n-close local fields F and E, the module $\mathcal{S}(G(F)/H(F))^{K_\ell(F)}$ over the algebra $\mathcal{H}_\ell(G(F))$ is isomorphic to the module $\mathcal{S}(G(E)/H(E))^{K_\ell(E)}$ over the algebra $\mathcal{H}_\ell(G(E))$, where we identify $\mathcal{H}_\ell(G(F))$ and $\mathcal{H}_\ell(G(E))$ using Kazhdan's isomorphism.

In fact, we prove a more general theorem, see §3. This implies the following corollary.

Corollary C. *Let (G, H) be a uniform spherical pair of reductive groups defined over $\mathbb{Z}$. Suppose that*

- *For any local field F, and any parabolic subgroup $P \subset G$, there is a finite number of double cosets $P(F) \setminus G(F)/H(F)$.*
- *For any local field F of characteristic zero the pair $(G(F), H(F))$ is a Gelfand pair, i.e. for any irreducible smooth representation ρ of $G(F)$ we have*

$$\dim \operatorname{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) \leq 1.$$

Then for any local field F the pair $(G(F), H(F))$ is a Gelfand pair.

In fact, we prove a more general theorem, see §3.

Remark. *In a similar way one can deduce an analogous corollary for cuspidal representations. Namely, suppose that the first two conditions of the last corollary hold and the third condition holds for all cuspidal representations ρ. Then for any local field F the pair $(G(F), H(F))$ is a cuspidal Gelfand pair: for any irreducible smooth cuspidal representation ρ of $G(F)$ we have*

$$\dim \operatorname{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) \leq 1.$$

Remark. *Originally, we included in the formulation of Theorem B an extra condition: we demanded that the module* $\mathcal{S}(G(F)/H(F))^{K_\ell(F)}$ *is finitely generated over the Hecke algebra* $\mathcal{H}_\ell(G(F))$ *for any* F *and* l. *This was our original motivation for Theorem A. Later we realized that this condition just follows from the definition of uniform spherical pair. However, we think that Theorem A and the technique we use in its proof have importance of their own.*

In the last part of the paper we apply our technique to show that $(\mathrm{GL}_{n+1}, \mathrm{GL}_n)$ is a strong Gelfand pair over any local field and $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ is a Gelfand pair over any local field of odd characteristic.

Theorem D. *Let* F *be any local field. Then* $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ *is a strong Gelfand pair, i.e. for any irreducible smooth representations* π *of* $\mathrm{GL}_{n+1}(F)$ *and* τ *of* $\mathrm{GL}_n(F)$ *we have*

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1.$$

Theorem E. *Let* F *be any local field. Suppose that* $\mathrm{char}\, F \neq 2$. *Then* $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ *is a Gelfand pair.*

We deduce these theorems from the zero characteristic case, which was proven in [AGRS] and [JR96] respectively. The proofs in [AGRS] and [JR96] cannot be directly adapted to the case of positive characteristic since they rely on Jordan decomposition which is problematic in positive characteristic, local fields of positive characteristic being non-perfect.

Remark. *In* [AGS08]*, a special case of Theorem D was proven for all local fields; namely the case when* τ *is one-dimensional.*

Remark. *In* [AG09a] *and (independently) in* [SZ]*, an analog of Theorem D was proven for Archimedean local fields. In* [AG09b]*, an analog of Theorem E was proven for Archimedean local fields.*

0.1. **Structure of the paper.**

In Section 1 we introduce notation and give some general preliminaries.

In Section 2 we prove Theorem A.

In Subsection 2.1 we collect a few general facts for the proof. One is a criterion, due to Bernstein, for finite generation of the space of K-invariant vectors in a representation of a reductive group G; the other facts concern homologies of l-groups. In Subsection 2.2 we prove the main inductive step in the proof of Theorem A, and in Subsection 2.3 we prove Theorem A. Subsection 2.4 is devoted to the proofs of some facts about the homologies of l-groups.

In Section 3 we prove Theorem B and derive Corollary C.

In Subsection 3.1 we introduce the notion of uniform spherical pair. In Subsection 3.2 we prove the theorem and the corollary.

We apply our results in Section 4. In Subsection 4.1 we prove that the pair $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ satisfies the assumptions of Corollary C over fields of characteristic different from 2. In Subsections 4.3 and 4.2 we prove that the pair $(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, \Delta\, \mathrm{GL}_n)$ satisfies the assumptions of Corollary C. These facts imply Theorems D and E.

0.2. **Acknowledgments.**

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1. Preliminaries and notation

Definition 1.0.1. *A local field is a locally compact complete non-discrete topological field. In this paper we will consider only non-Archimedean local fields. All such fields have discrete valuations.*

Remark 1.0.2. *Any local field of characteristic zero and residue characteristic p is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers and any local field of characteristic p is a finite extension of the field $\mathbb{F}_p((t))$ of formal Laurent series over the field with p elements.*

Notation 1.0.3. *For a local field F we denote by val_F its valuation, by O_F the ring of integers and by $\mathcal{P}_F$ its unique maximal ideal. For an algebraic group G defined over O_F we denote by $K_\ell(G,F)$ the kernel of the (surjective) morphism $G(O_F) \to G(O_F/\mathcal{P}_F^\ell)$. If $\ell > 0$ then we call $K_\ell(G,F)$ the ℓ-th congruence subgroup.*

We will use the terminology of l-spaces and l-groups introduced in [BZ76]. An l-space is a locally compact second countable totally disconnected topological space, an l-group is a l-space with a continuous group structure. For further background on l-spaces, l-groups and their representations we refer the reader to [BZ76].

Notation 1.0.4. *Let G be an l-group. Denote by $\mathcal{M}(G)$ the category of smooth complex representations of G.*

Define the functor of coinvariants $CI_G : \mathcal{M}(G) \to Vect$ by

$$CI_G(V) := V/(\mathrm{Span}\{v - gv \,|\, v \in V,\, g \in G\}).$$

Sometimes we will also denote $V_G := CI_G(V)$.

Notation 1.0.5. *For an l-space X we denote by $\mathcal{S}(X)$ the space of locally constant compactly supported complex valued functions on X. If X is an analytic variety over a non-Archimedean local field, we denote by $\mathcal{M}(X)$ the space of locally constant compactly supported measures on X.*

Notation 1.0.6. *For an l-group G and an open compact subgroup K we denote by $\mathcal{H}(G,K)$ or $\mathcal{H}_K(G)$ the Hecke algebra of G w.r.t. K, i.e. the algebra of compactly supported measures on G that are invariant w.r.t. both left and right multiplication by K.*

For a local field F and a reductive group G defined over O_F we will also denote $\mathcal{H}_\ell(G(F)) := \mathcal{H}_{K_\ell(G)}(G(F))$.

Notation 1.0.7. *By a reductive group over a ring R, we mean a smooth group scheme over $\mathrm{Spec}(R)$ all of whose geometric fibers are reductive and connected.*

2. Finite Generation of Hecke Modules

The goal of this section is to prove Theorem A.

In this section F is a fixed (non-Archimedean) local field of arbitrary characteristic. All the algebraic groups and algebraic varieties that we consider in this section are defined over F. In particular, reductive means reductive over F.

For the reader's convenience, we now give an overview of the argument. In Lemma 2.1.10 we present a criterion, due to Bernstein, for the finite generation of spaces of K-invariants. The proof of the criterion uses the theory of Bernstein Center. This condition is given in terms of all parabolic subgroups of G. We directly prove this condition when the parabolic is G (this is Step 1 in the proof of Theorem A).

The case of general parabolic is reduced to the case where the parabolic is G. For this, the main step is to show that the assumptions of the theorem imply similar assumptions for the Levi components of the parabolic subgroups of G. This is proved in Lemma 2.2.4 by stratifying the space $G(F)/P(F)$ according to the $H(F)$-orbits inside it. In the proof of this lemma we use two homological tools: Lemma 2.1.11 that which gives a criterion for finite dimensionality of the first homology of a representation and Lemma 2.1.12 which connects the homologies of a representation and of its induction.

2.1. **Preliminaries.**

Notation 2.1.1. *For l-groups $H < G$ we denote by $ind_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ the compactly supported induction functor and by $Ind_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ the full induction functor.*

Definition 2.1.2. *Let G be a reductive group, let $P < G$ be a parabolic subgroup with unipotent radical U, and let $M := P/U$. Such M is called a Levi subquotient of G. Note that every representation of $M(F)$ can be considered as a representation of $P(F)$ using the quotient morphism $P \twoheadrightarrow M$. Define:*

1. *The Jacquet functor $r_{GM} : \mathcal{M}(G(F)) \to \mathcal{M}(M(F))$ by $r_{GM}(\pi) := (\pi|_{P(F)})_{U(F)}$.*
2. *The parabolic induction functor $i_{GM} : \mathcal{M}(M(F)) \to \mathcal{M}(G(F))$ by $i_{GM}(\tau) := ind_{P(F)}^{G(F)}(\tau)$.*

Note that i_{GM} is right adjoint to r_{GM}. A representation π of $G(F)$ is called cuspidal if $r_{GM}(\pi) = 0$ for any Levi subquotient M of G.

Definition 2.1.3. *Let G be an l-group. A smooth representation V of G is called* ***compact*** *if for any $v \in V$ and $\xi \in \widetilde{V}$ the matrix coefficient function defined by $m_{v,\xi}(g) := \xi(gv)$ is a compactly supported function on G.*

Theorem 2.1.4 (Bernstein-Zelevinsky)**.** *Let G be an l-group. Then any compact representation of G is a projective object in the category $\mathcal{M}(G)$.*

Definition 2.1.5. *Let G be a reductive group.*
(i) Denote by G^1 the preimage in $G(F)$ of the maximal compact subgroup of $G(F)/[G,G](F)$.
(ii) Denote $G_0 := G^1 Z(G(F))$.
(iii) A complex character of $G(F)$ is called unramified if it is trivial on G^1. We denote the set of all unramified characters by Ψ_G. Note that $G(F)/G^1$ is a lattice and therefore we can identify Ψ_G with $(\mathbb{C}^\times)^n$. This defines a structure of algebraic variety on Ψ_G.
(iv) For any smooth representation ρ of $G(F)$ we denote $\Psi(\rho) := ind_{G^1}^G(\rho|_{G^1})$. Note that $\Psi(\rho) \simeq \rho \otimes \mathcal{O}(\Psi_G)$, where $G(F)$ acts only on the first factor, but this action depends on the second factor. This identification gives a structure of $\mathcal{O}(\Psi_G)$-module on $\Psi(\rho)$.

Remark 2.1.6. *The definition of unramified characters above is not the standard one, but it is more convenient for our purposes.*

Theorem 2.1.7 (Harish-Chandra)**.** *Let G be a reductive group and V be a cuspidal representation of $G(F)$. Then $V|_{G^1}$ is a compact representation of G^1.*

Corollary 2.1.8. *Let G be a reductive group and ρ be a cuspidal representation of $G(F)$. Then*
(i) $\rho|_{G^1}$ is a projective object in the category $\mathcal{M}(G^1)$.
(ii) $\Psi(\rho)$ is a projective object in the category $\mathcal{M}(G(F))$.

Proof. (i) is clear.
(ii) note that
$$Hom_G(\Psi(\rho), \pi) \cong Hom_{G/G_1}(\mathcal{O}(\Psi_M), Hom_{G^1}(\rho, \pi)),$$
for any representation π. Therefore the functor $\pi \mapsto Hom_G(\Psi(\rho), \pi)$ is a composition of two exact functors and hence is exact. $\square$

Definition 2.1.9. *Let G be a reductive group and $K < G(F)$ be a compact open subgroup. Denote*

$$\mathcal{M}(G,K) := \{V \in \mathcal{M}(G(F)) \,|V \text{ is generated by } V^K\}$$

and

$$\mathcal{M}(G,K)^{\perp} := \{V \in \mathcal{M}(G(F) \,|V^K = 0\}.$$

We call K a splitting subgroup if the category $\mathcal{M}(G(F))$ is the direct sum of the categories $\mathcal{M}(G,K)$ and $\mathcal{M}(G,K)^{\perp}$, and $\mathcal{M}(G,K) \cong \mathcal{M}(\mathcal{H}_K(G))$. Recall that an abelian category $\mathcal{A}$ is a direct sum of two abelian subcategories $\mathcal{B}$ and $\mathcal{C}$, if every object of $\mathcal{A}$ is isomorphic to a direct sum of an object in $\mathcal{B}$ and an object in $\mathcal{C}$, and, furthermore, that there are no non-trivial morphisms between objects of $\mathcal{B}$ and $\mathcal{C}$.

We will use the following statements from Bernstein's theory on the center of the category $\mathcal{M}(G)$. Let $P < G$ be a parabolic subgroup and M be the reductive quotient of P.

(1) The set of splitting subgroups defines a basis at 1 for the topology of $G(F)$. If G splits over O_F then, for any large enough ℓ, the congruence subgroup $K_\ell(G,F)$ is splitting.
(2) Let $\overline{P}$ denote the parabolic subgroup of G opposite to P, and let $\overline{r}_{GM} : \mathcal{M}(G(F)) \to \mathcal{M}(M(F))$ denote the Jacquet functor defined using $\overline{P}$. Then $\overline{r}_{GM}$ is right adjoint to i_{GM}. In particular, i_{GM} maps projective objects to projective ones and hence for any irreducible cuspidal representation ρ of $M(F)$, $i_{GM}(\Psi(\rho))$ is a projective object of $\mathcal{M}(G(F))$.
(3) Denote by $\mathcal{M}_\rho$ the subcategory of $\mathcal{M}(G(F))$ generated by $i_{GM}(\Psi(\rho))$. Then

$$\mathcal{M}(G,K) = \oplus_{(M,\rho)\in B_K}\mathcal{M}_\rho,$$

where B_K is some finite set of pairs consisting of a Levi subquotient of G and its cuspidal representation. Moreover, for any Levi subquotient $M < G$ and a cuspidal representation ρ of $M(F)$ such that $\mathcal{M}_\rho \subset \mathcal{M}(G,K)$ there exist $(M',\rho') \in B_K$ such that $\mathcal{M}_\rho = \mathcal{M}_{\rho'}$.
(4) $End(i_{GM}(\Psi(\rho)))$ is finitely generated over $\mathcal{O}(\Psi)$ which is finitely generated over the center of the ring $End(i_{GM}(\Psi(\rho)))$. The center of the ring $End(i_{GM}(\Psi(\rho)))$ is equal to the center $Z(\mathcal{M}_\rho)$ of the category $\mathcal{M}_\rho$.

For statements 1 see e.g. [BD84, pp. 15-16] and [vD, §2]. For statement 2 see [Ber87] or [Bus01, Theorem 3]. For statements 3,4 see [BD84, Proposition 2.10,2.11].

We now present a criterion, due to Bernstein, for finite generation of the space V^K, consisting of vectors in a representation V that are invariant with respect to a compact open subgroup K.

Lemma 2.1.10. *Let V be a smooth representation of $G(F)$. Suppose that for any parabolic $P < G$ and any irreducible cuspidal representation ρ of $M(F)$ (where M denotes the reductive quotient of P), $\mathrm{Hom}_{G(F)}(i_{GM}(\Psi(\rho)),V)$ is a finitely generated module over $\mathcal{O}(\Psi_M)$. Then V^K is a finitely generated module over $Z(\mathcal{H}_K(G(F)))$, for any compact open subgroup $K < G(F)$.*

Proof.
Step 1. Proof for the case when K is splitting and $V = i_{GM}(\Psi(\rho))$ for some Levi subquotient M of G and an irreducible cuspidal representation ρ of $M(F)$. Let P denote the parabolic subgroup that defines M and U denote its unipotent radical. Denote $K_M := K/(U(F) \cap K) < M(F)$. If $V^K = 0$ there is nothing to prove. Otherwise $\mathcal{M}_\rho$ is a direct summand of $\mathcal{M}(G,K)$. Now

$$V^K = \Psi(\rho)^{K_M} = \rho^{K_M} \otimes \mathcal{O}(\Psi).$$

Hence V^K is finitely generated over $Z(\mathcal{M}_\rho)$. Hence V^K is finitely generated over $Z(\mathcal{M}(G,K)) = Z(\mathcal{H}_K(G))$.

Step 2. Proof for the case when K is splitting and $V \in \mathcal{M}_\rho$ for some Levi subquotient $M < G$ and an irreducible cuspidal representation ρ of $M(F)$.

Let

$$\phi : i_{GM}(\Psi(\rho)) \otimes \mathrm{Hom}(i_{GM}(\Psi(\rho)),V) \twoheadrightarrow V$$

be the natural epimorphism. We are given that $\mathrm{Hom}(i_{GM}(\Psi(\rho)), V)$ is finitely generated over $\mathcal{O}(\Psi)$. Hence it is finitely generated over $Z(\mathcal{M}(\rho))$. Choose some generators $\alpha_1, ..., \alpha_n \in \mathrm{Hom}(i_{GM}(\Psi(\rho))$. Let

$$\psi : i_{GM}(\Psi(\rho))^n \hookrightarrow i_{GM}(\Psi(\rho)) \otimes \mathrm{Hom}(i_{GM}(\Psi(\rho)), V)$$

be the corresponding morphism. $Im(\phi \circ \psi)$ is $Z(\mathcal{M}(\rho))$-invariant and hence coincides with $Im(\phi)$. Hence $\phi \circ \psi$ is onto. The statement now follows from the previous step.

Step 3. Proof for the case when K is splitting.

Let $W < V$ be the subrepresentation generated by V^K. By definition $W \in \mathcal{M}(G, K)$ and hence $W = \oplus_{i=1}^n W_i$ where $W_i \in \mathcal{M}_{\rho_i}$ for some ρ_i. The lemma now follows from the previous step.

Step 4. General case

Let K' be a splitting subgroup s.t. $K' < K$. Let $v_1 ... v_n \in V^{K'}$ be the generators of $V^{K'}$ over $Z(\mathcal{H}_{K'}(G(F)))$ given by the previous step. Define $w_i := e_K v_i \in V^K$ where $e_K \in \mathcal{H}_K(G(F))$ is the normalized Haar measure of K. Let us show that w_i generate V^K over

$Z(\mathcal{H}_K(G(F)))$. Let $x \in V^K$. We can represent x as a sum $\sum h_i v_i$, where $h_i \in Z(\mathcal{H}_{K'}(G(F)))$. Now

$$x = e_K x = \sum e_K h_i v_i = \sum e_K e_K h_i v_i = \sum e_K h_i e_K v_i = \sum e_K h_i e_K e_K v_i = \sum e_K h_i e_K w_i.$$

□

Finally, in this subsection, we state two facts about homologies of l-groups. The proofs and relevant definitions are in Subsection 2.4.

Lemma 2.1.11. *Let G be an algebraic group and U be its unipotent radical. Let ρ be an irreducible cuspidal representation of $(G/U)(F)$. We treat ρ as a representation of $G(F)$ with trivial action of $U(F)$.*

Let $H < G$ be an algebraic subgroup. Suppose that the space of coinvariants $\rho_{H(F)}$ is finite dimensional. Then $\dim \mathrm{H}_1(H(F), \rho) < \infty$.

We will also use the following version of Shapiro Lemma.

Lemma 2.1.12. *Let G be an l-group that acts transitively on an l-space X. Let $\mathcal{F}$ be a G-equivariant sheaf over X. Choose a point $x \in X$, let $\mathcal{F}_x$ denote the stalk of $\mathcal{F}$ at x and G_x denote the stabilizer of x. Then*

$$\mathrm{H}_i(G, \mathcal{F}(X)) = \mathrm{H}_i(G_x, \mathcal{F}_x).$$

2.2. **Descent Of Finite Multiplicity.**

Definition 2.2.1. *We call a pair (G, H) consisting of a reductive group G and an algebraic subgroup H* ***an F-spherical pair*** *if for any parabolic subgroup $P \subset G$, there is a finite number of double cosets in $P(F) \setminus G(F)/H(F)$.*

Remark 2.2.2. *If $charF = 0$ and G is quasi-split over F then (G, H) is an F-spherical pair if and only if it is a spherical pair of algebraic groups. However, we do not know whether this is true if $charF > 0$.*

Notation 2.2.3. *Let G be a reductive group and H be a subgroup. Let $P < G$ be a parabolic subgroup and M be its Levi quotient. We denote by H_M the image of $H \cap P$ under the projection $P \twoheadrightarrow M$.*

The following Lemma is the main step in the proof of Theorem A

Lemma 2.2.4. *Let (G, H) be an F-spherical pair. Let $P < G$ be a parabolic subgroup and M be its Levi quotient. Then*
(i) (M, H_M) is also an F-spherical pair.
(ii) Suppose also that for any smooth irreducible representation ρ of $G(F)$ we have

$$\dim \mathrm{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) < \infty.$$

Then for any irreducible cuspidal representation σ of $M(F)$ we have

$$\dim \mathrm{Hom}_{H_M(F)}(\sigma|_{H_M(F)}, \mathbb{C}) < \infty.$$

Remark 2.2.5. *One can show that the converse of (ii) is also true. Namely, if* $\dim \mathrm{Hom}_{H_M(F)}(\sigma|_{H_M(F)}, \mathbb{C}) < \infty$ *for any irreducible cuspidal representation* σ *of* $M(F)$ *for any Levi subquotient* M *then* $\dim \mathrm{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) < \infty$ *for any smooth irreducible representation* ρ *of* $G(F)$. *We will not prove this since we will not use this.*

We will need the following lemma.

Lemma 2.2.6. *Let* M *be an l-group and* V *be a smooth representation of* M. *Let* $0 = F^0V \subset ... \subset F^{n-1}V \subset F^nV = V$ *be a finite filtration of* V *by subrepresentations. Suppose that for any* i, *either*

$$\dim(F^iV/F^{i-1}V)_M = \infty$$

or

$$\textit{both } \dim(F^iV/F^{i-1}V)_M < \infty \textit{ and } \dim \mathrm{H}_1(M, (F^iV/F^{i-1}V)) < \infty.$$

Suppose also that $\dim V_M < \infty$. *Then* $\dim(F^iV/F^{i-1}V)_M < \infty$ *for any* i.

Proof. We prove by a decreasing induction on i that $\dim(F^iV)_M < \infty$, and, therefore, $\dim(F^iV/F^{i-1}V)_M < \infty$. Consider the short exact sequence

$$0 \to F^{i-1}V \to F^iV \to F^iV/F^{i-1}V \to 0,$$

and the corresponding long exact sequence

$$... \to \mathrm{H}_1(M, (F^iV/F^{i-1}V)) \to (F^{i-1}V)_M \to (F^iV)_M \to (F^iV/F^{i-1}V)_M \to 0.$$

In this sequence $\dim \mathrm{H}_1(M, (F^iV/F^{i-1}V)) < \infty$ and $\dim(F^iV)_M < \infty$, and hence $\dim(F^{i-1}V)_M < \infty$. ☐

Now we are ready to prove Lemma 2.2.4.

Proof of Lemma 2.2.4.
(i) is trivial.
(ii) Let $P < G$ be a parabolic subgroup, M be the Levi quotient of P and let ρ be a cuspidal representation of $M(F)$. We know that $\dim(i_{GM}\rho)_{H(F)} < \infty$ and we have to show that $\dim \rho_{H_M(F)} < \infty$.

Let $\mathcal{I}$ denote the natural $G(F)$-equivariant locally constant sheaf of complex vector spaces on $G(F)/P(F)$ such that $i_{GM}\rho \cong \mathcal{S}(G(F)/P(F), \mathcal{I})$. Let Y_j denote the $H(F)$ orbits on $G(F)/P(F)$. We know that there exists a natural filtration on $\mathcal{S}(G(F)/P(F), \mathcal{I})|_{H(F)}$ with associated graded components isomorphic to $\mathcal{S}(Y_j, \mathcal{I}_j)$, where $\mathcal{I}_j$ are $H(F)$- equivariant sheaves on Y_j corresponding to $\mathcal{I}$. For any j choose a representative $y_j \in Y_j$. Do it in such a way that there exists j_0 such that $y_{j_0} = [1]$. Let $P_j := G_{y_j}$ and M_j be its Levi quotient. Note that $P_{j_0} = P$ and $M_{j_0} = M$. Let ρ_j be the stalk of $\mathcal{I}_j$ at the point y_j. Clearly ρ_j is a cuspidal irreducible representation of $M_j(F)$ and $\rho_{j_0} = \rho$. By Shapiro Lemma (Lemma 2.1.12)

$$\mathrm{H}_i(H(F), \mathcal{S}(Y_j, \mathcal{I}_j)) \cong \mathrm{H}_i((H \cap P_j)(F), \rho_j).$$

By Lemma 2.1.11 either $\dim \mathrm{H}_0((H \cap P_j)(F), \rho_j) = \infty$ or both $\dim \mathrm{H}_0((H \cap P_j)(F), \rho_j) < \infty$ and $\dim \mathrm{H}_1((H \cap P_j)(F), \rho_j) < \infty$. Hence by Lemma 2.2.6 $\dim \mathrm{H}_0((H \cap P_j)(F), \rho_j) < \infty$ and hence $\dim \rho_{H_M(F)} < \infty$. ☐

2.3. **Proof of Theorem A.**

In this subsection we prove Theorem A. Let us remind its formulation.

Theorem 2.3.1. *Let* (G, H) *be an* F*-spherical pair. Suppose that for any irreducible smooth representation* ρ *of* $G(F)$ *we have*

$$\dim \mathrm{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) < \infty. \tag{2}$$

Then for any compact open subgroup $K < G(F)$, $\mathcal{S}(G(F)/H(F))^K$ *is a finitely generated module over the Hecke algebra* $\mathcal{H}_K(G(F))$.

Remark 2.3.2. *Conjecturally, any F-spherical pair satisfies the condition (2). In* [Del] *and in* [SV] *this is proven for wide classes of spherical pairs, which include all symmetric pairs over fields of characteristic different from 2.*

We will need several lemmas and definitions.

Lemma 2.3.3. *Let (G,H) be an F-spherical pair, and denote $\widetilde{H} = H(F)Z(G(F)) \cap G^1$. Suppose that for any smooth (respectively cuspidal) irreducible representation ρ of $G(F)$ we have $\dim \mathrm{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) < \infty$. Then for any smooth (respectively cuspidal) irreducible representation ρ of $G(F)$ and for every character $\widetilde{\chi}$ of $\widetilde{H}$ whose restriction to $H(F) \cap G^1$ is trivial, we have*

$$\dim \mathrm{Hom}_{\widetilde{H}}(\rho|_{\widetilde{H}}, \widetilde{\chi}) < \infty.$$

Proof. Let ρ be a smooth (respectively cuspidal) irreducible representation of $G(F)$, and let $\widetilde{\chi}$ be a character of $\widetilde{H}$ whose restriction to $H(F) \cap G^1$ is trivial.

$$\mathrm{Hom}_{\widetilde{H}}\left(\rho|_{\widetilde{H}}, \widetilde{\chi}\right) = \mathrm{Hom}_{(H(F)Z(G(F)))\cap G_0}\left(\rho|_{(H(F)Z(G(F)))\cap G_0}, Ind_{\widetilde{H}}^{(H(F)Z(G(F)))\cap G_0}\widetilde{\chi}\right).$$

Since

$$H(F)Z(G(F)) \cap G_0 = \widetilde{H}Z(G(F)) \cap G_0 = \widetilde{H}Z(G(F)),$$

the subspace of $Ind_{\widetilde{H}}^{(H(F)Z(G(F)))\cap G_0}\widetilde{\chi}$ that transforms under $Z(G(F))$ according to the central character of ρ is at most one dimensional. If this subspace is trivial, then the lemma is clear. Otherwise, denote it by τ. Since $H(F) \cap G^1$ is normal in $H(F)Z(G(F))$, we get that the restriction of $Ind_{\widetilde{H}}^{(H(F)Z(G(F)))\cap G_0}\widetilde{\chi}$ to $H(F) \cap G^1$ is trivial, and hence that $\tau|_{H(F)\cap G^1}$ is trivial. Hence $\mathrm{Hom}_{\widetilde{H}}\left(\rho|_{\widetilde{H}}, \widetilde{\chi}\right)$ is equal to

$$\mathrm{Hom}_{(H(F)Z(G(F)))\cap G_0}\left(\rho|_{(H(F)Z(G(F)))\cap G_0}, \tau\right) = \mathrm{Hom}_{H(F)\cap G_0}\left(\rho|_{H(F)\cap G_0}, \tau|_{H(F)\cap G_0}\right) =$$
$$= \mathrm{Hom}_{H(F)}\left(\rho|_{H(F)}, Ind_{H(F)\cap G_0}^{H(F)}\tau|_{H(F)\cap G_0}\right).$$

Since $H(F)/H(F) \cap G_0$ is finite and abelian, the representation $Ind_{H(F)\cap G_0}^{H(F)}\tau|_{H(F)\cap G_0}$ is a finite direct sum of characters of $H(F)$, the restrictions of all to $H(F)\cap G^1$ are trivial. Any character θ of $H(F)$ whose restriction to $H(F) \cap G^1$ is trivial can be extended to a character of $G(F)$, because $H(F)/(H(F) \cap G^1)$ is a sub-lattice of $G(F)/G^1$. Denoting the extension by Θ, we get that

$$\mathrm{Hom}_{H(F)}\left(\rho|_{H(F)}, \theta\right) = \mathrm{Hom}_{H(F)}\left((\rho \otimes \Theta^{-1})|_{H(F)}, \mathbb{C}\right),$$

but $\rho \otimes \Theta^{-1}$ is again smooth (respectively cuspidal) irreducible representation of $G(F)$, so this last space is finite-dimensional.

□

Lemma 2.3.4. *Let A be a commutative unital Noetherian algebra without zero divisors and let K be its field of fractions. Let $K^{\mathbb{N}}$ be the space of all sequences of elements of K. Let V be a finite dimensional subspace of $K^{\mathbb{N}}$ and let $M := V \cap A^{\mathbb{N}}$. Then M is finitely generated.*

Proof. Since A does not have zero divisors, M injects into $K^{\mathbb{N}}$. There is a number n such that the projection of V to $K^{\{1,\cdots n\}}$ is injective. Therefore, M injects into $A^{\{1,\cdots n\}}$, and, since A is Noetherian, M is finitely generated. □

Lemma 2.3.5. *Let M be an l-group, let $L \subset M$ be a closed subgroup, and let $L' \subset L$ be an open normal subgroup of L such that L/L' is a lattice. Let ρ be a smooth representation of M of countable dimension. Suppose that for any character χ of L whose restriction to L' is trivial we have*

$$\dim \mathrm{Hom}_L(\rho|_L, \chi) < \infty.$$

Consider $\mathrm{Hom}_{L'}(\rho, \mathcal{S}(L/L'))$ as a representation of L, where L acts by $((hf)(x))([y]) = (f(x))([yh])$. Then this representation is finitely generated.

Proof. By assumption, the action of L on $\mathrm{Hom}_{L'}(\rho, \mathcal{S}(L/L'))$ factors through L/L'. Since L/L' is discrete, $\mathcal{S}(L/L')$ is the group algebra $\mathbb{C}[L/L']$. We want to show that $\mathrm{Hom}_{L'}(\rho, \mathbb{C}[L/L'])$ is a finitely generated module over $\mathbb{C}[L/L']$.

Let $\mathbb{C}(L/L')$ be the fraction field of $\mathbb{C}[L/L']$. Choosing a countable basis for the vector space of ρ, we can identify any $\mathbb{C}$-linear map from ρ to $\mathbb{C}[L/L']$ with an element of $\mathbb{C}[L/L']^{\mathbb{N}}$. Moreover, the condition that the map intertwines the action of L/L' translates into a collection of linear equations that the tuple in $\mathbb{C}[L/L']^{\mathbb{N}}$ should satisfy. Hence, $\mathrm{Hom}_{L'}(\rho, \mathbb{C}[L/L'])$ is the intersection of the $\mathbb{C}(L/L')$-vector space $\mathrm{Hom}_{L'}(\rho, \mathbb{C}(L/L'))$ and $\mathbb{C}[L/L']^{\mathbb{N}}$. By Lemma 2.3.4, it suffices to prove that $\mathrm{Hom}_{L'}(\rho, \mathbb{C}(L/L'))$ is finite dimensional over $\mathbb{C}(L/L')$.

Since L' is separable, and ρ is smooth and of countable dimension, there are only countably many linear equations defining $\mathrm{Hom}_{L'}(\rho, \mathbb{C}(L/L'))$; denote them by $\phi_1, \phi_2, \ldots \in \left(\mathbb{C}(L/L')^{\mathbb{N}}\right)^*$. Choose a countable subfield $K \subset \mathbb{C}$ that contains all the coefficients of the elements of $\mathbb{C}(L/L')$ that appear in any of the ϕ_i's. If we define W as the $K(L/L')$-linear subspace of $K(L/L')^{\mathbb{N}}$ defined by the ϕ_i's, then $\mathrm{Hom}_{L'}(\rho, \mathbb{C}(L/L')) = W \otimes_{K(L/L')} \mathbb{C}(L/L')$, so $\dim_{\mathbb{C}(L/L')} \mathrm{Hom}_{L'}(\rho, \mathbb{C}(L/L')) = \dim_{K(L/L')} W$.

Since L/L' is a lattice generated by, say, $g_1, \ldots, g_n$, we get that $K(L/L') = K(t_1^{\pm 1}, \ldots, t_n^{\pm 1}) = K(t_1, \ldots, t_n)$. Choosing elements $\pi_1, \ldots, \pi_n \in \mathbb{C}$ such that $tr.deg_K(K(\pi_1, \ldots, \pi_n)) = n$, we get an injection ι of $K(L/L')$ into $\mathbb{C}$. As before, we get that if we denote the $\mathbb{C}$-vector subspace of $\mathbb{C}^{\mathbb{N}}$ cut by the equations $\iota(\phi_i)$ by U, then $\dim_{K(L/L')} W = \dim_{\mathbb{C}} U$. However, U is isomorphic to $\mathrm{Hom}_{L'}(\rho, \chi)$, where χ is the character of L/L' such that $\chi(g_i) = \pi_i$. By assumption, this last vector space is finite dimensional. □

Now we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. By Lemma 2.1.10 it is enough to show that for any parabolic $P < G$ and any irreducible cuspidal representation ρ of $M(F)$ (where M denotes the Levi quotient of P), $\mathrm{Hom}(i_{GM}(\Psi(\rho)), \mathcal{S}(G(F)/H(F)))$ is a finitely generated module over $\mathcal{O}(\Psi_M)$.

Step 1. Proof for the case $P = G$.
We have

$$\mathrm{Hom}_{G(F)}(i_{GM}(\Psi(\rho)), \mathcal{S}(G(F)/H(F))) = \mathrm{Hom}_{G(F)}(\Psi(\rho), \mathcal{S}(G(F)/H(F))) = \mathrm{Hom}_{G^1}(\rho, \mathcal{S}(G(F)/H(F))).$$

Here we consider the space $\mathrm{Hom}_{G^1}(\rho, \mathcal{S}(G(F)/H(F)))$ with

the natural action of G. Note that G^1 acts trivially and hence this action gives rise to an action of G/G^1, which gives the $\mathcal{O}(\Psi_G)$ - module structure.

Now consider the subspace

$$V := \mathrm{Hom}_{G^1}(\rho, \mathcal{S}(G^1/(H(F) \cap G^1))) \subset \mathrm{Hom}_{G^1}(\rho, \mathcal{S}(G(F)/H(F))).$$

It generates $\mathrm{Hom}_{G^1}(\rho, \mathcal{S}(G(F)/H(F)))$ as a representation of $G(F)$, and therefore also as an $\mathcal{O}(\Psi_G)$ - module. Note that V is $H(F)$ invariant. Therefore it is enough to show that V is finitely generated over $H(F)$. Denote $H' := H(F) \cap G^1$ and $H'' := (H(F)Z(G(F))) \cap G^1$. Note that

$$\mathcal{S}(G^1/H') \cong ind_{H''}^{G^1}(\mathcal{S}(H''/H')) \subset Ind_{H''}^{G^1}(\mathcal{S}(H''/H')).$$

Therefore V is canonically embedded into $\mathrm{Hom}_{H''}(\rho, \mathcal{S}(H''/H'))$. The action of H on V is naturally extended to an action Π on $\mathrm{Hom}_{H''}(\rho, \mathcal{S}(H''/H'))$ by

$$((\Pi(h)(f))(v))([k]) = f(h^{-1}v)([h^{-1}kh]).$$

Let Ξ be the action of H'' on $\mathrm{Hom}_{H''}(\rho, \mathcal{S}(H''/H'))$ as described in Lemma 2.3.5, i.e.

$$((\Xi(h)(f))(v))([k]) = f(v)([kh]).$$

By Lemmas 2.3.5 and 2.3.3 it is enough to show that for any $h \in H''$ there exist an $h' \in H$ and a scalar α s.t.

$$\Xi(h) = \alpha \Pi(h').$$

In order to show this let us decompose h to a product $h = zh'$ where $h' \in H$ and $z \in Z(G(F))$. Now

$$((\Xi(h)(f))(v))([k]) = f(v)([kh]) = f(h^{-1}v)([h^{-1}kh]) = f(h^{'-1}z^{-1}v)([h^{'-1}kh']) = \\ = \alpha f(h'^{-1}v)([h^{'-1}kh']) = \alpha((\Pi(h')(f))(v))([k]),$$

where α is the scalar with which z^{-1} acts on ρ.

Step 2. Proof in the general case.

$$\mathrm{Hom}_{G(F)}(i_{GM}(\Psi(\rho)), \mathcal{S}(G(F)/H(F))) = \mathrm{Hom}_{M(F)}(\Psi(\rho), \overline{r}_{MG}(\mathcal{S}(G(F)/H(F)))) = \\ = \mathrm{Hom}_{M(F)}(\Psi(\rho), ((\mathcal{S}(G(F)/H(F)))|_{\overline{P}(F)})_{\overline{U}(F)}),$$

where $\overline{U}$ is the unipotent radical of $\overline{P}$, the parabolic opposite to P. Let $\{Y_i\}_{i=1}^n$ be the orbits of $\overline{P}(F)$ on $G(F)/H(F)$. We know that there exists a filtration on $(\mathcal{S}(G(F)/H(F)))|_{\overline{P}(F)}$ such that the associated graded components are isomorphic to $\mathcal{S}(Y_i)$. Consider the corresponding filtration on $((\mathcal{S}(G(F)/H(F)))|_{\overline{P}(F)})_{\overline{U}(F)}$. Let V_i be the associated graded components of this filtration. We have a natural surjection $\mathcal{S}(Y_i)_{\overline{U}} \twoheadrightarrow V_i$. In order to prove that $\mathrm{Hom}_{M(F)}(\Psi(\rho), ((\mathcal{S}(G(F)/H(F)))|_{\overline{P}(F)})_{\overline{U}(F)})$ is finitely generated it is enough to prove that $\mathrm{Hom}_{M(F)}(\Psi(\rho), V_i)$ is finitely generated. Since $\Psi(\rho)$ is a projective object of $\mathcal{M}(M(F))$ (by Corollary 2.1.8), it is enough to show that $\mathrm{Hom}_{M(F)}(\Psi(\rho), \mathcal{S}(Y_i)_{\overline{U}(F)})$ is finitely generated. Denote $Z_i := \overline{U}(F) \setminus Y_i$. It is easy to see that $Z_i \cong M(F)/((H_i)_M(F))$, where H_i is some conjugation of H. Now the assertion follows from the previous step using Lemma 2.2.4. □

2.4. **Homologies of l-groups.**

The goal of this subsection is to prove Lemma 2.1.11 and Lemma 2.1.12.

We start with some generalities on abelian categories.

Definition 2.4.1. *Let $\mathcal{C}$ be an abelian category. We call a family of objects $\mathcal{A} \subset Ob(\mathcal{C})$* ***generating*** *family if for any object $X \in Ob(\mathcal{C})$ there exists an object $Y \in \mathcal{A}$ and an epimorphism $Y \twoheadrightarrow X$.*

Definition 2.4.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a right-exact additive functor. A family of objects $\mathcal{A} \subset Ob(\mathcal{C})$ is called* ***$\mathcal{F}$-adapted*** *if it is generating , closed under direct sums and for any exact sequence $0 \to A_1 \to A_2 \to ...$ with $A_i \in \mathcal{A}$, the sequence $0 \to \mathcal{F}(A_1) \to \mathcal{F}(A_2) \to ...$ is also exact.*

For example, a generating, closed under direct sums system consisting of projective objects is $\mathcal{F}$-adapted for any right-exact functor $\mathcal{F}$. For an l-group G the system of objects consisting of direct sums of copies of $\mathcal{S}(G)$ is an example of such system.

The following results are well-known.

Theorem 2.4.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a right-exact additive functor. Suppose that there exists an $\mathcal{F}$-adapted family $\mathcal{A} \subset Ob(\mathcal{C})$. Then $\mathcal{F}$ has derived functors.*

Lemma 2.4.4. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ be right-exact additive functors. Suppose that both $\mathcal{F}$ and $\mathcal{G}$ have derived functors.*

(i) Suppose that $\mathcal{F}$ is exact. Suppose also that there exists a class $\mathcal{E} \subset Ob(\mathcal{A})$ which is $\mathcal{G} \circ \mathcal{F}$-adapted and such that $\mathcal{F}(X)$ is $\mathcal{G}$-acyclic for any $X \in \mathcal{E}$. Then the functors $L^i(\mathcal{G} \circ \mathcal{F})$ and $L^i\mathcal{G} \circ \mathcal{F}$ are isomorphic.

(ii) Suppose that there exists a class $\mathcal{E} \subset Ob(\mathcal{A})$ which is $\mathcal{G} \circ \mathcal{F}$-adapted and $\mathcal{F}$-adapted and such that $\mathcal{F}(X)$ is $\mathcal{G}$-acyclic for any $X \in \mathcal{E}$. Let $Y \in \mathcal{A}$ be an $\mathcal{F}$-acyclic object. Then $L^i(\mathcal{G} \circ \mathcal{F})(Y)$ is (naturally) isomorphic to $L^i\mathcal{G}(\mathcal{F}(Y))$.

(iii) Suppose that $\mathcal{G}$ is exact. Suppose that there exists a class $\mathcal{E} \subset Ob(\mathcal{A})$ which is $\mathcal{G} \circ \mathcal{F}$-adapted and $\mathcal{F}$-adapted. Then the functors $L^i(\mathcal{G} \circ \mathcal{F})$ and $\mathcal{G} \circ L^i\mathcal{F}$ are isomorphic.

Definition 2.4.5. *Let G be an l-group. For any smooth representation V of G denote* $\mathrm{H}_i(G,V) := L^i CI_G(V)$. *Recall that CI_G denotes the coinvariants functor.*

Proof of Lemma 2.1.12. Note that $\mathcal{F}(X) = ind_{G_x}^G \mathcal{F}_x$. Note also that $ind_{G_x}^G$ is an exact functor, and $CI_{G_x} = CI_G \circ ind_{G_x}^G$. The lemma follows now from Lemma 2.4.4(i). □

Lemma 2.4.6. *Let L be a lattice. Let V be a linear space. Let L act on V by a character. Then*

$$\mathrm{H}_1(L,V) = \mathrm{H}_0(L,V) \otimes_{\mathbb{C}} (L \otimes_{\mathbb{Z}} \mathbb{C}).$$

The proof of this lemma is straightforward.

Lemma 2.4.7. *Let L be an l-group and $L' < L$ be a subgroup. Then*
(i) for any representation V of L we have

$$\mathrm{H}_i(L',V) = L^i\mathcal{F}(V),$$

where $\mathcal{F} : \mathcal{M}(L) \to Vect$ is the functor defined by $\mathcal{F}(V) = V_{L'}$.
(ii) Suppose that L' is normal. Let $\mathcal{F}' : \mathcal{M}(L) \to \mathcal{M}(L/L')$ be the functor defined by $\mathcal{F}'(V) = V_{L'}$. Then for any representation V of L we have $\mathrm{H}_i(L',V) = L^i\mathcal{F}'(V)$.

Proof. (i) Consider the restriction functor $Res^L_{L'} : \mathcal{M}(L) \to \mathcal{M}(L')$. Note that it is exact. Consider also the functor $\mathcal{G} : \mathcal{M}(L') \to Vect$ defined by $\mathcal{G}(\rho) := \rho_{L'}$. Note that $\mathcal{F} = \mathcal{G} \circ Res^L_{L'}$. The assertion follows now from Lemma 2.4.4(i) using the fact that $\mathcal{S}(L)$ is a projective object in $\mathcal{M}(L')$.
(ii) follows from (i) in a similar way, but using part (iii) of Lemma 2.4.4 instead part (i). □

Lemma 2.4.8. *Let G be a reductive group and $H < G$ be a subgroup. Consider the functor*

$$\mathcal{F} : \mathcal{M}(G(F)) \to \mathcal{M}(H(F)/(H(F) \cap G^1)) \text{ defined by } \mathcal{F}(V) = V_{H(F)\cap G^1}.$$

Then any finitely generated cuspidal representation of $G(F)$ is an $\mathcal{F}$-acyclic object.

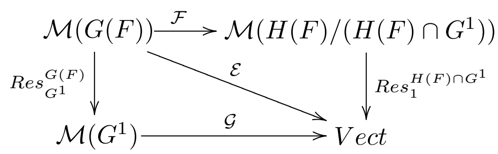
Proof. Consider the restriction functors

$$Res_1^{H(F)/(H(F)\cap G^1)} : \mathcal{M}(H(F)/(H(F) \cap G^1)) \to Vect$$

and

$$Res_{G^1}^{G(F)} : \mathcal{M}(G(F)) \to \mathcal{M}(G^1).$$

Note that they are exact. Consider also the functor $\mathcal{G} : \mathcal{M}(G^1) \to Vect$ defined by $\mathcal{G}(\rho) := \rho_{G^1\cap H(F)}$. Denote $\mathcal{E} := \mathcal{G} \circ Res_{G^1}^{G(F)}$. Note that $\mathcal{E} = Res_1^{H(F)/(H(F)\cap G^1)} \circ \mathcal{F}$.



Let π be a cuspidal finitely generated representation of $G(F)$. By Corollary 2.1.8, $Res_{G^1}^{G(F)}(\pi)$ is projective and hence $\mathcal{G}$-acyclic. Hence by Lemma 2.4.4(ii) π is $\mathcal{E}$-acyclic. Hence by Lemma 2.4.4(iii) π is $\mathcal{F}$-acyclic.

□

Lemma 2.4.9. *Let L be an l-group and $L' < L$ be a normal subgroup. Suppose that* $\mathrm{H}_i(L',\mathbb{C}) = 0$ *for all $i > 0$. Let ρ be a representation of L/L'. Denote by $Ext(\rho)$ the natural representation of L obtained from ρ. Then* $\mathrm{H}_i(L/L',\rho) = \mathrm{H}_i(L, Ext(\rho))$.

Proof. Consider the coinvariants functors $\mathcal{E} : \mathcal{M}(L) \to Vect$ and $\mathcal{F} : \mathcal{M}(L/L') \to Vect$ defined by $\mathcal{E}(V) := V_L$ and $\mathcal{F}(V) := V_{L/L'}$. Note that $\mathcal{F} = \mathcal{E} \circ Ext$ and Ext is exact. By Shapiro Lemma (Lemma 2.1.12), $\mathcal{S}(L/L')$ is acyclic with respect to both $\mathcal{E}$ and $\mathcal{F}$. The lemma follows now from Lemma 2.4.4(ii). $\square$

Remark 2.4.10. *Recall that if $L' = N(F)$ where N is a unipotent algebraic group, then $\mathrm{H}_i(L') = 0$ for all $i > 0$.*

Now we are ready to prove Lemma 2.1.11

Proof of Lemma 2.1.11. By Lemma 2.4.9 we can assume that G is reductive.

Let $\mathcal{F} : \mathcal{M}(G(F)) \to Vect$ be the functor defined by $\mathcal{F}(V) := V_{H(F)}$. Let

$$\mathcal{G} : \mathcal{M}(G(F)) \to \mathcal{M}(H(F)/(H(F) \cap G^1))$$

be defined by

$$\mathcal{G}(V) := V_{H(F) \cap G^1}.$$

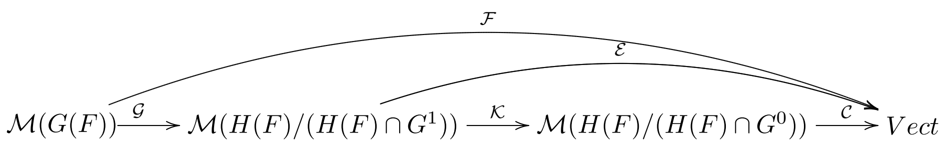
Let

$$\mathcal{E} : \mathcal{M}(H(F)/(H(F) \cap G^1)) \to Vect$$

be defined by

$$\mathcal{E}(V) := V_{H(F)/(H(F) \cap G^1)}.$$

Clearly, $\mathcal{F} = \mathcal{E} \circ \mathcal{G}$. By Lemma 2.4.8, ρ is $\mathcal{G}$-acyclic. Hence by Lemma 2.4.4(ii), $L^i\mathcal{F}(\rho) = L^i\mathcal{E}(\mathcal{G}(\rho))$.



Consider the coinvariants functors $\mathcal{K} : \mathcal{M}(H(F)/(H(F) \cap G^1)) \to \mathcal{M}(H(F)/(H(F) \cap G^0))$ and $\mathcal{C} : \mathcal{M}(H(F)/(H(F) \cap G^0)) \to Vect$ defined by $\mathcal{K}(\rho) := \rho_{(H(F) \cap G_0)/(H(F) \cap G^1)}$ and $\mathcal{C}(\rho) := \rho_{H(F)/(H(F) \cap G^1)}$. Note that $\mathcal{E} = \mathcal{C} \circ \mathcal{K}$.

Note that $\mathcal{C}$ is exact since the group $H(F)/(H(F) \cap G^1)$ is finite. Hence by Lemma 2.4.4(iii), $L^i\mathcal{E} = \mathcal{C} \circ L^i\mathcal{K}$.

Now, by Lemma 2.4.7,

$$\mathrm{H}_i(H(F), \rho) = L^i\mathcal{F}(\rho) = L^i\mathcal{E}(\mathcal{G}(\rho)) = \mathcal{C}(L^i\mathcal{K}(\mathcal{G}(\rho))) = \mathcal{C}(\mathrm{H}_i((H(F) \cap G_0)/(H(F) \cap G^1), \mathcal{G}(\rho))).$$

Hence, by Lemma 2.4.6, if $\mathrm{H}_0(H(F), \rho)$ is finite dimensional then $\mathrm{H}_1(H(F), \rho)$ is finite dimensional. $\square$

3. Uniform Spherical Pairs

In this section we introduce the notion of uniform spherical pair and prove Theorem B.

We follow the main steps of [Kaz86], where the author constructs an isomorphism between the Hecke algebras of a reductive group over close enough local fields. First, he constructs a linear isomorphism between the Hecke algebras, using Cartan decomposition. Then, he shows that for two given elements of the Hecke algebra there exists m such that if the fields are m-close then the product of those elements will be mapped to the product of their images. Then he uses the fact that the Hecke algebras are finitely generated and finitely presented to deduce the theorem.

Roughly speaking, we call a pair $H < G$ of reductive groups a uniform spherical pair if it possesses a relative analog of Cartan decomposition, i.e. a "nice" description of the set of double cosets $K_0(G, F) \setminus G(F)/H(F)$ which in some sense does not depend on F. We give the precise definition in the first subsection and prove Theorem B in the second subsection.

3.1. **Definitions.**

Let R be a complete and smooth local ring, let m denote its maximal ideal, and let π be an element in $m \setminus m^2$. A good example to keep in mind is the ring $\mathbb{Z}_p[[\pi]]$. An (R,π)-local field is a local field F together with an epimorphism of rings $R \to O_F$, such that the image of π (which we will continue to denote by π) is a uniformizer. Denote the collection of all (R,π)-local fields by $\mathcal{F}_{R,\pi}$.

Suppose that G is a reductive group defined and split over R. Let T be a fixed split torus, and let $X_*(T)$ be the coweight lattice of T. For every $\lambda \in X_*(T)$ and every (R,π)-local field F, we write $\pi^\lambda = \lambda(\pi) \in T(F) \subset G(F)$. We denote the subgroup $G(O_F)$ by $K_0(F)$, and denote its ℓ'th congruence subgroup by $K_\ell(F)$.

Definition 3.1.1. *Let F be a local field. Let $X \subset \mathbb{A}^n_{O_F}$ be a closed subscheme. For any $x, y \in X(F)$, define the valuative distance between x and y to be $val_F(x,y) := \min\{val_F(x_i - y_i)\}$. Also, for any $x \in X(F)$, define $val_F(x) := \min\{val_F(x_i)\}$. The ball of valuative radius ℓ around a point x in $X(F)$ will be denoted by $B(x,\ell)(F)$.*

Definition 3.1.2. *Let G be a split reductive group defined over R and let $H \subset G$ be a smooth reductive subgroup defined over R. We say that the pair (G,H) is uniform spherical if there are*

- *An R-split torus $T \subset G$,*
- *An affine embedding $G/H \hookrightarrow \mathbb{A}^n$.*
- *A finite subset $\mathfrak{X} \subset G(R)/H(R)$.*
- *A subset $\Upsilon \subset X_*(T)$.*

such that

(1) *The map $x \mapsto K_0(F)x$ from $\pi^\Upsilon \mathfrak{X}$ to $K_0(F)\backslash G(F)/H(F)$ is onto for every $F \in \mathcal{F}_{R,\pi}$.*
(2) *For every $x, y \in \pi^\Upsilon \mathfrak{X} \subset (G/H)(R[\pi^{-1}])$, the closure in G of the $R[\pi^{-1}]$-scheme*
$$T_{x,y} := \{g \in G \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[\pi^{-1}]) | gx = y\}$$
is smooth over R. We denote this closure by $S_{x,y}$.
(3) *For every $x \in \pi^\Upsilon \mathfrak{X}$, the valuation $val_F(x)$ does not depend on $F \in \mathcal{F}_{R,\pi}$.*
(4) *There exists l_0 s.t. for any $l > l_0$, for any $F \in \mathcal{F}_{R,\pi}$ and for every $x \in \mathfrak{X}$ and $\alpha \in \Upsilon$ we have $K_l \pi^\alpha K_l x = K_l \pi^\alpha x$.*

If G, H are defined over $\mathbb{Z}$, we say that the pair (G,H) is uniform spherical if, for every R as above, the pair $(G \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R), H \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R))$ is uniform spherical.

In Section 4 we give two examples of uniform spherical pairs. We will list now several basic properties of uniform spherical pairs. In light of the recent developments in the structure theory of symmetric and spherical pairs (e.g. [Del], [SV]), we believe that the majority of symmetric pairs and many spherical pairs defined over local fields are specializations of appropriate uniform spherical pairs.

From now and until the end of the section we fix a uniform spherical pair (G,H). First note that, since H is smooth, the fibers of $G \to G/H$ are smooth. Hence the map $G \to G/H$ is smooth.

Lemma 3.1.3. *Let (G,H) be a uniform spherical pair. Let $x, y \in \pi^\Upsilon \mathfrak{X}$. Let F be an (R,π)-local field. Then*
$$S_{x,y}(O_F) = T_{x,y}(F) \cap G(O).$$

Proof. The inclusion $S_{x,y}(O_F) \subset T_{x,y}(F) \cap G(O_F)$ is evident. In order to prove the other inclusion we have to show that any map $\psi : \mathrm{Spec}(O_F) \to G \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, O_F$ such that $\mathrm{Im}(\psi|_{\mathrm{Spec}\, F}) \subset T_{x,y} \times_{\mathrm{Spec}\, R[\pi^{-1}]} \mathrm{Spec}\, F$ satisfies $\mathrm{Im}\, \psi \subset S_{x,y} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, O_F$.

This holds since $S_{x,y} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, O_F$ lies in the closure of $T_{x,y} \times_{\mathrm{Spec}\, R[\pi^{-1}]} \mathrm{Spec}\, F$ in $G \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, O_F$. $\square$

Lemma 3.1.4. *If (G,H) is uniform spherical, then there is a subset $\Delta \subset \pi^\Upsilon \mathfrak{X}$ such that, for every $F \in \mathcal{F}_{R,\pi}$, the map $x \mapsto K_0(F)x$ is a bijection between Δ and $K_0(F)\backslash G(F)/H(F)$.*

Proof. It is enough to show that for any $F, F' \in \mathcal{F}_{R,\pi}$ and for any $x, y \in \pi^\Upsilon \mathfrak{X}$, the equality $K_0(F)x = K_0(F)y$ is equivalent to $K_0(F')x = K_0(F')y$.

The scheme $S_{x,y} \otimes O_F$ is smooth over R, and hence it is smooth over O_F. Therefore, it is formally smooth. This implies that the map $S_{x,y}(O_F) \to S_{x,y}(\mathbb{F}_q)$ is onto and hence $\{g \in G(O_F) | gx = y\} \neq \emptyset$ if and only if $S_{x,y}(\mathbb{F}_q) \neq \emptyset$.

Hence, the two equalities $K_0(F)x = K_0(F)y$ and $K_0(F')x = K_0(F')y$ are equivalent to $S_{x,y}(\mathbb{F}_q) \neq \emptyset$. □

From now untill the end of the section we fix Δ as in the lemma.

Proposition 3.1.5. *If (G,H) is uniform spherical, then for every $x \in \pi^\Upsilon \mathfrak{X}$ and every $\ell \in \mathbb{N}$, there is $M \in \mathbb{N}$ such that for every $F \in \mathcal{F}_{R,\pi}$, the set $K_\ell(F)x$ contains a ball of radius M around x.*

Proof. Since, for every $\delta \in X_*(T)$ and every $\ell \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $K_n(F) \subset \pi^\delta K_\ell(F)\pi^{-\delta}$ for every F, we can assume that $x \in \mathfrak{X}$. The claim now follows from the following version of the implicit function theorem.

Lemma 3.1.6. *Let F be a local field. Let X and Y be affine schemes defined over O_F. Let $\psi : X \to Y$ be a smooth morphism defined over O_F. Let $x \in X(O_F)$ and $y := \psi(x)$. Then $\psi(B(x,\ell)(F)) = B(y,\ell)(F)$ for any natural number l.*

Proof. The inclusion $\psi(B(x,\ell)(F)) \subset B(y,\ell)(F)$ is clear. We prove the inclusion $\psi(B(x,\ell)(F)) \supset B(y,\ell)(F)$.
Case 1: X and Y are affine spaces and ψ is etale. The proof is standard.
Case 2: $X = \mathbb{A}^m$, ψ is etale: We can assume that $Y \subset \mathbb{A}^{m+n}$ is defined by $f_1, \ldots, f_n$ with independent differentials, and that ψ is the projection. The proof in this case follows from Case 1 by considering the map $F : \mathbb{A}^{m+n} \to \mathbb{A}^{m+n}$ given by $F(x_1, \ldots, x_{m+n}) = (x_1, \ldots, x_m, f_1, \ldots, f_n)$.
Case 3: ψ is etale: Follows from Case 2 by restriction from the ambient affine spaces.
Case 4: In general, a smooth morphism is a composition of an etale morphism and a projection, for which the claim is trivial. □

□

Lemma 3.1.7. *For every $\lambda \in X_*(T)$ and $x \in \pi^\Upsilon \mathfrak{X}$, there is a finite subset $B \subset \pi^\Upsilon \mathfrak{X}$ such that $\pi^\lambda K_0(F)x \subset \bigcup_{y \in B} K_0(F)y$ for all $F \in \mathcal{F}_{R,\pi}$.*

Proof. By Lemma 3.1.4, we can assume that the sets $K_0(F)\pi^\lambda x_0$ for $\lambda \in \Upsilon$ are disjoint. There is a constant C such that for every F and for every $g \in \pi^\lambda K_0(F)\pi^\delta$, $val_F(gx_0) \geq C$. Fix F and assume that $g \in K_0(F)\pi^\lambda K_0(F)\pi^\delta$. From the proof of Proposition 3.1.5, it follows that $K_0(F)gx_0$ contains a ball whose radius depends only on λ, δ. Since F is locally compact, there are only finitely many disjoint such balls in the set $\{x \in G(F)/H(F) \,|\, val_F(x) \geq C\}$, so there are only finitely many $\eta \in \Upsilon$ such that $val_F(\pi^\lambda x_0) \geq C$. By definition, this finite set, S, does not depend on the field F. Therefore, $\pi^\lambda K_0(F)\pi^\delta x_0 \subset \bigcup_{\eta \in S} K_0(F)\pi^\eta x_0$. □

Notation 3.1.8.

- *Denote by $\mathcal{M}_\ell(G(F)/H(F))$ the space of $K_\ell(F)$-invariant compactly supported measures on $G(F)/H(F)$.*
- *For a K_l invariant subset $U \subset G(F)/H(F)$ we denote by $1_U \in \mathcal{M}_\ell(G(F)/H(F))$ the Haar measure on $G(F)/H(F)$ multiplied by the characteristic function of U and normalized s.t. its integral is 1. We define in a similar way $1_V \in \mathcal{H}_\ell(G,F)$ for a K_l-double invariant subset $V \subset G(F)$.*

Proposition 3.1.9. *If (G,H) is uniform spherical then $\mathcal{M}_\ell(G(F)/H(F))$ is finitely generated over $\mathcal{H}_\ell(G,F)$ for any ℓ.*

Proof. As in step 4 of Lemma 2.1.10, it is enough to prove the assertion for large enough l. Thus we may assume that for every $x \in \mathfrak{X}$ and $\alpha \in \Upsilon$ we have $K_l\pi^\alpha K_l x = K_l \pi^\alpha x$. Therefore, $1_{K_l\pi^\alpha K_l}1_{K_l x} = 1_{K_l\pi^\alpha x}$. Hence for any $g \in K_0/K_l$ we have $(g1_{K_l\pi^\alpha K_l})1_{K_l x} = 1_{gK_l\pi^\alpha x}$. Now, the elements $1_{gK_l\pi^\alpha x}$ span $\mathcal{M}_\ell(G(F)/H(F))$ by condition 1 in definition 3.1.2. This implies that the elements $1_{K_l x}$ generate $\mathcal{M}_\ell(G(F)/H(F))$. □

3.2. **Close Local Fields.**

Definition 3.2.1. *Two (R,π)-local fields $F,E \in \mathcal{F}_{R,\pi}$, are n-close if there is an isomorphism $\phi_{E,F}$: $O_F/\pi^n \to O_E/\pi^n$ such that the two maps $R \to O_F \to O_F/\pi^n \to O_E/\pi^n$ and $R \to O_E \to O_E/\pi^n$ coincide. In this case, ϕ is unique.*

Theorem 3.2.2 ([Kaz86])**.** *Let F be an (R,π) local field. Then, for any ℓ, there exists n such that, for any $E \in \mathcal{F}_{R,\pi}$, which is n-close to F, there exists a unique isomorphism $\Phi_{\mathcal{H},\ell}$ between the algebras $\mathcal{H}_\ell(G,F)$ and $\mathcal{H}_\ell(G,E)$ that maps the Haar measure on $K_\ell(F)\pi^\lambda K_\ell(F)$ to the Haar measure on $K_\ell(E)\pi^\lambda K_\ell(E)$, for every $\lambda \in X_*(T)$, and intertwines the actions of the finite group $K_0(F)/K_\ell(F) \overset{\phi_{F,E}}{\cong} K_0(E)/K_\ell(E)$.*

In this section we prove the following refinement of Theorem B from the Introduction:

Theorem 3.2.3. *Let (G,H) be a uniform spherical pair. Then, for any $\ell \in \mathbb{N}$ and $F \in \mathcal{F}_{R,\pi}$, there exists n such that, for any $E \in \mathcal{F}_{R,\pi}$ that is n-close to F, there exists a unique map*

$$\mathcal{M}_\ell(G(F)/H(F)) \to \mathcal{M}_\ell(G(E)/H(E))$$

which is an isomorphism of modules over the Hecke algebra

$$\mathcal{H}(G(F),K_\ell(F)) \overset{\Phi_{\mathcal{H},\ell}}{\cong} \mathcal{H}(G(E),K_\ell(E))$$

that maps the Haar measure on $K_\ell(F)x$ to the Haar measure on $K_\ell(E)x$, for every $x \in \Delta \subset \pi^\lambda\Upsilon$, and intertwines the actions of the finite group $K_0(F)/K_\ell(F) \overset{\phi_{F,E}}{\cong} K_0(E)/K_\ell(E)$.

For the proof we will need notation and several lemmas.

Notation 3.2.4. *For any valued field F with uniformizer π and any integer $m \in \mathbb{Z}$, we denote by $res_m : F \to F/\pi^m O$ the projection. Note that the groups $\pi^n O$ are naturally isomorphic for all n. Hence if two local fields $F,E \in \mathcal{F}_{R,\pi}$ are n-close, then for any m we are given an isomorphism, which we also denote by $\phi_{F,E}$ between $\pi^{m-n}O_F/\pi^m O_F$ and $\pi^{m-n}O_E/\pi^m O_E$, which are subgroups of $F/\pi^m O_F$ and $E/\pi^m O_E$.*

Lemma 3.2.5. *Suppose that (G,H) is a uniform spherical pair, and suppose that $F,E \in \mathcal{F}_{R,\pi}$ are ℓ-close. Then for all $\delta \in \Delta$,*

$$\phi_{F,E}(\mathrm{Stab}_{K_0(F)/K_\ell(F)}\, K_\ell(F)\delta) = \mathrm{Stab}_{K_0(E)/K_\ell(E)}\, K_\ell(E)\delta.$$

Proof. The stabilizer of $K_\ell(F)\delta$ in K_0/K_ℓ is the projection of the stabilizer of δ in K_0 to K_0/K_ℓ. In other words, it is the image of $S_{\delta,\delta}(O_F)$ in $S_{\delta,\delta}(O_F/\pi^\ell)$. Since $S_{\delta,\delta}$ is smooth over R, it is smooth over O_F. Hence $S_{\delta,\delta}$ is formally smooth, and so this map is onto. The same applies to the stabilizer of $K_\ell(E)\delta$ in $K_0(E)/K_\ell(E)$, but $\phi_{F,E}(S_{\delta,\delta}(O/\pi^\ell)) = S_{\delta,\delta}(O'/\pi'^\ell)$. □

Corollary 3.2.6. *Let $\ell \in \mathbb{N}$.*

Then, for any $F,E \in \mathcal{F}_{R,\pi}$ that are ℓ-close, there exists a unique morphism of vector spaces

$$\Phi_{\mathcal{M},\ell} : \mathcal{M}_\ell(G(F)/H(F)) \to \mathcal{M}_\ell(G(E)/H(E))$$

that maps the Haar measure on $K_\ell(F)x$ to the Haar measure on $K_\ell(E)x$, for every $x \in \Delta$, and intertwines the actions of the finite group $K_0(F)/K_\ell(F) \overset{\phi_{F,E}}{\cong} K_0(E)/K_\ell(E)$. Moreover, this morphism is an isomorphism.

Proof. The uniqueness is evident. By Lemma 3.2.5 and Lemma 3.1.4, the map between $K_\ell(F)\backslash G(F)/H(F)$ and $K_\ell(E)\backslash G(E)/H(E)$ given by

$$K_\ell(F)g\delta \mapsto K_\ell(E)g'\delta,$$

where $g \in K_0(F)$ and $g' \in K_0(E)$ satisfy that $\phi_{F,E}(res_\ell(g)) = res_\ell(g')$, is a bijection. This bijection gives the required isomorphism. □

Remark 3.2.7. *A similar construction can be applied to the pair $(G \times G, \Delta G)$. In this case, the main result of* [Kaz86] *is that the obtained linear map $\Phi_{\mathcal{H},\ell}$ between the Hecke algebras $\mathcal{H}(G(F), K_\ell(F))$ and $\mathcal{H}(G(E), K_\ell(E))$ is an isomorphism of algebras if the fields F and E are close enough.*

The following Lemma is evident:

Lemma 3.2.8. *Let $P(x) \in R[\pi^{-1}][x_1, \ldots, x_d]$ be a polynomial. For any natural numbers M and k, there is N such that, if $F, E \in \mathcal{F}_{R,\pi}$ are N-close, and $x_0 \in \pi^{-k}O_F^d, y_0 \in \pi^{-k}O_E^d$ satisfy that $P(x_0) \in \pi^{-k}O_F$ and $\phi_{F,E}(res_N(x_0)) = res_N(y_0)$, then $P(y_0) \in \pi^{-k}O_E$ and $\phi_{F,E}(res_M(P(x_0))) = res_M(P(y_0))$.*

Corollary 3.2.9. *Suppose that (G, H) is a uniform spherical pair. Fix an embedding of G/H to an affine space $\mathbb{A}^d$. Let $\lambda \in X_*(T)$, $x \in \pi^{\Upsilon}\mathfrak{X}$, $F \in \mathcal{F}_{R,\pi}$, and $k \in G(O_F)$. Choose m such that $\pi^\lambda kx \in \pi^{-m}O_F^d$. Then, for every M, there is $N \geq M + m$ such that, for any $E \in \mathcal{F}_{R,\pi}$ that is N-close to F, and for any $k' \in G(O_E)$ such that $\phi_{F,E}(res_N(k)) = res_N(k')$,*

$$\pi^\lambda k'x \in G(E)/H(E) \cap \pi^{-m}O_E^d \text{ and } \phi_{F,E}(res_M(\pi^\lambda kx)) = res_M(\pi^\lambda k'x).$$

Corollary 3.2.10. *Suppose that (G, H) is a uniform spherical pair. Fix an embedding of G/H to an affine space $\mathbb{A}^d$. Let m be an integer. For every M, there is N such that, for any $F, E \in \mathcal{F}_{R,\pi}$ that are N-close, any $x \in G(F)/H(F) \cap \pi^{-m}O_F^d$ and any $y \in G(E)/H(E) \cap \pi^{-m}O_E^d$, such that $\phi_{F,E}(res_{N-m}(x)) = res_{N-m}(y)$, we have $\Phi_{\mathcal{M}}(1_{K_M(F)}x) = 1_{K_M(E)}y$.*

Proof. Let $k_F \in G(O_F)$ and $\delta \in \Delta$ such that $x = k_F\delta$. By Proposition 3.1.5, there is an l such that, for any $L \in \mathcal{F}_{R,\pi}$ and any $k_L \in G(O_L)$, we have $K_M(L)k_L\delta$ contains a ball of radius l.

Using the previous corollary, choose an integer N such that, for any F and E that are N-close and any $k_E \in G(O_E)$, such that $\phi_{F,E}(res_N(k_F)) = res_N(k_E)$, we have

$$k_E\delta \in (G(E)/H(E)) \cap \pi^{-m}O_E^d \text{ and } \phi_{F,E}(res_l(x)) = res_l(k_E\delta).$$

Choose such $k_E \in G(O_E)$ and let $z = k_E\delta$. Since $res_l(z) = \phi_{F,E}(res_l(x)) = res_l(y)$, we have that $z \in B(y, l)$, and hence $z \in K_M(E)y$. Hence

$$1_{K_M(E)}y = 1_{K_M(E)}z = \Phi_{\mathcal{M}}(1_{K_M(F)}x).$$

□

From the last two corollaries we obtain the following one.

Corollary 3.2.11. *Given $\ell \in \mathbb{N}$, $\lambda \in X_*(T)$, and $\delta \in \Delta$, there is n such that if $F, E \in \mathcal{F}_{R,\pi}$ are n-close, and $g_F \in G(O_F), g_E \in G(O_E)$ satisfy that $\phi_{F,E}(res_n(g_F)) = res_n(g_E)$, then $\Phi_{\mathcal{M},\ell}(1_{K_\ell(F)}\pi^\lambda g_F\delta) = 1_{K_\ell(E)}\pi^\lambda g_E\delta$.*

Proposition 3.2.12. *Let $F \in \mathcal{F}_{R,\pi}$. Then for every ℓ, and every two elements $f \in \mathcal{H}_\ell(F)$ and $v \in \mathcal{M}_\ell(F)$, there is n such that, if $E \in \mathcal{F}_{R,\pi}$ is n-close to F, then $\Phi_{\mathcal{M},\ell}(f \cdot v) = \Phi_{\mathcal{H},\ell}(f) \cdot \Phi_{\mathcal{M},\ell}(v)$.*

Proof. By linearity, we can assume that $f = 1_{K_\ell(F)} k_1 \pi^\lambda k_2 1_{K_\ell(F)}$ and that $v = 1_{K_\ell(F)} k_3 \delta$, where $k_1, k_2, k_3 \in K_0(F)$. Choose $N \geq l$ big enough so that $\pi^\lambda K_N(F) \pi^{-\lambda} \subset K_\ell(F)$.

Choose $k_i' \in G(O_E)$ such that $\phi_{F,E}(res_N(k_i)) = res_N(k_i')$. Since $\Phi_{\mathcal{M},\ell}$ and $\Phi_{\mathcal{H},\ell}$ intertwine left multiplication by $1_{K_\ell(F)} k_1 1_{K_\ell(F)}$ to left multiplication by $1_{K_\ell(E)} k_1' 1_{K_\ell(E)}$, we can assume that $k_1 = 1 = k_1'$. Also, since k_2 normalizes $K_\ell(F)$, we can assume that $k_2 = 1 = k_2'$. Let $K_\ell(F) = \bigcup_{i=1}^s K_N(F) g_i$ be a decomposition of $K_\ell(F)$ into cosets. Choose $g_i' \in K_\ell(E)$ such that $\phi_{F,E}(res_N(g_i)) = res_N(g_i')$. Then

$$1_{K_\ell(F)} = c \sum_{i=1}^s 1_{K_N(F)} g_i \text{ and } 1_{K_\ell(E)} = c \sum_{i=1}^s 1_{K_N(E)} g_i'$$

where $c = |K_\ell(F)/K_N(F)| = |K_\ell(E)/K_N(E)|$. Hence

$$fv = 1_{K_\ell(F)} \pi^\lambda 1_{K_\ell(F)} k_3 \delta = c \sum_{i=1}^s 1_{K_\ell(F)} \pi^\lambda 1_{K_N(F)} g_i k_3 \delta = c \sum_{i=1}^s 1_{K_\ell(F)} \pi^\lambda g_i k_3 \delta.$$

and

$$\Phi_{\mathcal{H},\ell}(f) \Phi_{\mathcal{M},\ell}(v) = 1_{K_\ell(E)} \pi^\lambda 1_{K_\ell(E)} k_3' \delta = c \sum_{i=1}^s 1_{K_\ell(E)} \pi^\lambda 1_{K_N(E)} g_i k_3' \delta = c \sum_{i=1}^s 1_{K_\ell(E)} \pi^\lambda g_i' k_3' \delta.$$

The proposition follows now from Corollary 3.2.11. ☐

Now we are ready to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. We have to show for any ℓ there exists n such that if $F, E \in \mathcal{F}_{R,\pi}$ are n-close then the map $\Phi_{\mathcal{M},l}$ constructed in Corollary 3.2.6 is an isomorphism of modules over $\mathcal{H}(G(F), K_\ell(F)) \overset{\Phi_{\mathcal{H},\ell}}{\cong} \mathcal{H}(G(E), K_\ell(E))$.

Since $\mathcal{H}(G(F), K_\ell(F))$ is Noetherian, $\mathcal{M}_\ell(G(F)/H(F))$ is generated by a finite set $v_1, \ldots, v_n$ satisfying a finite set of relations $\sum_i f_{i,j} v_i = 0$. Without loss of generality we may assume that for any $x \in \mathfrak{X}$ the Haar measure on $K_\ell(F)x$ is contained in the set $\{v_i\}$.

By Proposition 3.2.12, if E is close enough to F, then $\Phi_{\mathcal{M},\ell}(v_i)$ satisfy the above relations.

Therefore there exists a homomorphism of Hecke modules $\Phi' : \mathcal{M}_\ell(G(F)/H(F)) \to \mathcal{M}_\ell(G(E)/H(E))$ given on the generators v_i by $\Phi'(v_i) := \Phi_{\mathcal{M},\ell}(v_i)$.

Φ' intertwines the actions of the finite group $K_0(F)/K_\ell(F) \overset{\phi_{F,E}}{\cong} K_0(E)/K_\ell(E)$. Therefore, by Corollary 3.2.6, in order to show that Φ' coincides with $\Phi_{\mathcal{M},\ell}$ it is enough to check that Φ' maps the normalized Haar measure on $K_\ell(F)x$ to the normalized Haar measure on $K_\ell(E)x$ for every $x \in \Delta$. In order to do this let us decompose $x = \pi^\alpha x_0$ where $x_0 \in \mathfrak{X}$ and $\alpha \in \Upsilon$. Now, since (G, H) is uniformly spherical we have

$$1_{K_n(F)x} = 1_{K_n(F)\pi^\alpha K_n(F)} 1_{K_n(F)x_0}$$

and

$$1_{K_n(E)x} = 1_{K_n(E)\pi^\alpha K_n(E)} 1_{K_n(E)x_0}.$$

Therefore, since Φ' is a homomorphism, we have

$$\Phi'(1_{K_n(F)x}) = \Phi'(1_{K_n(F)\pi^\alpha K_n(F)} 1_{K_n(F)x_0}) = 1_{K_n(E)\pi^\alpha K_n(E)} 1_{K_n(E)x_0} = 1_{K_n(F)x}.$$

Hence the linear map $\Phi_{\mathcal{M},\ell} : \mathcal{M}_\ell(G(F)/H(F)) \to \mathcal{M}_\ell(G(E)/H(E))$ is a homomorphism of Hecke modules. Since it is a linear isomorphism, it is an isomorphism of Hecke modules. ☐

Now we obtain the following generalization of Corollary C:

Corollary 3.2.13. *Let (G, H) be a uniform spherical pair. Suppose that*

- *For any $F \in \mathcal{F}_{R,\pi}$, the pair (G, H) is F-spherical.*

- *For any $E \in \mathcal{F}_{R,\pi}$ and natural number n, there is a field $F \in \mathcal{F}_{R,\pi}$ such that E and F are n-close and the pair $(G(F), H(F))$ is a Gelfand pair, i.e. for any irreducible smooth representation ρ of $G(F)$ we have*
$$\dim \operatorname{Hom}_{H(F)}(\rho|_{H(F)}, \mathbb{C}) \leq 1.$$

Then $(G(F), H(F))$ is a Gelfand pair for any $F \in \mathcal{F}_{R,\pi}$.

Remark 3.2.14. *Fix a prime power $q = p^k$. Let F be the unramified extension of $\mathbb{Q}_p$ of degree k, let W be the ring of integers of F, and let $R = W[[\pi]]$. Then $\mathcal{F}_{R,\pi}$ includes all local fields with residue field $\mathbb{F}_q$, and so Corollary 3.2.13 implies Corollary C.*

Corollary 3.2.13 follows from Theorem 3.2.3, Theorem 2.3.1, and the following lemma.

Lemma 3.2.15. *Let F be a local field and $H < G$ be a pair of reductive groups defined over F. Suppose that G is split over F. Then $(G(F), H(F))$ is a Gelfand pair if and only if for any large enough $l \in \mathbb{Z}_{>0}$ and any simple module ρ over $\mathcal{H}_l(G(F))$ we have*
$$\dim \operatorname{Hom}_{\mathcal{H}_l(G(F))}(\mathcal{M}_l(G(F)/H(F)), \rho) \leq 1.$$

This lemma follows from statement (1) formulated in Subsection 2.1.

4. Applications

In this section we prove that the pair $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ is a Gelfand pair for any local field F of characteristic different from 2 and the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a strong Gelfand pair for any local field F. We use Corollary 3.2.13 to deduce those results from the characteristic zero case which were proven in [JR96] and [AGRS] respectively. Let $R = W[[\pi]]$.

To verify condition (2) in Definition 3.1.2, we use the following straightforward lemma:

Lemma 4.0.1. *Let $G = (\mathrm{GL}_{n_1})_R \times \cdots \times (\mathrm{GL}_{n_k})_R$ and let $C < G \otimes_R R[\pi^{-1}]$ be a sub-group scheme defined over $R[\pi^{-1}]$. Suppose that C is defined by equations of the following type:*
$$\sum_{i=1}^{l} \epsilon_i a_{\mu_i} \pi^{\lambda_i} = \pi^{\nu},$$
or
$$\sum_{i=1}^{l} \epsilon_i a_{\mu_i} \pi^{\lambda_i} = 0,$$
where $\epsilon_i = \pm 1$, $a_1, ..., a_{n_1^2+...+n_k^2}$ are entries of matrices, $1 \leq \mu_i \leq n_1^2 + ... + n_k^2$ are some indices, and ν, λ_i are integers. Suppose also that the indices μ_i are distinct for all the equations. Then the closure $\overline{C}$ of C in G is smooth over R.

To verify condition (4) in Definition 3.1.2, we use the following straightforward lemma:

Lemma 4.0.2. *Suppose that there exists a natural number ℓ_0 such that, for any $F \in \mathcal{F}_{R,\pi}$ and any $\ell > \ell_0$, there is a subgroup $P_\ell < K_\ell(G, F)$ satisfying that for every $x \in \mathfrak{X}$*
1. *For any $\alpha \in \Upsilon$ we have $\pi^\alpha P_\ell \pi^{-\alpha} \subset K_\ell$.*
2. *$K_\ell x = P_\ell x$.*

Then condition (4) in Definition 3.1.2 is satisfied.

In our applications, we use the following to show that the pairs we consider are F-spherical.

Proposition 4.0.3. *Let F be an infinite field, and consider $G = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ embedded in the standard way in $M = \mathrm{Mat}_{n_1} \times \cdots \times \mathrm{Mat}_{n_k}$. Let $A, B \subset G \otimes F$ be two F-subgroups whose closures in M are affine subspaces M_A, M_B.*

(1) *For any $x, y \in G(F)$, if the variety $\{(a, b) \in A \times B | axb = y\}$ is non-empty, then it has an F-rational point.*
(2) *If (G, A) is a spherical pair, then it is also an F-spherical pair.*

Proof. (1) Denote the projections $G \to \mathrm{GL}_{n_j}$ by π_j. Assume that $x, y \in G(F)$, and there is a pair $(\overline{a}, \overline{b}) \in (A \times B)(\overline{F})$ such that $\overline{a} x \overline{b} = y$. Let $L \subset M_A \times M_B$ be the affine subspace $\{(\alpha, \beta) | \alpha x = y\beta\}$, defined over F. By assumption, the functions $(\alpha, \beta) \mapsto \det \pi_j(\alpha)$ and $(\alpha, \beta) \mapsto \det \pi_j(\beta)$, for $j = 1, \ldots, k$, are non-zero on $L(\overline{F})$. Hence there is $(a, b) \in L(F) \cap G$, which means that $axb^{-1} = y$.

(2) Applying (1) to A and any parabolic subgroup $B \subset G$, any $(A \times B)(\overline{F})$-orbit in $G(\overline{F})$ contains at most one $(A \times B)(F)$-orbit. Since there are only finitely many $(A \times B)(\overline{F})$-orbits in $G(\overline{F})$, the pair (G, A) is F-spherical.

□

4.1. The Pair $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$.

In this subsection we assume $p \neq 2$ and consider only local fields of characteristic different from 2.

Let $G := (\mathrm{GL}_{n+k})_R$ and $H := (\mathrm{GL}_n)_R \times (\mathrm{GL}_k)_R < G$ be the subgroup of block matrices. Note that H is a symmetric subgroup since it consists of fixed points of conjugation by $\epsilon = \begin{pmatrix} Id_k & 0 \\ 0 & -Id_n \end{pmatrix}$. We prove that (G, H) is a Gelfand pair using Corollary C. The pair (G, H) is a symmetric pair, hence it is a spherical pair and therefore by Proposition 4.0.3 it is F-spherical. The second condition of Corollary C is [JR96, Theorem 1.1]. It remains to prove that (G, H) is a uniform spherical pair.

Proposition 4.1.1. *The pair (G, H) is uniform spherical.*

Proof. Without loss of generality suppose that $n \geq k$. Let $\mathfrak{X} = \{x_0\}$, where

$$x_0 := \begin{pmatrix} Id_k & 0 & Id_k \\ 0 & Id_{n-k} & 0 \\ 0 & 0 & Id_k \end{pmatrix} H \text{ and } \Upsilon = \{(\mu_1, ..., \mu_k, 0, ..., 0) \in X_*(T) \,|\, \mu_1 \leq ... \leq \mu_k \leq 0\}.$$

To show the first condition we show that every double coset in $K_0 \backslash G / H$ includes an element of the form
$\begin{pmatrix} Id_k & 0 & diag(\pi^{\mu_1}, ..., \pi^{\mu_k}) \\ 0 & Id_{n-k} & 0 \\ 0 & 0 & Id_k \end{pmatrix}$ s.t. $\mu_1 \leq ... \leq \mu_k \leq 0$. Take any $g \in G$. By left multiplication by K_0 we can bring it to upper triangular form. By right multiplication by H we can bring it to a form $\begin{pmatrix} Id_n & A \\ 0 & Id_k \end{pmatrix}$. Conjugating by a matrix $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K_0 \cap H$ we can replace it by $\begin{pmatrix} Id_n & k_1 A k_2^{-1} \\ 0 & Id_k \end{pmatrix}$. Hence we can bring A to be a k-by-$(n-k)$ block of zero, followed by the a diagonal matrix of the form $diag(\pi^{\mu_1}, ..., \pi^{\mu_k})$ s.t. $\mu_1 \leq ... \leq \mu_k$. Multiplying by an element of K_0 of the form $\begin{pmatrix} Id_k & 0 & k \\ 0 & Id_{n-k} & 0 \\ 0 & 0 & Id_k \end{pmatrix}$ we can bring A to the desired form.

As for the second condition, we first compute the stabilizer G_{x_0} of x_0 in G. Note that the coset $x_0 \in G/H$ equals

$$\left\{ \begin{pmatrix} g_1 & g_2 & h \\ g_3 & g_4 & 0 \\ 0 & 0 & h \end{pmatrix} \,|\, \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in (\mathrm{GL}_n)_R,\, h \in (\mathrm{GL}_k)_R \right\}$$

and

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \begin{pmatrix} Id_k & 0 & Id_k \\ 0 & Id_{n-k} & 0 \\ 0 & 0 & Id_k \end{pmatrix} = \begin{pmatrix} A & B & A+C \\ D & E & D+F \\ G & H & G+I \end{pmatrix}.$$

Hence

$$G_{x_0} = \left\{ \begin{pmatrix} g_1 & g_2 & h-g_1 \\ g_3 & g_4 & -g_3 \\ 0 & 0 & h \end{pmatrix} \mid \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in (\mathrm{GL}_n)_R,\ h \in (\mathrm{GL}_k)_R \right\}.$$

Therefore, for any $\delta_1 = (\lambda_{1,1}, ..., \lambda_{1,k}, 0, ..., 0), \delta_2 = (\lambda_{2,1}, ..., \lambda_{2,k}, 0, ..., 0) \in \Upsilon$,

$$G(F)_{\pi^{\lambda_1} x_0, \pi^{\lambda_2} x_0} = \left\{ \begin{pmatrix} \pi^{\lambda_2} g_1 \pi^{-\lambda_1} & \pi^{\lambda_2} g_2 & \pi^{\lambda_2}(h-g_1) \\ g_3 \pi^{-\lambda_1} & g_4 & -g_3 \\ 0 & 0 & h \end{pmatrix} \mid \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in (\mathrm{GL}_n)_R,\ h \in (\mathrm{GL}_k)_R \right\} =$$

$$= \left\{ \begin{pmatrix} A & B & C \\ D & E & F \\ 0 & 0 & I \end{pmatrix} \in GL_{n+k}, | D = -F\pi^{-\lambda_1}, C = \pi^{\lambda_2} I - A\pi^{\lambda_1} \right\}.$$

The second condition of Definition 3.1.2 follows now from Lemma 4.0.1.

As for the third condition, we use the embedding $G/H \to G$ given by $g \mapsto g\epsilon g^{-1}\epsilon$. It is easy to see that $val_F(\pi^\mu x_0) = \mu_1$, which is independent of F.

Let us now prove the last condition using Lemma 4.0.2. Take $l_0 = 1$ and

$$P := \left\{ \begin{pmatrix} Id & 0 & 0 \\ D & E & F \\ G & H & I \end{pmatrix} \in GL_{n+k} \right\}.$$

Let $P_l := P(F) \cap K_l(GL_{n+k}, F)$. The first condition of Lemma 4.0.2 obviously holds. To show the second condition, we have to show that for any F, any $l \geq 1$ and any $g \in K_l(GL_{n+k}, F)$ there exist $p \in P_l$ and $h \in H(F)$ such that $gx_0 = px_o h$. In other words, we have to solve the following equation:

$$\begin{pmatrix} Id_k + A & B & Id_k + A + C \\ D & Id_{n-k} + E & D + F \\ G & H & Id_k + G + I \end{pmatrix} = \begin{pmatrix} Id_k & 0 & Id_k \\ D' & Id_{n-k} + E' & D' + F' \\ G' & H' & Id_k + G' + I' \end{pmatrix} \begin{pmatrix} Id_k + x & y & 0 \\ z & Id_k + w & 0 \\ 0 & 0 & Id_k + h \end{pmatrix},$$

where all the capital letters denote matrices of appropriate sizes with entries in $\pi^l \mathcal{O}_F$, and the matrices in the left hand side are parameters and matrices in the right hand side are unknowns.

The solution is given by:

$$x = A, \quad y = B, \quad z = D, \quad w = E, \quad h = A + C$$
$$D' = 0, \quad E' = 0, \quad F' = (D+F)(Id_k + A + C)^{-1},$$
$$H' = (H - G(Id_k + A)^{-1}B)(-D(Id_k + A)^{-1}B + Id_{n-k} + E)^{-1}$$
$$G' = (G - H'D)(Id_k + A)^{-1}, \quad I' = (G + I - A - C)(Id_k + A + C)^{-1} - G'$$

□

4.2. **Structure of the spherical space** $(\mathrm{GL}_{n+1}\times\mathrm{GL}_n)/\Delta\,\mathrm{GL}_n$**.** Consider the embedding $\iota:\mathrm{GL}_n\hookrightarrow\mathrm{GL}_{n+1}$ given by

$$A\mapsto\begin{pmatrix}1&0\\0&A\end{pmatrix}.$$

Denote $G=\mathrm{GL}_{n+1}(F)\times\mathrm{GL}_n(F)$ and $H=\Delta\,\mathrm{GL}_n(F)$. The quotient space G/H is isomorphic to $(\mathrm{GL}_{n+1})_R$ via the map $(g,h)\mapsto g\iota(h^{-1})$. Under this isomorphism, the action of G on G/H becomes $(g,h)\cdot X=gX\iota(h^{-1})$.

The space G/H is spherical. Indeed, let $B\subset G$ be the Borel subgroup consisting of pairs (b_1,b_2), where b_1 is lower triangular and b_2 is upper triangular, and let $x_0\in G/H$ be the point represented by the matrix

$$x_0=\begin{pmatrix}1&e\\0&I\end{pmatrix},$$

where e is a row vector of 1's. We claim that Bx_0 is open in G/H. Let $\mathfrak{b}$ be the Lie algebra of B. It consists of pairs (X,Y) where X is lower triangular and Y is upper triangular. The infinitesimal action of $\mathfrak{b}$ on X at x_0 is given by $(X,Y)\mapsto Xx_0-x_0d\iota(Y)$. To show that the image is Mat_{n+1}, it is enough to show that the images of the maps $X\mapsto Xx_0$ and $Y\mapsto x_0d\iota(Y)$ have trivial intersection. Suppose that $Xx_0=x_0d\iota(Y)$. Then $X=x_0d\iota(Y)x_0^{-1}$, i.e.

$$X=\begin{pmatrix}1&e\\&I\end{pmatrix}\begin{pmatrix}0&0\\0&Y\end{pmatrix}\begin{pmatrix}1&-e\\&I\end{pmatrix}=\begin{pmatrix}0&eY\\0&Y\end{pmatrix}.$$

Since X is lower triangular and Y is upper triangular, both have to be diagonal. But $eY=0$ implies that $Y=0$, and hence also X=0. Proposition 4.0.3 implies that the pair (G,H) is F-spherical.

The following describes the quotient $G(O_F)\backslash G(F)/H(F)$.

Lemma 4.2.1. *For every matrix $A\in\mathrm{Mat}_{n+1}(F)$ there are $k_1\in\mathrm{GL}_{n+1}(O)$ and $k_2\in\mathrm{GL}_n(O)$ such that*

$$k_1A\iota(k_2)=\begin{pmatrix}\pi^a&\pi^{b_1}&\pi^{b_2}&\dots&\pi^{b_n}\\&\pi^{c_1}&&&\\&&\pi^{c_2}&&\\&&&\ddots&\\&&&&\pi^{c_n}\end{pmatrix},\tag{3}$$

where the numbers a,b_i,c_i satisfy that if $i<j$ then $c_i-c_j\le b_i-b_j\le 0$ and $b_1\le c_1$.

Proof. Let a be the minimal valuation of an element in the first column of A. There is an integral matrix w_1 such that the first column of the matrix w_1A is $\pi^a,0,0,\dots,0$. Let C be the $n\times n$ lower-right sub-matrix of w_1A. By Cartan decomposition, there are integral matrices w_2,w_3 such that $w_2Cw_3^{-1}$ is diagonal, and its diagonal entries are π^{c_i} for a non-decreasing sequence c_i. Finally, there are integral and diagonal matrices d_1,d_2 such that the matrix $d_1\iota(w_2)w_1A\iota(w_3^{-1})\iota(d_2^{-1})$ has the form (3).

Suppose that $i<j$ and $b_i>b_j$. Then adding the j'th column to the i'th column and subtracting $\pi^{c_j-c_i}$ times the i'th row to the j'th row, we can change the matrix (3) so that $b_i=b_j$. Similarly, if $i<j$ and $b_i-b_j<c_i-c_j$, then adding $\pi^{b_j-b_i-1}$ times the i'th column to the j'th column, and subtracting $\pi^{c_i+b_j-b_i-1-c_j}$ times the j'th row to the i'th row changes the matrix (3) so that b_i becomes smaller in 1. Finally, if $c_1<b_1$ than adding the second row to the first changes the matrix so that $c_1=b_1$. ☐

Let $T\subset G$ be the torus consisting of pairs (t_1,t_2) such that t_i are diagonal. The co-character group of T is the group $\mathbb{Z}^{n+1}\times\mathbb{Z}^n$. The positive Weyl chamber of T that is defined by B[1] is the set $\Delta\subset X_*(T)$ consisting of pairs (μ,ν) such that the μ_i's are non-decreasing and the ν_i's are non-increasing. Lemma

[1]The positive Weyl chamber defined by the Borel B is the subset of co-weights λ such that $\pi^\lambda B(O)\pi^{-\lambda}\subset B(O)$

4.2.1 implies that the set $\{\pi^\lambda x_0\}_{\lambda\in\Delta}$ is a complete set of orbit representatives for $G(O)\backslash G(F)/H(F)$.

We are ready to prove that (G, H) is uniform spherical.

Proposition 4.2.2. *The pair* $((\mathrm{GL}_{n+1})_R \times (\mathrm{GL}_n)_R, \Delta(\mathrm{GL}_n)_R)$ *is uniform spherical.*

Proof. Let $\Upsilon \subset X_*(T)$ be the positive Weyl chamber and let $\mathfrak{X} := \{x_0\}$. By the above, the first condition of Definition 3.1.2 holds. As for the second condition, an easy computation shows that if $a, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{Z}$, $a', b'_1, \ldots, b'_n, c'_1, \ldots, c'_n \in \mathbb{Z}$ satisfy the conclusion of Lemma 4.2.1, and $(k_1, k_2) \in G(O)$ satisfy that

$$k_1 \begin{pmatrix} \pi^a & \pi^{b_1} & \pi^{b_2} & \ldots & \pi^{b_n} \\ & \pi^{c_1} & & & \\ & & \pi^{c_2} & & \\ & & & \ddots & \\ & & & & \pi^{c_n} \end{pmatrix} \iota(k_1) = \begin{pmatrix} \pi^{a'} & \pi^{b'_1} & \pi^{b'_2} & \ldots & \pi^{b'_n} \\ & \pi^{c'_1} & & & \\ & & \pi^{c'_2} & & \\ & & & \ddots & \\ & & & & \pi^{c'_n} \end{pmatrix},$$

then $a = a'$, $c_i = c'_i$, k_1 has the form $\begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix}$, where B is a $1 \times n$ matrix and D is an $n \times n$ matrix that satisfy the equations $D = \pi^c k_2 \pi^{-c}$ and $B\pi^c = \pi^b - \pi^{b'} k_2$, where π^c denotes the diagonal matrix with entries $\pi^{c_1}, \ldots, \pi^{c_n}$, π^b denotes the row vector with entries π^{b_i}, and $\pi^{b'}$ denotes the row vector with entries $\pi^{b'_i}$. The second condition of Definition 3.1.2 holds by Lemma 4.0.1.

The third condition follows because, using the affine embedding as above, $\pi^\lambda x_0$ has the form (3) and so $val_F(\pi^\lambda x_0)$ is independent of F.

Finally it is left to verify the last condition. In the following, we will distinguish between the ℓth congruence subgroup in $\mathrm{GL}_{n+1}(F)$, which we denote by $K_\ell(\mathrm{GL}_{n+1}(F))$, the ℓth congruence subgroup in $\mathrm{GL}_n(F)$, which we denote by $K_\ell(\mathrm{GL}_n(F))$, and the ℓth congruence subgroup in $G = \mathrm{GL}_{n+1}(F) \times \mathrm{GL}_n(F)$, which we denote by K_ℓ. By lemma 4.0.2 it is enough to show that $(B \cap K_l)x_0 = K_l x_0$. It is easy to see that $K_l x_0 = x_0 + \pi^l Mat_n(O_F)$. Let $y \in x_0 + \pi^l Mat_n(O_F)$. We have to show that $y \in (B \cap K_l)x_0$. In order to do this let us represent y as a block matrix

$$y = \begin{pmatrix} a & b \\ c & D \end{pmatrix},$$

where a is a scalar and D is $n \times n$ matrix. Using left multiplication by lower triangular matrix from $K_l(\mathrm{GL}_{n+1}(F))$ we may bring y to the form $\begin{pmatrix} 1 & b' \\ 0 & D' \end{pmatrix}$. We can decompose $D' = LU$, where $L, U \in K_l(\mathrm{GL}_{n+1}(F))$ and L is lower triangular and U is upper triangular. Therefore by action of an element from $B \cap K_l$ we may bring y to the form $\begin{pmatrix} 1 & b'' \\ 0 & Id \end{pmatrix}$. Using right multiplication by diagonal matrix from $K_l(\mathrm{GL}_{n+1}(F))$ (with first entry 1) we may bring y to the form $\begin{pmatrix} 1 & e \\ 0 & D'' \end{pmatrix}$, where e is a row vector of 1's and D'' is a diagonal matrix. Finally, using left multiplication by diagonal matrix from $K_l(\mathrm{GL}_{n+1}(F))$ we may bring y to be x_0. □

4.3. **The Pair** $(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, \Delta\, \mathrm{GL}_n)$**.**

In this section we prove Theorem D which states that $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a strong Gelfand pair for any local field F, i.e. for any irreducible smooth representations π of $\mathrm{GL}_{n+1}(F)$ and τ of $\mathrm{GL}_n(F)$ we have

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1.$$

It is well known (see e.g. [AGRS, section 1]) that this theorem is equivalent to the statement that $(\mathrm{GL}_{n+1}(F) \times \mathrm{GL}_n(F), \Delta\,\mathrm{GL}_n(F))$, where $\Delta\,\mathrm{GL}_n$ is embedded in $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ by the map $\iota \times Id$, is a Gelfand pair.

By Corollary C this statement follows from Proposition 4.2.2, and the following theorem:

Theorem 4.3.1 ([AGRS], Theorem 1)**.** *Let F be a local field of characteristic 0. Then $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a strong Gelfand pair.*

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UNIQUENESS OF SHALIKA FUNCTIONALS (THE ARCHIMEDEAN CASE)

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Abstract. Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let (π, V) be an irreducible admissible smooth Fréchet representation of $GL_{2n}(F)$. A Shalika functional $\phi : V \to \mathbb{C}$ is a continuous linear functional such that for any $g \in GL_n(F)$, $A \in \mathrm{Mat}_{n\times n}(F)$ and $v \in V$ we have

$$\phi\left[\pi\begin{pmatrix} g & A \\ 0 & g\end{pmatrix})v\right] = \exp(2\pi i\,\mathrm{Re}(\mathrm{Tr}(g^{-1}A)))\phi(v).$$

In this paper we prove that the space of Shalika functionals on V is at most one dimensional.

For non-Archimedean F (of characteristic zero) this theorem was proven in [JR96].

Contents

1. Introduction

Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let (π, V) be an admissible smooth Fréchet representation of $GL_{2n}(F)$. We assume that V is the canonical completion of an irreducible Harish-Chandra $(\mathfrak{g}, K)$- module in the sense of Casselman-Wallach (see e.g. [Wal92], chapter 11). A **Shalika functional** $\phi : V \to \mathbb{C}$ is a continuous linear functional such that for any $g \in GL_n(F)$, $A \in Mat_{n\times n}(F)$ and $v \in V$ we have

$$\phi\left[\pi\begin{pmatrix} g & A \\ 0 & g\end{pmatrix}v\right] = \exp\left(2\pi i\,\mathrm{Re}(\mathrm{Tr}(g^{-1}A))\right)\phi(v).$$

In this paper we prove the following theorem.

Theorem 1.1. *Let (π, V) be an irreducible admissible smooth Fréchet representation of $GL_{2n}(F)$. Then the space of Shalika functionals on V is at most one dimensional.*

For non-Archimedean F (of characteristic zero) this theorem was proven in [JR96]. The proof in [JR96] is based on the fact that $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$ is a Gelfand pair, which was also proven in [JR96], and the method of [FJ93, Section 3] of integration of Shalika functionals.

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In the Archimedean case those two ingredients also exist. Namely, [FJ93, Section 3] is valid also in the Archimedean case, and the fact that $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$ is a Gelfand pair is proven in [AG08b].

The proof that we present here is similar to the proof in [JR96]. The main difierence is that we have to prove the continuity of a certain linear form.

1.1. Structure of the proof.

We construct a linear map from the space of Shalika functionals to the space of linear periods (linear functionals on V that are invariant by $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$) and prove that the map is injective. Hence the uniqueness of the linear periods implies uniqueness of the Shalika functionals. The uniqueness of linear periods, i.e. the fact that $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$ is a Gelfand pair, is proven in [AG08b].

1.2. Structure of the paper.

In §2 we fix notation and terminology. In §3 we describe a way of obtaining a linear period from a Shalika functional by integration, as in [FJ93, Section 3]. In §4 we investigate the properties of the obtained period. In §5 we explain how this implies the uniqueness of Shalika functionals.

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2. Preliminaries and notation

2.1. Notation.

- Henceforth we fix an Archimedean field F (i.e. F is $\mathbb{R}$ or $\mathbb{C}$).
- For a group G acting on a vector space V we denote by V^G the space of G-invariant vectors in V. For a character χ of G we denote by $V^{G,\chi}$ the space of (G,χ)-equivariant vectors in V.
- For a smooth real algebraic variety M we denote by $\mathcal{S}(M)$ the space of Schwartz functions on M, i.e. the space of smooth functions that are rapidly decreasing as well as all their derivatives. For precise definition see e.g. [AG08a].
- We fix a natural number n and denote $G := GL_{2n}(F)$.
- We fix a norm on G by
$$||g|| := \sum_{1 \le i,j \le 2n} |g_{ij}|^2 + \sum_{1 \le i,j \le 2n} |(g^{-1})_{ij}|^2.$$
- We denote
$$G_1 := \left\{ \begin{pmatrix} g & 0 \\ 0 & Id \end{pmatrix} \mid g \in GL_n(F) \right\} \subset G$$
- We denote by $\nu : GL_n(F) \to G_1$ the isomorphism defined by
$$\nu(g) := \begin{pmatrix} g & 0 \\ 0 & Id \end{pmatrix}.$$
Note that for any $X \in \mathrm{Mat}(n \times n, F)$, $d\nu(X) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$.
- We denote
$$H := \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \in GL_n(F) \right\} \subset G$$
- We denote
$$U := \left\{ \begin{pmatrix} Id & A \\ 0 & Id \end{pmatrix} \mid A \in Mat_{n\times n}(F) \right\} \subset G$$

- We denote by $\mu : \mathrm{Mat}(n \times n, F) \to U$ the isomorphism defined by
$$\mu(A) := \begin{pmatrix} Id & A \\ 0 & Id \end{pmatrix}.$$
Note that for any $X \in \mathrm{Mat}(n \times n, F)$, $d\mu(X) = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$.
- We denote by $\tau : U \to F$ the homomorphism given by
$$\tau(\mu(A)) := \mathrm{Tr}(A).$$
- We let ψ be the additive character of F defined by $\psi(x) := e^{2\pi \mathrm{i} \,\mathrm{Re}\, x}$. We define an homomorphism $\Psi : U \to F^\times$ by
$$\Psi := \psi \circ \tau.$$
We extend Ψ to an homomorphism $\Psi : HU \to F^\times$ trivial on H.
- We denote by K the standard maximal compact subgroup of G. Thus $K = O(2n)$ if $F = \mathbb{R}$ and $K = U(2n)$ if $F = \mathbb{C}$.

2.2. **Admissible representations.**
In this paper we consider admissible smooth Fréchet representations of G, i.e. smooth admissible representations (π, V) of G such that V is a Fréchet space and, for any continuous semi-norm α on V, there exist another continuous semi-norm β on V and a natural number M such that for any $g \in G$,
$$\alpha(\pi(g)v) \leq \beta(v)||g||^M.$$

By Casselman - Wallach theorem (see e.g. [Wal92], chapter 11), V may be regarded as the canonical model of an irreducible Harish-Chandra $(\mathfrak{g}, K)-$module. By Casselman embedding theorem ([Cas80]), V can be realized as a closed subspace of a principal series representation. We denote by $\widetilde{V}$ the canonical model of the contragredient Harish-Chandra $(\mathfrak{g}, K)-$module. It is a subspace of the topological dual V^* of V.

3. Integration of Shalika functionals

In this section we fix:

- an irreducible admissible smooth Fréchet representation (π, V) of G
- a Shalika functional λ on V, i.e. $\lambda \in (V^*)^{HU,\Psi}$.

Theorem 3.1. *There exists $M \in \mathbb{R}$ such that for any $v \in v$ and over the region of $s \in \mathbb{C}$ with $\mathrm{Re}(s) > M$, the integral*
$$L_{\lambda,v}(s) := \int_{g \in G_1} \lambda(\pi(g)v)|\det(g)|^{s-\frac{1}{2}} dg$$
converges absolutely and is a holomorphic function of s.

Moreover, $L_{\lambda,v}(s)$ has meromorphic continuation to the complex plane and is a holomorphic multiple of the L-function L_π of the representation π. Finally, for any $\lambda \neq 0$ there exists $v \in V$ such that $L_{\lambda,v} = L_\pi$.

In [FJ93, Proposition 3.1] this theorem is proven under the following assumption:
(*) There exists a continuous semi-norm β on V such that $|\lambda(\pi(g)v)| \leq \beta(v)$ for any $g \in G$.
This may not be true in general. However, we have the following result.

Lemma 3.2. *There exist $M > 0$ and a continuous semi-norm β on V such that $|\lambda(\pi(g)v)| \leq |\det g|^{-M}\beta(v)$ for any $g \in G_1$.*

Before proving the Lemma, we check that, with the help of this Lemma, the proof of Theorem 3.1 is still valid. Indeed, the functions $g \mapsto \lambda(\pi(g)v)$ are bounded in [FJ93] and satisfy a sharper estimate ([FJ93, Lemma 3.1]). Here they satisfy the following estimates.

Lemma 3.3. *There is a continuous semi-norm γ on V such that, for any $v \in V$,*

$$|\lambda(\pi(g)v)| \leq |\det b^{-1}a|^{-M}\gamma(v)$$

for

$$g = u \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k$$

with $a, b \in GL(n, F)$, $u \in U$, $k \in K$. Furthermore, for any $v \in V$, there is $\Phi_v \in \mathcal{S}(\mathrm{Mat}(n \times n, F))$ such that

$$|\lambda(\pi(g)v)| \leq \Phi(b^{-1}a)|\det b^{-1}a|^{-M},$$

for g of the above form.

Proof. For the first assertion, we have

$$\lambda(\pi(g)v) = \Psi(u)\lambda(\pi(\nu(b^{-1}a))\pi(k)v).$$

Hence

$$|\lambda(\pi(g)v)| \leq |\det b^{-1}a|^{-M}\beta(\pi(k)v).$$

There is another continuous semi-norm γ such that, for all $k \in K$,

$$\beta(\pi(k)v) \leq \gamma(v).$$

The first assertion follows.

For the second assertion, we go through the proof of [FJ93, Lemma 3.1] (which is the above estimate with $M = 0$) and arrive at once at the present estimate. □

The proof of Theorem 3.1 is still valid. The only modification is that we need to check that, under our weaker assumption, two integrals in [FJ93] which depend on $s \in \mathbb{C}$, are still absolutely convergent for $\mathrm{Re}\, s >> 0$.

The first integral is integral [FJ93, 45]:

$$\int \lambda(\pi(g)v)\Phi(g)|\det g|^{s+n-\frac{1}{2}}d^\times g$$

where $\Phi \in \mathcal{S}(\mathrm{Mat}(2n \times 2n, F))$. We write

$$g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k.$$

Then

$$d^\times g = |\det a|^{-n}d^\times a d^\times b dx dk.$$

By Lemma 3.3, the integral of the absolute value is bounded by

$$\int |\det a|^{\mathrm{Re}\, s-M-\frac{1}{2}}|\det b|^{\mathrm{Re}\, s+M+n-\frac{1}{2}}\left|\Phi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right]\right| d^\times a d^\times b dx dk.$$

This does converge absolutely for $\mathrm{Re}\, s >> 0$.

The second integral is integral [FJ93, 48]. It has the form

$$\int \lambda\left[\pi\begin{pmatrix} a & 0 \\ 0 & \mathrm{Id} \end{pmatrix}\pi(x)v\right]|\det a|^{s-\frac{1}{2}}d^\times a d\mu(x)$$

where μ is the measure on $SL(2n, F)$ defined by

$$\int f(x)d\mu(x) =$$

$$\int f\left[\begin{pmatrix} b^{-1} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & u \\ 0 & \mathrm{Id} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & b \end{pmatrix} k\right]\Upsilon(u, b^{-1}, b; k)\ |\det b|^n d^\times b du dk$$

In this formula k is integrated over $K' = K \cap SL(2n, F)$ and the function Υ is in $\mathcal{S}(\mathrm{Mat}(n \times n, F)^3 \times K')$. The integral of the absolute value of the integrand is bounded by

$$\int \Phi_v(ab^{-2})|\det ab^{-2}|^{-M}\ |\Upsilon|(u, b^{-1}, b; k)|\det b|^n d^\times b du dk\ |\det a|^{\mathrm{Re}\, s-\frac{1}{2}}d^\times a.$$

After changing a to ab^2, the integral decomposes into a product:

$$\int \Phi_v(a)|\det a|^{\operatorname{Re} s-M-\frac{1}{2}} d^\times a \times \int |\Upsilon|(u,b^{-1},b;k)|\det b|^{n+2\operatorname{Re} s-1} d^\times b du dk.$$

The first integral converges for $\operatorname{Re} s >> 0$. The second integral converges for all s.

It remains to prove Lemma 3.2. We will prove the following more general lemma.

Lemma 3.4. *There exists $M_0 > 0$ such that, for any polynomial P on the real vector space $\operatorname{Mat}(n \times n, F)$, there exists a continuous semi-norm β_P on V such that for any $g \in \operatorname{GL}_n(F)$ we have*

$$|\lambda\left(\pi(\nu(g))v\right)| \leq \beta_P(v)\frac{1}{|P(g)|}|\det g|^{-M_0}.$$

Proof. We have

$$\lambda(\pi(\mu(X))v) = \psi(\operatorname{Tr} X)\lambda(v) \quad \forall X \in \operatorname{Mat}(n \times n, F).$$

We have then

$$\lambda(d\pi(d\mu(X))v) = 2\pi \mathrm{i} \operatorname{Re}\operatorname{Tr}(X)\lambda(v) \quad \forall X \in \operatorname{Mat}(n \times n, F)$$

and hence

$$\lambda(\pi(\nu(g))d\pi(d\mu(X))v) = 2\pi \mathrm{i} \operatorname{Re}\operatorname{Tr}(gX)\lambda(\pi(\nu(g))v) \quad \forall X \in \operatorname{Mat}(n \times n, F) \text{ and } g \in \operatorname{GL}_n(F).$$

Similarly, if Q is a polynomial on the real vector space $\operatorname{Mat}(n \times n, F)$, there is an element X_Q of the enveloping algebra of $\mathfrak{gl}_{2n}(F)$ such that

$$\lambda(\pi(\nu(g))d\pi(X_Q)v) = Q(g)\lambda(\pi(\nu(g))v) \quad \forall g \in \operatorname{GL}_n(F).$$

We know that there exist a continuous semi-norm β on V and a natural number M such that $|\lambda(\pi(g)v)| \leq \beta(v)||g||^M$ for any $g \in G$. Therefore for any $g \in \operatorname{GL}_n(F)$ we have

$$|Q(g)\lambda(\pi(\nu(g))v)| = |\lambda(\pi(\nu(g))d\pi(\mu(X_Q))v)| \leq \beta(d\pi(X_Q)v)||\nu(g)||^M.$$

Note that $||\nu(g)||^M = P_0(g)|\det g|^{-2M}$ for a suitable polynomial P_0 on the real vector space $\operatorname{Mat}(n \times n, F)$. Therefore, we have, with $M_0 = 2M$,

$$|\lambda(\pi(\nu(g))v)| \leq \beta(d\pi(X_Q)v)\frac{P_0(g)}{|Q(g)|}|\det g|^{-M_0}.$$

We may take Q of the form $Q = P_0 P$ where P is another polynomial. Since $v \mapsto \beta(d\pi(X_Q)v)$ is a continuous semi-norm the Lemma follows. ☐

4. Properties of $L_{\lambda,v}$

Theorem 4.1. *Let (π, V) be an irreducible admissible smooth Fréchet representation of G. Fix a Shalika functional $\lambda \in (V^*)^{HU,\Psi}$ and a vector $v \in V$. Then, for any polynomial p, the product $p(s)L_{\lambda,v}(s)$ is bounded at infinity on every vertical strip of finite width.*

In [FJ93, §§3.3] the following statement is proven.

Lemma 4.2. *For $\operatorname{Re}(s)$ large enough, $L_{\lambda,v}(s)$ is a finite sum of functions of the type*

$$\mathcal{L}_{u,\xi,\Phi}(s) := \int_{g \in G} \Phi(g)\xi(\pi(g)u)|\det g|^{s+n-\frac{1}{2}} dg,$$

where $\Phi \in \mathcal{S}(\operatorname{Mat}(2n \times 2n, F))$, $u \in V$, $\xi \in V^*$.

Now Theorem 4.1 follows from the following one.

Theorem 4.3. *Let (π, V) be an irreducible admissible smooth Fréchet representation of G. Let $\Phi \in \mathcal{S}(\operatorname{Mat}(2n \times 2n, F))$, $u \in V$ and $\xi \in \widetilde{V}$. Then $\mathcal{L}_{u,\xi,\Phi}(s)$ has a meromorphic continuation to $\mathbb{C}$ whose product by any polynomial is bounded at infinity on any vertical strip. The continuation is a holomorphic multiple of $L_\pi(s) = L(s,\pi)$. It satisfies the functional equation*

$$\int \widehat{\Phi}(g)\xi(\pi({}^t g^{-1})u)|\det g|^{1-s+n-\frac{1}{2}} dg = \gamma(s,\pi,\psi)\mathcal{L}_{u,\xi,\Phi}(s)$$

where

$$\gamma(s,\pi,\psi) := \varepsilon(s,\pi,\psi)\frac{L(1-s,\widetilde{\pi})}{L(s,\pi)}.$$

and

$$\widehat{\Phi}(X) := \int_{\mathrm{Mat}(2n\times 2n,F)} \Phi(Y)\psi(\mathrm{tr}(XY^t))dY.$$

Finally, these assertions remain true if ξ is in V^* (topological dual of V).

This theorem is proven in [GJ72] in slightly narrower generality: the vectors u and ξ are $K-$finite and the function Φ is the product of a Gaussian function and a polynomial. For the convenience of the reader we indicate how to extend the results of [GJ72].

We will need the following lemma.

Lemma 4.4. *Let $T \subset G$ be the torus of diagonal matrices. We will also regard T as the subset $(F^\times)^{2n}$ of F^{2n}. Let $\chi : T \to \mathbb{C}^\times$ be a multiplicative character. Let (π, V) be the corresponding representation of principal series of G. Let $v \in V$ and $\xi \in \widetilde{V}$. Let Φ be a Schwartz function on $\mathrm{Mat}(2n \times 2n, F)$.*

Then there exists a Schwartz function $\phi \in \mathcal{S}(F^{2n})$ such that

$$\int_{g\in G} \Phi(g)\xi(\pi(g)v)|\det g|^{s+n-\frac{1}{2}}dg = \int_{t\in T} \phi(t)\chi(t)|\det t|^s dt$$

for any $s \in \mathbb{C}$ such that the integral on the right converges absolutely.

Proof. Let N denote the group of upper triangular matrices with unit diagonal. Let $B = TN$ and δ_B be the module of the group B. Realize V as the space of smooth functions on G that satisfy

$$f(tg) = \chi(t)f(g)\delta_B^{1/2}(t) \text{ and } f(ug) = f(g) \text{ for any } t \in T \text{ and } u \in N.$$

Realize also $\widetilde{V}$ in the corresponding way. Then

$$\xi(\pi(g)v) = \int_{k\in K} v(kg)\xi(k)dk\,,$$

where K is the standard maximal compact subgroup. Now

$$\int_G \Phi(g)\xi(\pi(g)v)|\det g|^{s+\frac{n-1}{2}}dg = \int_G\int_K \Phi(g)v(kg)\xi(k)|\det g|^{s\frac{n-1}{2}}dgdk = \int_G\int_K \Phi(k^{-1}g)v(g)\xi(k)|\det g|^{s+\frac{n-1}{2}}dgdk$$

To compute this integral we set

$$g = \begin{pmatrix} a_1 & u_{1,2} & \cdots & u_{1,2n} \\ 0 & a_2 & \cdots & u_{2,2n} \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & a_{2n} \end{pmatrix} k'\,.$$

Then

$$dg = |a_1|^{1-2n}|a_2|^{2-2n}\cdots|a_{2n-1}|^{-1} \otimes d^\times a_i \otimes du_{i,j}dk'$$

We set

$$\phi(a_1,...,a_{2n}) := \int \Phi\left[k^{-1}\begin{pmatrix} a_1 & u_{1,2} & \cdots & u_{1,2n} \\ 0 & a_2 & \cdots & u_{2,2n} \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & a_{2n} \end{pmatrix} k'\right] v(k')\xi(k)dkdk' \otimes du_{i,j}\,.$$

Clearly ϕ is a Schwartz function on F^{2n} and

$$\int_{g\in G} \Phi(g)\xi(\pi(g)v)|\det g|^{s+\frac{n-1}{2}}dg = \int_{t\in T} \phi(t)\chi(t)|\det t|^s dt$$

for any $s \in \mathbb{C}$ such that the integral on the right converges. □

Now we can prove Theorem 4.3.

Proof of Theorem 4.3. The representation (π, V) is a sub-representation of a principal series representation determined by a character χ of T and the representation $(\widetilde{\pi}, \widetilde{V})$ is then a quotient of the representation determined by χ^{-1}. For $u \in V$ and $\xi \in \widetilde{V}$ (or ξ in the principal series determined by χ^{-1}) we have

$$\mathcal{L}_{u,\xi,\Phi}(s) = \int_{(a_1,..,a_{2n}) \in F^{\times 2n}} \phi(a_1, a_2, ..., a_{2n}) \chi_1(a_1)|a_1|^s ... \chi_{2n}(a_{2n})|a_{2n}|^s d^\times a_1 ... d^\times a_{2n} \,.$$

The right hand side extends to a meromorphic function of s and the product of this function by any polynomial is bounded at infinity in any vertical strip. Moreover, the function ϕ depends continuously on $\Phi, u \in V, \xi \in \widetilde{V}$. Therefore the analytic continuation depends continuously on $\Phi, v \in V, \xi \in \widetilde{V}$. By continuity, it has the properties stated in the Theorem. To extend further to the case where ξ is in the topological dual V^* we appeal to the Dixmier Malliavin Lemma ([DM78]) applied to the representation of $SL_{2n}(F)$ on $\mathcal{S}(\mathrm{Mat}_{2n\times 2n}(F))$ defined by

$$g_1\Phi(X) := \Phi(g_1^{-1}X)$$

Thus we may assume Φ is of the form

$$\Phi(X) = \int_{SL_{2n}(F)} \Phi_1(g_1^{-1}X) f(g_1) dg_1$$

where f_1 is a C^∞ function of compact support on $SL_{2n}(F)$. Then

$$\mathcal{L}_{u,\xi,\Phi} = \mathcal{L}_{u,\xi_1,\Phi_1}$$

where

$$\xi_1(v) := \xi(\pi(f_1)v) \,.$$

Now ξ_1 is in $\widetilde{V}$ and our assertion follows.

□

Remark 4.5. *The previous result with $\xi \in V^*$ is used without comment in* [FJ93], *formula (57). This is why we included a sketch of the proof.*

Theorem 4.6. *There exists $M > 0$ such that for any even integer $M' \geq 2$ and any polynomial p on $\mathbb{C}$, there exists a semi-norm β on V such that $|pL_{\lambda,v}|_{M'+M+\mathrm{i}\mathbb{R}}| \leq \beta(v)$.*

First, we will prove the following lemma.

Lemma 4.7. *There exists $M > 0$ such that, for any even integer $M' \geq 2$, there exists a continuous semi-norm β on V such that $|L_{\lambda,v}(s)| \leq \beta(v)$ for $\mathrm{Re}\, s = M' + M$.*

Proof. By Lemma 3.4 there exists $M_0 > 0$, and for any polynomial P on $\mathrm{Mat}(n \times n, F)$, a continuous semi-norm β_P such that

$$\lambda\left(\pi(\nu(g))v\right) \leq \beta_P(v) \frac{1}{|P(g)|} |\det g|^{-M_0}.$$

Let $M := 1/2 + n^2 + M_0$. Let $M' \geq 2$ be an even integer and let $s \in \mathbb{C}$ with $\mathrm{Re}(s) = M + M'$. Let

$$P(X) = |\det X|^{M'} \prod_{i,j} (1 + X_{ij}\overline{X}_{ij}).$$

Let

$$\beta(v) := \beta_P(v) \int_{X \in \mathrm{Mat}(n\times n,F)} \frac{dX}{\prod_{i,j}(1 + X_{ij}\overline{X}_{ij})}.$$

Now

$$|L_{\lambda,v}| = \left| \int_{GL_n(F)} \lambda(\pi(\nu(g))v) |\det g|^{s-1/2} dg \right| \leq \int_{GL_n(F)} \beta_P(v) \frac{1}{|P(g)|} |\det g|^{-M_0} |\det g|^{n^2+M'+M_0} dg =$$

$$\int_{GL_n(F)} \beta_P(v) \frac{1}{|P(g)|} |\det g|^{n^2+M'} dg = \int_{X \in \mathrm{Mat}(n\times n,F)} \beta_P(v) \frac{1}{|P(X)|} |\det X|^{M'} dX = \beta(v).$$

□

Proof of Theorem 4.6. For any $g \in GL_n(F)$ we have

$$L_{\lambda,\pi(\nu(g))v}(s) = |\det(g)|^{1/2-s} L_{\lambda,v}(s).$$

We can apply this to $g = \exp(tX)$, with $t \in \mathbb{R}$ and $X \in \mathrm{Mat}(n \times n, F)$. We get

$$L_{\lambda,\pi(\nu(g))v}(s) = |\det(\exp(tX))|^{\frac{1}{2}-s} L_{\lambda,v}(s).$$

Differentiating this identity with respect to t at $t = 0$, we get

$$L_{\lambda,d\pi(d\nu(X))v}(s) = (\frac{1}{2} - s)c(X)L_{\lambda,v}(s),$$

where $c(X) = \mathrm{Tr}\, X$ if $F = \mathbb{R}$ and $c(X) = 2\,\mathrm{Re}\,\mathrm{Tr}\, X$ if $F = \mathbb{C}$. Similarly, for any polynomial p on $\mathbb{C}$ there exists X_p in the universal enveloping algebra of $\mathfrak{gl}_{2n}(F)$ such that

$$L_{\lambda,d\pi(X_p)v}(s) = p(s)L_{\lambda,v}(s).$$

The theorem follows now from Lemma 4.7. ☐

Notation 4.8. *Define another representation π^θ on the same space V by $\pi^\theta(g) := \pi((g^t)^{-1})$. Recall that $\pi^\theta \cong \widetilde{\pi}$.*

For any Shalika functional $\lambda : \pi \to \mathbb{C}$ we define $\lambda^\theta : \pi^\theta \to \mathbb{C}$ by

$$\lambda^\theta(v) := \lambda\left(\pi\begin{pmatrix} 0_{nn} & Id_{nn} \\ -Id_{nn} & 0_{nn}\end{pmatrix} v\right).$$

It is easy to see that λ^θ is a Shalika functional for the rerpresentation π^θ.

Theorem 4.9 ([FJ93], Proposition 3.3)**.**

$$\gamma(s,\pi,\psi)L^{\pi}_{\lambda,v}(s) = L^{\pi^\theta}_{\lambda^\theta,v}(1-s).$$

Using Theorem 4.6 we obtain the following corollary.

Corollary 4.10. *There exists $N < 0$ such that for any odd integer $N' \leq -1$ and any polynomial p on $\mathbb{C}$ there exists a semi-norm β on V such that $|pL_{\lambda,v}|_{N'+N+i\mathbb{R}}| \leq \beta(v)$.*

5. Uniqueness of Shalika functionals

Theorem 5.1. *Let (π, V) an irreducible admissible representation of G. Let λ be a Shalika functional. Then the functional $L(\lambda) : V \to \mathbb{C}$ defined by*

$$L(\lambda)(v) := \frac{L_{\lambda,v}}{L_\pi}(\frac{1}{2})$$

is continuous.

Proof. By Theorem 4.6 we choose $M > 1$ such that for any polynomial p there exists a semi-norm β on V such that $|pL_{\lambda,v}|_{M+i\mathbb{R}}| \leq \beta(v)$. By Corollary 4.10 we choose $N < 0$ such for any polynomial p there exists a semi-norm β' on V such that $|pL_{\lambda,v}|_{N+i\mathbb{R}}| \leq \beta'(v)$. Let q be a polynomial such that the multiset of poles of $1/q$ (with multiplicities) coincides with the multiset of poles of $L_\pi|_{[N,M]+i\mathbb{R}}$. Here, $[N, M] + i\mathbb{R}$ denotes the strip $N \leq \mathrm{Re}(s) \leq M$. It is enough to show that the map $L'(\lambda)$ defined by $L'(\lambda)(v) := L_{\lambda,v}q(\frac{1}{2})$ is continuous. Now there exists a semi-norm α on V such that, for any $v \in V$,

$$|qL_{\lambda,v}|_{M+i\mathbb{R}}| \leq \alpha(v) \text{ and } |qL_{\lambda,v}|_{N+i\mathbb{R}}| \leq \alpha(v).$$

By Theorem 4.1, for any $v \in V$, there exists Δ such that $|qL_{\lambda,v}(s)| \leq \alpha(v)$ if $s \in [N, M] + i\mathbb{R}$ and $|\mathrm{Im}\, s| \geq \Delta$. Now by maximal modulus principle $L'(\lambda)(v) \leq \alpha(v)$ for any $v \in V$. ☐

Definition 5.2. *Let (π, V) an irreducible admissible representation of G. We define a map*

$$L : (V^*)^{HU,\Psi} \to (V^*)^{HG_1}$$

by

$$L(\lambda)(v) = \frac{L_{\lambda,v}}{L_\pi}(\frac{1}{2})$$

Theorem 3.1 implies

Proposition 5.3. *L is a monomorphism.*

Now we use the following theorem from [AG08b].

Theorem 5.4 (see [AG08b], Theorem I)**.** *The pair $(GL_{2n}, GL_n \times GL_n)$ is a Gelfand pair. Namely,*

$$\dim(V^*)^{\mathrm{GL}_n(F)\times \mathrm{GL}_n(F)} \leq 1.$$

Corollary 5.5. *Theorem 1.1 holds. Namely,*

$$\dim(V^*)^{HU,\Psi} \leq 1.$$

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SMOOTH TRANSFER OF KLOOSTERMAN INTEGRALS (THE ARCHIMEDEAN CASE)

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Abstract. We establish the existence of a transfer, which is compatible with Kloosterman integrals, between Schwartz functions on $\mathrm{GL}_n(\mathbb{R})$ and Schwartz functions on the variety of non-degenerate Hermitian forms. Namely, we consider an integral of a Schwartz function on $\mathrm{GL}_n(\mathbb{R})$ along the orbits of the two sided action of the groups of upper and lower unipotent matrices twisted by a non-degenerate character. This gives a smooth function on the torus. We prove that the space of all functions obtained in such a way coincides with the space that is constructed analogously when $\mathrm{GL}_n(\mathbb{R})$ is replaced with the variety of non-degenerate hermitian forms. We also obtain similar results for $\mathfrak{gl}_n(\mathbb{R})$.

The non-Archimedean case is done in [Jac03a] and our proof follows the same lines. However we have to face additional difficulties that appear only in the Archimedean case.

Contents

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1. Introduction

Let N^n be the subgroup of upper triangular matrices in GL_n with unit diagonal, and let A^n be the group of invertible diagonal matrices. We define a character $\theta : N^n(\mathbb{R}) \to \mathbb{C}^\times$ by

$$\theta(u) = \exp(i \sum_{k=1}^{n-1} u_{k,k+1}).$$

Let $\mathcal{S}(GL_n(\mathbb{R}))$ be the space of Schwartz functions on $GL_n(\mathbb{R})$. We define a map $\Omega : \mathcal{S}(GL_n(\mathbb{R})) \to C^\infty(A^n)$ by

$$\Omega(\Phi)(a) := \int_{(u_1,u_2)\in N^n(\mathbb{R})\times N^n(\mathbb{R})} \Phi(u_1^t a u_2)\theta(u_1u_2)du_1du_2.$$

Similarly, we let $S^n(\mathbb{C})$ be the space of non-degenerate Hermitian matrices $n \times n$. We define a map $\Omega : \mathcal{S}(S^n(\mathbb{C})) \to C^\infty(A^n)$ by

$$\Omega(\Psi)(a) := \int_{u\in N^n(\mathbb{C})} \Psi(\overline{u}^t au)\theta(u\overline{u})du.$$

We say that $\Phi \in \mathcal{S}(GL_n(\mathbb{R}))$ *matches* $\Psi \in \mathcal{S}(S^n(\mathbb{C}))$ if for every $a \in A^n(F)$, we have

$$\Omega(\Phi)(a) = \gamma(a)\Omega(\Psi)(a),$$

where

$$\gamma(a) := sign(a_1)sign(a_1a_2)...sign(a_1a_2,...,a_{n-1}) \text{ for } a = diag(a_1,a_2,...,a_n).$$

The main theorem of this paper is

Theorem A. *For every $\Phi \in \mathcal{S}(GL_n(\mathbb{R}))$ there is a matching $\Psi \in \mathcal{S}(S^n(\mathbb{C}))$, and conversely.*

We also prove a similar theorem for $\mathfrak{gl}_n$.

We also consider non-regular orbital integrals and prove that if two functions match then their non-regular orbital integrals are also equal (up to a suitable transfer factor). This implies in particular that regular orbital integrals are dense in all orbital integrals.

The non-Archimedean counterpart of this paper is done in [Jac03a, Jac03b] and our proof follows the same lines. However we have to face additional difficulties that appear only in the Archimedean case.

For the motivation of this problem we refer the reader to [Jac03a].

In the case of $\mathrm{GL}(2,\mathbb{R})$ Theorem A was proven in [Jac05], using different methods.

1.1. A sketch of the proof.

First we show that the theorem for $\mathfrak{gl}_n$ implies the theorem for GL_n. Then we prove the theorem for $\mathfrak{gl}_n$ by induction. We construct certain open sets $O_i \subset \mathfrak{gl}_n(\mathbb{R})$ (for their definition see §§3.1) and use the *intermediate Kloosterman integrals* in order to describe $\Omega(\mathcal{S}(O_i))$ in terms of $\Omega(\mathcal{S}(GL_i(\mathbb{R})))$ and $\Omega(\mathcal{S}(gl_{n-i}(\mathbb{R})))$. This gives a smooth matching for $\mathcal{S}(O_i)$ by the induction hypothesis. We denote $U := \bigcup O_i$ and $Z := \mathfrak{gl}_n(\mathbb{R}) - U$ and obtain by partition of unity smooth matching for $\mathcal{S}(U)$.

Then we use an important fact. Namely, if Φ matches Ψ then the Fourier transform of Φ matches the Fourier transform of Ψ multiplied by a constant. This is proven in [Jac03a] in the non-Archimedean case and the same proof holds in the Archimedean case. The proof of this fact is based on an explicit formula for the Kloosterman integral of the Fourier transform of Φ in terms of the Kloosterman integral of Φ (see Theorem 3.2.4).

In order to complete the proof of the main theorem we prove the following Key Lemma.

Lemma B. *Let $N^n \times N^n$ act on $\mathfrak{gl}_n$ by $x \mapsto u_1^t x u_2$. Let χ denote the character of $N^n \times N^n$ defined by $\chi(u_1,u_2) = \theta(u_1u_2)$.*

Then any function in $\mathcal{S}(\mathfrak{gl}_n(\mathbb{R}))$ can be written as a sum $f + g + h$ s.t. f is a Schwartz function on U, the Fourier transform of g is a Schwartz function on U and h is a function that annihilates any $(N^n \times N^n, \chi)-$ equivariant distribution on $\mathfrak{gl}_n(\mathbb{R})$ and in particular $\Omega(h) = 0$.

1.2. **The spaces of functions considered.**

Since the proof relies on Fourier transform, in the Archimedean case it would not be appropriate to consider the space of smooth compactly supported functions. Therefore we had to work with Schwartz functions. Theories of Schwartz functions were developed by various authors in various generalities. We chose for this problem the version developed in [AG08, AG] in the generality of Nash (i.e. smooth semi-algebraic) manifolds. In Appendix A of the present paper we develop further the tools for working with Schwartz functions from [AG08, AG] and [AG09, Appendix B], for the purposes of this paper.

1.3. **Difficulties that we encounter in the Archimedean case.**

Roughly speaking, most of the additional difficulties in the Archimedean case come from the fact that the space of Schwartz functions in the Archimedean case is a topological vector space unlike the space of Schwartz functions in the non-Archimedean case which is just a vector space. Part of those difficulties are technical and can be overcome using the theory of nuclear Fréchet spaces. However there are more essential difficulties in the Key Lemma. Namely, in the non-Archimedean case the Key lemma is equivalent to the following one

Lemma C. *Any $(N^n \times N^n, \chi)$-equivariant distribution on $\mathfrak{gl}_n(\mathbb{R})$ supported on Z, whose Fourier transform is also supported on Z, vanishes.*

Note that even this lemma is harder in the Archimedean since we have to deal with transversal derivatives. However, this difficulty is overcome using the fact that the transversal derivatives are controlled by the action of stabilizer of a point on the normal space to its orbit. This action is rather simple since it is an algebraic action of a unipotent group.

The main difficulty, though, is that in the Archimedean case Lemma C in not equivalent to Lemma B but only to the following weak version of it

Lemma D. *Any function in $\mathcal{S}(\mathfrak{gl}_n(\mathbb{R}))$ can be approximated by a sum $f + g + h$ s.t. f is a Schwartz function on U, the Fourier transform of g is a Schwartz function on U and h is a function that annihilates any $(N^n \times N^n, \chi)-$ equivariant distribution on $\mathfrak{gl}_n(\mathbb{R})$ and in particular $\Omega(f) = 0$.*

We believe that the reason that the Key Lemma holds is a part of a general phenomenon. To describe this phenomenon note that a statement concerning equivariant distributions can be reformulated to a statement concerning closure of subspaces of Schwartz functions. The phenomenon is that in many cases this statement holds without the need to consider the closure. We discuss two manifestations of this phenomenon in §§§2.2.2 and 2.2.3, and prove them in appendices A.2 and A.3. The proofs there remind in their spirit the proof of the classical Borel Lemma.

1.4. **Contents of the paper.**

In §2 we fix notational conventions and list the basic facts on Schwartz functions and nuclear Fréchet spaces that we will use.

In §3 we prove the main result. In §§3.1 we introduce the notation that we will use to discuss our problem, and reformulate the main result in this notation. In §§3.2 we introduce the main ingredients of the proof: description of $\Omega(\mathcal{S}(O_i))$ using intermediate Kloosterman integrals, inversion formula that connects Fourier transform to Kloosterman integrals, and the Key lemma. In §§3.3 we deduce the main result, Theorem A, from the main ingredients.

In §4 we prove the inversion formula.

In §5 we prove the Key lemma.

In §6 we consider non-regular orbital integrals, define matching for them and prove that if two functions match then their non-regular orbital integrals also match.

In appendix A we give some complementary facts about Nash manifolds and Schwartz functions on them and prove an analog of Dixmier - Malliavin Theorem and prove dual versions of special cases of uncertainty principle and localization principle. Those versions are two manifestations of the phenomenon described above.

1.5. **Acknowledgments.**

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2. Preliminaries

2.1. General notation.

- All the algebraic varieties and algebraic groups we consider in this paper are real.
- For a group G acting on a set X and a point $x \in X$ we denote by Gx or by $G(x)$ the orbit of x, by G_x the stabilizer of x and by X^G the set of G-fixed points in X.
- For Lie groups G or H we will usually denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{h}$.
- An action of a Lie algebra $\mathfrak{g}$ on a (smooth, algebraic, etc) manifold M is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Note that an action of a (Lie, algebraic, etc) group on M defines an action of its Lie algebra on M.
- For a Lie algebra $\mathfrak{g}$ acting on M, an element $\alpha \in \mathfrak{g}$ and a point $x \in M$ we denote by $\alpha(x) \in T_xM$ the value at point x of the vector field corresponding to α. We denote by $\mathfrak{g}x \subset T_xM$ or by $\mathfrak{g}(x)$ the image of the map $\alpha \mapsto \alpha(x)$ and by $\mathfrak{g}_x \subset \mathfrak{g}$ its kernel.
- For a Lie algebra (or an associative algebra) $\mathfrak{g}$ acting on a vector space V and a subspace $L \subset V$, we denote by $\mathfrak{g}L \subset V$ the image of the action map $\mathfrak{g} \otimes L \to V$.
- For a representation V of a Lie algebra $\mathfrak{g}$ we denote by $V^{\mathfrak{g}}$ the space of $\mathfrak{g}$-invariants and by $V_{\mathfrak{g}} := V/\mathfrak{g}V$ the space of $\mathfrak{g}$-coinvariants.
- For manifolds $L \subset M$ we denote by $N_L^M := (T_M|_L)/T_L$ the normal bundle to L in M.
- Denote by $CN_L^M := (N_L^M)^*$ the conormal bundle.
- For a point $y \in L$ we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space.
- By bundle we always mean a vector bundle.
- For a manifold M we denote by $C^\infty(M)$ the space of infinitely differentiable functions on M, equipped with the standard topology.

2.2. Schwartz functions on Nash manifolds.

We will require a theory of Schwartz functions on Nash manifolds as developed e.g. in [AG08]. Nash manifolds are smooth semi-algebraic manifolds but in the present work, except of Appendix A, only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic* in the body of the paper.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG08]. We will use the following notations.

Notation 2.2.1. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.*

We will need several properties of Schwartz functions from [AG08].

Property 2.2.2 ([AG08], Theorem 4.1.3)**.** *$\mathcal{S}(\mathbb{R}^n)$ = Classical Schwartz functions on $\mathbb{R}^n$.*

Property 2.2.3 ([AG08], Theorem 5.4.3)**.** *Let $U \subset M$ be an open Nash submanifold, then*

$$\mathcal{S}(U) \cong \{\phi \in \mathcal{S}(M) | \quad \phi \text{ is 0 on } M \setminus U \text{ with all derivatives}\}.$$

In this paper we will consider $\mathcal{S}(U)$ as a subspace of $\mathcal{S}(X)$.

Property 2.2.4 (see [AG08], §5)**.** *Let M be a Nash manifold. Let $M = \bigcup_{i=1}^n U_i$ be a finite cover of M by open Nash submanifolds. Then a function f on M is a Schwartz function if and only if it can be written as $f = \sum_{i=1}^{n} f_i$ where $f_i \in \mathcal{S}(U_i)$ (extended by zero to M).*

Moreover, there exists a smooth partition of unity $1 = \sum_{i=1}^{n} \lambda_i$ such that for any Schwartz function $f \in \mathcal{S}(M)$ the function $\lambda_i f$ is a Schwartz function on U_i (extended by zero to M).

Property 2.2.5 (see [AG08], §5). *Let $Z \subset M$ be a Nash closed submanifold. Then restriction maps $\mathcal{S}(M)$ onto $\mathcal{S}(Z)$.*

Property 2.2.6 ([AG09], Theorem B.2.4)**.** *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a Nash bundle over N. Fix Nash measures μ on M and ν on N.*

Then
(i) there exists a unique continuous linear map $\phi_* : \mathcal{S}(M) \to \mathcal{S}(N)$ such that for any $f \in \mathcal{S}(N)$ and $g \in \mathcal{S}(M)$ we have

$$\int_{x\in N} f(x)\phi_* g(x) d\nu = \int_{x\in M} (f(\phi(x)))g(x) d\mu.$$

In particular, we mean that both integrals converge.
(ii) If ϕ is surjective then ϕ_* is surjective.

In fact

$$\phi_* g(x) = \int_{z\in\phi^{-1}(x)} g(z) d\rho$$

for an appropriate measure ρ.

We will need the following analog of Dixmier - Malliavin theorem.

Property 2.2.7. *Let $\phi : M \to N$ be a Nash map of Nash manifolds. Then multiplication defines an onto map $\mathcal{S}(M) \otimes \mathcal{S}(N) \twoheadrightarrow \mathcal{S}(M)$.*

For proof see Theorem A.1.1.
We will also need the following notion.

Notation 2.2.8. *Let $\phi : M \to N$ be a Nash map of Nash manifolds. We call a function $f \in C^\infty(M)$ Schwartz along the fibers of ϕ if for any Schwartz function $g \in \mathcal{S}(N)$, we have $(g \circ \phi)f \in \mathcal{S}(M)$.*

We denote the space of such functions by $\mathcal{S}^{\phi,N}(M)$. If there is no ambiguity we will sometimes denote it by $\mathcal{S}^\phi(M)$ or by $\mathcal{S}^N(M)$. We define the topology on $\mathcal{S}^\phi(M)$ using the following system of semi-norms: for any seminorms α on $\mathcal{S}(N)$ and β on $\mathcal{S}(M)$ we define

$$\mathfrak{N}^\alpha_\beta(f) := \sup_{g\in\mathcal{S}(N)|\alpha(g)<1} \beta(f(g\circ\phi)).$$

We will use the following corollary of Property 2.2.6.

Corollary 2.2.9. *Let $\phi : M \to N$ be a Nash map and $\psi : L \to M$ be a Nash submersion. Fix Nash measures on L and M. Then there is a natural continuous linear map $\phi_* : \mathcal{S}^N(L) \to \mathcal{S}^N(M)$.*

Remark 2.2.10. *Let $\phi : M \to N$ be a Nash map of Nash manifolds. Let $V \subset N$ be a dense open Nash submanifold. Let $U := \phi^{-1}(V)$. Suppose that U is dense in M. Then we have embeddings*

$$\mathcal{S}(M) \hookrightarrow \mathcal{S}^{\phi,N}(M) \hookrightarrow \mathcal{S}^{\phi,V}(U).$$

In this paper we will view $\mathcal{S}(M)$ and $\mathcal{S}^{\phi,N}(M)$ as subspaces of $\mathcal{S}^{\phi,V}(U)$.

2.2.1. *Fourier transform.*

Notation 2.2.11. *Let V be a finite dimensional real vector space. Let B be a non-degenerate bilinear form on V and ψ be a non-trivial additive character of $\mathbb{R}$. Then B and ψ define Fourier transform with respect to the self-dual Haar measure on V. We denote it by $\mathcal{F}_{B,\psi} : \mathcal{S}(V) \to \mathcal{S}(V)$. If there is no ambiguity, we will omit B and ψ. We will also denote by $\mathcal{F}^*_{B,\psi} : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ the dual map.*

We will use the following trivial observation.

Lemma 2.2.12. *Let V be a finite dimensional real vector space. Let a Nash group G act linearly on V. Let B be a G-invariant non-degenerate symmetric bilinear form on V. Let ψ be a non-trivial additive character of $\mathbb{R}$. Then $\mathcal{F}_{B,\psi}$ commutes with the action of G.*

2.2.2. *Dual uncertainty principle.*

Theorem 2.2.13. *Let V be a finite dimensional real vector space. Let B be a non-degenerate bilinear form on V and ψ be a non-trivial additive character of $\mathbb{R}$. Let $L \subset V$ be a subspace. Suppose that $L^{\perp} \nsubseteq L$. Then*

$$\mathcal{S}(V - L) + \mathcal{F}(\mathcal{S}(V - L)) = \mathcal{S}(V).$$

For proof see Appendix A.3.

Remark 2.2.14. *It is much easier to prove that*

$$\overline{\mathcal{S}(V - L) + \mathcal{F}(\mathcal{S}(V - L))} = \mathcal{S}(V)$$

since this is equivalent to the fact that there are no distributions on V supported in L with Fourier transform supported in L.

2.2.3. *Coinvariants in Schwartz functions.*

Theorem 2.2.15. *Let a connected algebraic group G act on a real algebraic manifold X. Let Z be a G-invariant Zariski closed subset of X. Let $\mathfrak{g}$ be the Lie algebra of G. Let χ be a unitary character of G. Suppose that for any $\mathfrak{z} \in Z$ and $k \in \mathbb{Z}_{\geq 0}$ we have*

$$(\chi \otimes \mathrm{Sym}^k(CN_{z,Gz}^X) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z}))_{\mathfrak{g}_z} = 0.$$

Then

$$\mathcal{S}(X) = \mathcal{S}(X - Z) + \mathfrak{g}(\mathcal{S}(X) \otimes \chi).$$

For proof see Appendix A.2.

Corollary 2.2.16. *Let a unipotent group G act on a real algebraic manifold X. Let χ be a unitary character of G.*

Let $Z \subset X$ be a Zariski closed G-invariant subset. Suppose also that for any point $z \in Z$ the restriction $\chi|_{G_z}$ is non-trivial. Then

$$\mathcal{S}(X) \otimes \chi = \mathcal{S}(X - Z) \otimes \chi + \mathfrak{g}(\mathcal{S}(X) \otimes \chi),$$

where $\mathfrak{g}$ is the Lie algebra of G.

Proof. The action of G_z on $\mathrm{Sym}^k(CN_{z,Gz}^X) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z})$ is algebraic and hence if G is unipotent this action is unipotent and therefore if $(\chi)_{\mathfrak{g}_z} = 0$ then

$$(\chi \otimes \mathrm{Sym}^k(CN_{z,Gz}^X) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z}))_{\mathfrak{g}_z} = 0.$$

□

Remark 2.2.17. *Note that the statement that $\mathcal{S}(X) \otimes \chi = \overline{\mathcal{S}(X - Z) \otimes \chi + \mathfrak{g}(\mathcal{S}(X) \otimes \chi)}$ is equivalent to the statement that any G-invariant distribution on X which is supported on Z vanishes, which is a generalization of a result from* [KV96].

2.3. **Nuclear Fréchet spaces.**

A good exposition on nuclear Fréchet spaces can be found in Appendix A of [CHM00].

We will need the following well-known facts from the theory of nuclear Fréchet spaces.

Proposition 2.3.1 (see e.g. [CHM00], Appendix A)**.**
Let V be a nuclear Fréchet space and W be a closed subspace. Then both W and V/W are nuclear Fréchet spaces.

Proposition 2.3.2 (see e.g. [CHM00], Appendix A)**.**

Let $0 \to V \to W \to U \to 0$ be an exact sequence of nuclear Fréchet spaces. Suppose that the embedding $V \to W$ is closed. Let L be a nuclear Fréchet space. Then the sequence $0 \to V\widehat{\otimes}L \to W\widehat{\otimes}L \to U\widehat{\otimes}L \to 0$ is exact and the embedding $V\widehat{\otimes}L \to W\widehat{\otimes}L$ is closed.

Corollary 2.3.3.
Let $V \to W$ be onto map between nuclear Fréchet spaces and L be a nuclear Fréchet space. Then the map $V\widehat{\otimes}L \to W\widehat{\otimes}L$ is onto.

Corollary 2.3.4. *Let $\phi_i : V_i \to W_i$ $i = 1, 2$ be onto maps between nuclear Fréchet spaces. Then the map $\phi_1\widehat{\otimes}\phi_2 : V_1\widehat{\otimes}V_2 \to W_1\widehat{\otimes}W_2$ is onto.*

Proposition 2.3.5 (see e.g. [AG], Corollary 2.6.2)**.**
Let M be a Nash manifold. Then $\mathcal{S}(M)$ is a nuclear Fréchet space.

Proposition 2.3.6 (see e.g. [AG], Corollary 2.6.3)**.**
Let M_i, $i = 1, 2$ be Nash manifolds Then

$$\mathcal{S}(M_1 \times M_2) = \mathcal{S}(M_1)\widehat{\otimes}\mathcal{S}(M_2).$$

Definition 2.3.7. *By a subspace of a topological vector space V we mean a linear subspace $L \subset V$ equipped with a topology such that the embedding $L \subset V$ is continuous.*

Note that by Banach open map theorem if L and V are nuclear Fréchet spaces and L is closed in V then the topology of L is the induced topology from V.

By an image of a continuous linear map between topological vector spaces we mean the image equipped with the quotient topology. Similarly for a continuous linear map between topological vector spaces $\phi : V_1 \to V_2$ and a subspace $L \subset V_1$ we the image $\phi(L)$ to be equipped with the quotient topology.

*Similarly a **sum** of two subspaces will be considered with the quotient topology of the direct sum.*

Remark 2.3.8. *Note that by Proposition 2.3.1, sum of nuclear Fréchet spaces and image of a nuclear Fréchet space are nuclear Fréchet spaces.*

Note also the operations of taking sum of subspaces and image of subspace commute.

Finally note that if L and L' are two nuclear Fréchet subspaces of a complete locally convex topological vector space V which coincide as linear subspaces then they are the same. Indeed, by Banach open map theorem they are both the same as $L + L'$.

Notation 2.3.9. *Let V_i, $i = 1, 2$ be locally convex complete topological vector spaces. Let $L_i \subset V_i$ be subspaces. We denote by $\mathcal{M}^{V_1,V_2}_{L_1,L_2} : L_1\widehat{\otimes}L_2 \to V_1\widehat{\otimes}V_2$ the natural map.*

From Corollary 2.3.4 we obtain the following corollary.

Corollary 2.3.10. *Let V_i, $i = 1, 2$ be locally convex complete topological vector spaces. Let L_i, $i = 1, 2$ be nuclear Fréchet spaces. Let $\phi_i : L_i \to V_i$ be continuous linear maps. Then*

$$\mathrm{Im}(\phi_1\widehat{\otimes}\phi_2) = \mathrm{Im}(\mathcal{M}^{V_1,V_2}_{\mathrm{Im}(\phi_1),\mathrm{Im}(\phi_2)}).$$

Notation 2.3.11. *Let M_i, $i = 1, 2$ be smooth manifolds. We denote by $\mathcal{M}_{M_1,M_2} : C^\infty(M_1)\widehat{\otimes}C^\infty(M_2) \to C^\infty(M_1 \times M_2)$ the product map. For two subspaces $L_i \subset C^\infty(M_i)$ we denote by $\mathcal{M}_{L_1,L_2} : L_1\widehat{\otimes}L_2 \to C^\infty(M_1 \times M_2)$ the composition $\mathcal{M}_{M_1,M_2} \circ \mathcal{M}^{C^\infty(M_1),C^\infty(M_2)}_{L_1,L_2}$.*

3. Proof of the main result

3.1. **Notation.**

In this paper we let D be a semi-simple 2-dimensional algebra over $\mathbb{R}$, i.e. $D = \mathbb{C}$ or $D = \mathbb{R} \oplus \mathbb{R}$. Let $a \mapsto \overline{a}$ denote the non-trivial involution of D, i.e. complex conjugate or swap. Let n be a natural number. Let $\psi : \mathbb{R} \to \mathbb{C}^\times$ be a nontrivial character. The following notation will be used throughout the body of the paper. In case when there is no ambiguity we will omit from the notations the n, the D and the ψ.

- Denote by $H^n(D)$ the space of hermitian matrices of size n.
- Denote $S^n(D) := H(D) \cap GL_n(D)$.
- Denote by $\Delta^n_i : H \to \mathbb{R}$ the main i-minor.

- Let $O_i^n(D) \subset H$ be the subset of matrices with $\Delta_i \neq 0$.
- Let $U^n(D) := \bigcup_{i=1}^{n-1} O_i$ and $Z^n(D) := H - U$.
- Let $N^n(D) < GL_n(D)$ be the subgroup consisting of upper unipotent matrices.
- Let $\mathfrak{n}^n(D)$ denote the Lie algebra of N^n.
- We define a character $\chi_\psi : N \to \mathbb{C}^\times$ by $\chi_\psi(x) := \psi(\sum_{i=1}^{n-1}(x_{i,i+1} + \overline{x_{i,i+1}}))$.
- Let the group N act on H by $x \mapsto \overline{u}^t xu$.
- Fix a symmetric $\mathbb{R}$-bilinear form B_D^n on H by $B(x,y) := \mathrm{Tr}_\mathbb{R}(xwyw)$, where $w := w_n$ is the longest element in the Weyl group of GL_n.
- Denote by $A^n < GL_n(\mathbb{R})$ the subgroup of diagonal matrices. We will also view A^n as a subset of $S^n(D)$.
- Define $\Omega_D^{n,\psi} : \mathcal{S}^{\det,\mathbb{R}^\times}(S^n(D)) \to C^\infty(A^n)$ by
$$\Omega_D^{n,\psi}(\Psi)(a) := \int_N \Psi(\overline{u}^t au)\chi(u)du.$$
Here, du is the standard Haar measure on N.
For proof that the integral converges absolutely, depends smoothly on a and defines a continuous map $\mathcal{S}^{\det}(S^n(D)) \to C^\infty(A^n)$ see Proposition 3.1.1. By Remark 2.2.10 $\Omega_D^{n,\psi}$ defines in particular a continuous map $\mathcal{S}(H^n(D)) \to C^\infty(A^n)$.
- Denote by $N_i^n(D) < N^n(D)$ the subgroup defined by
$$N_i^n(D) := \left\{ \begin{pmatrix} Id_i & * \\ 0 & Id_{n-i} \end{pmatrix} \right\}.$$
- Define $\Omega_{D,i}^{n,\psi} : \mathcal{S}^{\Delta_i}(O_i^n) \to \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(S^i \times H^{n-i})$, where $S^i \times H^{n-i}$ is considered as a subspace of H_n, in the following way
$$\Omega_{D,i}^{n,\psi}(\Psi)(a) := \int_{N_i^n} \Psi(\overline{u}^t au)\chi(u)du.$$
Here, du is the standard Haar measure on N_i^n.
For proof that the integral converges absolutely, depends smoothly on a and defines a continuous map $\mathcal{S}^{\Delta_i}(O_i^n) \to \mathcal{S}^{\Delta_i}(S^i \times H^{n-i})$ see Proposition 3.1.1.
- Define a character $\eta_D : \mathbb{R}^\times \to \{\pm 1\}$ by $\eta_D = 1$ if $D = \mathbb{R} \oplus \mathbb{R}$ and $\eta_D = \mathrm{sign}$ if $D = \mathbb{C}$.
- Define $\sigma : H^n(D) \to \mathbb{R}$ by $\sigma(x) := \prod_{i=1}^{n-1} \Delta_i^n(x)$.
- Define $\widetilde{\Omega}_D^{n,\psi} : \mathcal{S}^{\det,\mathbb{R}^\times}(S^n) \to C^\infty(A^n)$ by
$$\widetilde{\Omega}_D^n(\Psi)(a) := \eta(\sigma(a))|\sigma(a)|\Omega(\Psi)(a)$$
- Define $\widetilde{\Omega}_{D,i}^{n,\psi} : \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(O_i^n) \to \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(S^i \times H^{n-i})$, in the following way
$$\widetilde{\Omega}_{D,i}^{n,\psi}(\Psi)(a) := \eta(\Delta_i(a))^{n-i}|\Delta_i(a)|^{n-i}\Omega_i^n.$$
- We define $\Omega_D^{n_1,\dots,n_k,\psi} : \mathcal{S}^{\det\times\dots\times\det}(S^{n_1}(D) \times \dots \times S^{n_k}(D)) \to C^\infty(A^{n_1} \times \dots \times A^{n_k})$ in a similar way to $\Omega_D^{n,\psi}$. Analogously we define $\widetilde{\Omega}_D^{n_1,\dots,n_k,\psi}$.

Proposition 3.1.1.
(i) The integral $\Omega_D^{n,\psi}$ converges absolutely and defines a continuous map $\mathcal{S}^{\det}(S^n(D)) \to C^\infty(A^n)$.
(ii) The integral $\Omega_{D,i}^{n,\psi}$ converges absolutely and defines a continuous map $\mathcal{S}^{\Delta_i}(O_i^n) \to \mathcal{S}^{\Delta_i}(S^i \times H^{n-i})$.

Proof.
(i) Consider the map $\beta : H \to \mathbb{R}^n$ defined by $\beta = (\Delta_1, ..., \Delta_n)$. Consider A to be embedded in $\mathbb{R}^n$ by $(t_1, ...t_n) \mapsto (t_1, t_1t_2, ..., t_1t_2...t_n)$. Let $V := \beta^{-1}(A) \subset H$. Let $p_n : \mathbb{R}^n \to \mathbb{R}$ denote the projection on the last coordinate. Note that the action map defines an isomorphism $N \times A \to V$. Let $\alpha : V \to N$ denote the projection. Let $\mathfrak{X} \in \mathcal{S}^{Id}(V)$ be defined by $\mathfrak{X}(v) := \chi(\alpha(v))$. Define $\Omega' : \mathcal{S}^{\beta,A}(V) \to \mathcal{S}^{Id}(A)$ by $\Omega'(f) := \beta_*(\mathfrak{X}f)$. Now, Ω is given by the following composition
$$\mathcal{S}^{\det,\mathbb{R}^\times}(S) \subset \mathcal{S}^{\beta,p_n^{-1}(\mathbb{R}^\times)}(S) \subset \mathcal{S}^{\beta,A}(V) \xrightarrow{\Omega'} \mathcal{S}^{Id}(A) \subset C^\infty(A).$$

(ii) Consider $S^i \times H^{n-i}$ as a subset in H^n. Denote it by B. Consider the action map $N_i \times B \to H$. Note that it is an open embedding and its image is O_i. We consider the standard Haar measures on B and N_i, and their multiplication on O_i. Consider the projections: $\alpha_i : O_i \to N_i$ and $\beta_i : O_i \to B$. Let $\mathfrak{X}_i \in \mathcal{S}^{Id}(O_i)$ be defined by $\mathfrak{X}_i(v) := \chi(\alpha_i(v))$. Consider $(\beta_i)_* : \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(O_i) \to \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(B)$. Now, $\Omega_i(f) = (\beta_i)_*(\mathfrak{X}_i f)$. □

The main theorem (Theorem A) can be reformulated now in the following way:

Theorem 3.1.2.
(i) $\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C})))$.
(ii) $\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(S(\mathbb{C})))$.

3.2. **Main ingredients.**
In this subsection we list three main ingredients of the proof of the main theorem.

3.2.1. *Intermediate Kloosterman Integrals.*

Proposition 3.2.1.
(i) The map $\widetilde{\Omega}_i^n$ *defines an onto map* $\mathcal{S}(O_i^n) \to \mathcal{S}(S^i \times H^{n-i})$.
(ii) $\widetilde{\Omega}^n = \widetilde{\Omega}^{i,n-i} \circ \widetilde{\Omega}_i^n$.

Proof. (i) follows from Property 2.2.6, since the map β_i from the proof of Proposition 3.1.1 is a surjective submersion.
(ii) is straightforward. □

Proposition 3.2.2. $\widetilde{\Omega}^{m,n}(\mathcal{S}(S^m \times H^n)) = \operatorname{Im}\mathcal{M}_{\widetilde{\Omega}^m(\mathcal{S}(S^m)),\widetilde{\Omega}^n(\mathcal{S}(H^n))}$.

Proof. Follows from the fact that $\widetilde{\Omega}^{m,n}|_{\mathcal{S}(S^m\times H^n)} = \widetilde{\Omega}^m|_{\mathcal{S}(S^m)}\widehat{\otimes}\widetilde{\Omega}^n|_{\mathcal{S}(H^n)} \circ \mathcal{M}_{S^m,H^n}$ and Corollary 2.3.10. □

From the last two propositions we obtain the following corollary.

Corollary 3.2.3. $\widetilde{\Omega}^n(\mathcal{S}(O_i^n)) = \operatorname{Im}\mathcal{M}_{\widetilde{\Omega}^{n-i}(\mathcal{S}(S^{n-i})),\widetilde{\Omega}^i(\mathcal{S}(H^i))}$.

3.2.2. *Inversion Formula.*

Theorem 3.2.4 (Jacquet)**.**

$$\widetilde{\Omega}^{\overline{\psi}}(\mathcal{F}(f))(diag(a_1,...,a_n)) =$$
$$= c^{n(n-1)/2}\int ... \int \widetilde{\Omega}^{\psi}(f)(diag(p_1,..p_n))\psi(-\sum_{i=1}^{n} a_{n+1-i}p_i + \sum_{i=1}^{n-1} 1/(a_{n-i}p_i))dp_n...dp_1.$$

Here, c is a constant, we will discuss it in §§4.2. The integral here is just an iterated integral. In particular we mean that the integral converges as an iterated integral.

The proof is essentially the same as in the p-adic case (see [Jac03a, Section 7]). For the sake of completeness we repeat it in §4.

3.2.3. *Key Lemma.*

Lemma 3.2.5 (Key Lemma)**.** *Consider the action of N on $\mathcal{S}(H)$ to be the standard action twisted by χ. Then*

$$\mathcal{S}(H) = \mathcal{S}(U) + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}\mathcal{S}(H).$$

For proof see §5.

3.3. **Proof of the main result.**

We prove Theorem 3.1.2 by induction. The base $n = 1$ is obvious. Thus, from now on we assume that $n \geq 2$ and that Theorem 3.1.2 holds for all dimensions smaller than n.

Proposition 3.3.1.

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(O_i(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(O_i(\mathbb{C}))).$$

Proof. Follows from Corollary 3.2.3 and the induction hypothesis. □

Corollary 3.3.2.

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(U(\mathbb{C}))).$$

Proof. Follows from the the previous proposition and partition of unity (property 2.2.4). □

Corollary 3.3.3. *Part (i) of Theorem 3.1.2 holds. Namely,* $\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C})))$.

Proof. By the previous Corollary and Theorem 3.2.4,

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{F}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R})))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}(U(\mathbb{C})))).$$

Clearly, $\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathfrak{n}\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathfrak{n}\mathcal{S}(H(\mathbb{C}))) = 0$. Hence, by Remark 2.3.8

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R})) + \mathcal{F}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R}))) + \mathfrak{n}\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(U(\mathbb{C})) + \mathcal{F}(\mathcal{S}(U(\mathbb{C}))) + \mathfrak{n}\mathcal{S}(H(\mathbb{C}))),$$

where we again consider the action of N on $\mathcal{S}(H)$ to be twisted by χ. Therefore, by the Key Lemma

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C}))).$$

□

It remains to prove part (ii) of Theorem 3.1.2.

Proof of part (ii) of Theorem 3.1.2. By Property 2.2.7,

$$\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R})) = \mathcal{S}(\mathbb{R}^\times)\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R})),$$

and hence

$$\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R})) = \mathcal{S}(\mathbb{R}^\times)\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R})),$$

where the action of $\mathcal{S}(\mathbb{R}^\times)$ on $\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))$ is given via $\det : H(\mathbb{R}\oplus\mathbb{R}) \to \mathbb{R}$.

Hence

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R}))) = \mathcal{S}(\mathbb{R}^\times)\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))).$$

By part (i)

$$\mathcal{S}(\mathbb{R}^\times)\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \mathcal{S}(\mathbb{R}^\times)\widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C}))).$$

As before,

$$\mathcal{S}(\mathbb{R}^\times)\widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(\mathbb{R}^\times)\mathcal{S}(H(\mathbb{C}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(S(\mathbb{C}))).$$

□

Remark 3.3.4. *One can give an alternative proof, that does not use Property 2.2.7, in the following way. Define maps* $\widetilde{\Omega}' : \mathcal{S}(H\times\mathbb{R}^\times) \to C^\infty(A\times\mathbb{R}^\times)$ *similarly to* $\widetilde{\Omega}$, *and not involving the second coordinate. From (i), using §2.3, we get that* $\operatorname{Im}\widetilde{\Omega}'_{\mathbb{C}} = \operatorname{Im}\widetilde{\Omega}'_{\mathbb{R}\oplus\mathbb{R}}$. *Using the graph of* det *we can identify* S *with a closed subset of* $H\times\mathbb{R}^\times$ *and* A *with a closed subset of* $A\times\mathbb{R}^\times$. *By Property 2.2.5, the restriction map* $\mathcal{S}(H\times\mathbb{R}^\times) \to \mathcal{S}(S)$ *is onto and hence* $\widetilde{\Omega}(\mathcal{S}(S)) = \operatorname{Im} res\circ\widetilde{\Omega}'$, *where* $res : C^\infty(A\times\mathbb{R}^\times) \to C^\infty(A)$ *is the restriction. This implies (ii).*

In fact, this alternative proof of (ii) is obtained from the previous proof by replacing Property 2.2.7 with its weaker version that states (in the notations of property 2.2.7) that the map $\mathcal{S}(M)\widehat{\otimes}\mathcal{S}(N) \to \mathcal{S}(M)$ *is onto. This is much simpler version since it follows directly from Property 2.2.5 and Proposition 2.3.6.*

4. Proof of the inversion formula

In this section we adapt the proof of Theorem 3.2.4 given in [Jac03a] to the Archimedean case. The proof is by induction. The induction step is based on analogous formula for the intermediate Kloostermann integral which is based on the Weil formula.

In §§4.1 we give notations for various Fourier transforms on H. In §§4.2 we recall the Weil formula and consider its special case which is relevant for us. In §§4.3 we introduce the Jacquet transform and the intermediate Jacquet transform which appears on the right hand side of the inversion formulas. In §§4.4 we prove the intermediate inversion formula. In §§4.5 we prove the inversion formula.

4.1. **Fourier transform.**

- We denote by $\mathcal{F}' := \mathcal{F}'_{H_n} : \mathcal{S}(H_n) \to \mathcal{S}(H_n)$ the Fourier transform w.r.t. the trace form.
- Note that $\mathcal{F}_{H_n} = ad(w) \circ \mathcal{F}'_{H_n} = \mathcal{F}'_{H_n} \circ ad(w)$.
- We denote by $\mathcal{F}'_{H_i \times H_{n-i}} : \mathcal{S}(H_n) \to \mathcal{S}(H_n)$ the partial Fourier transform w.r.t. the trace form on $H_i \times H_{n-i}$.
- We denote by $(H_i \times H_{n-i})^{\perp'} \subset H_n$ the orthogonal compliment to $H_i \times H_{n-i}$ w.r.t. the trace form.
- We denote by $\mathcal{F}'_{H_i \times H_{n-i}{}^{\perp'}} : \mathcal{S}(H_n) \to \mathcal{S}(H_n)$ the partial Fourier transform w.r.t. the trace form on $H_i \times H_{n-i}{}^{\perp'}$.
- Note that $\mathcal{F}'_{H_n} = \mathcal{F}'_{H_i \times H_{n-i}{}^{\perp'}} \circ \mathcal{F}'_{H_i \times H_{n-i}} = \mathcal{F}'_{H_i \times H_{n-i}} \circ \mathcal{F}'_{H_i \times H_{n-i}{}^{\perp'}}$.

4.2. **The Weil formula.**

Let ψ be a non-trivial additive character of $\mathbb{R}$. Recall the one dimensional Weil formula:

Proposition 4.2.1. *Let $a \in \mathbb{R}^\times$. Consider the function $\xi : D \to \mathbb{R}$ defined by $\xi(x) = \psi(ax\bar{x})$ as a distribution on D. Then $\mathcal{F}^*(\xi) = \zeta$, where ζ is a distribution defined by the function $\zeta(x) = |a|^{-1}\eta_D(a)c(D,\psi)\psi(-x\bar{x}/a)$.*

One can take this as a definition of $c(D, \psi)$.

The following proposition follows by a straightforward computation.

Proposition 4.2.2.
(i) $c(\mathbb{R} \oplus \mathbb{R}, \psi) = 1$
(ii) $c(\mathbb{C}, \psi)^2 = -1$
(iii) $c(\mathbb{C}, \psi)c(\mathbb{C}, \overline{\psi}) = 1$

Proposition 4.2.1 gives us the following corollary.

Corollary 4.2.3. *Let V be a free module over D equipped with a volume form. We have a natural Fourier transform $\mathcal{F}^* : \mathcal{S}^*(V) \to \mathcal{S}^*(V^*)$. Let Q be a hermitian form on V. Consider the function $\xi : V \to \mathbb{R}$ defined by $\xi(v) = \psi(Q(v))$ as a distribution on V. Let Q^{-1} be a hermitian norm on V^* which is the inverse of Q. Let $\det(Q)$ be the determinant of Q with respect to the volume form on V. Let ζ be a distribution defined by the function*

$$\zeta(x) = |\det(Q)|^{-1}(\eta_D(\det(Q)c(D,\psi))^{\dim V}\psi(-Q^{-1}(x)).$$

Then $\mathcal{F}^*(\xi) = \zeta$.

Corollary 4.2.4. *Let $(A, B) \in S^i \times S^{n-i}$. Consider the function $\xi : H_i \times H_{n-i}{}^{\perp'} \to \mathbb{R}$ defined by $\xi\left[\begin{pmatrix} 0 & \bar{u}^t \\ u & 0 \end{pmatrix}\right] = \psi(BuA\bar{u}^t)$ as a distribution on V. Consider also the function $\zeta : H_i \times H_{n-i}{}^{\perp'} \to \mathbb{R}$ defined by*

$$\zeta\left[\begin{pmatrix} 0 & \bar{u}^t \\ u & 0 \end{pmatrix}\right] = (\eta(\det A)/|\det A|)^{n-i}(\eta(\det B)/|\det B|)^i c(D,\psi)^{(n-i)i}\psi(B^{-1}\bar{u}^t A^{-1}u)$$

as a distribution on V.

Then $(\mathcal{F}'_{H_i \times H_{n-i}{}^{\perp'}})^*(\xi) = \zeta$.

4.3. **Jacquet transform.**

Definition 4.3.1. *Let ψ be a non-trivial additive character of $\mathbb{R}$. Let $0 \leq i \leq n$.*

- *We define $\mathcal{J}'_i : C^\infty(S^i \times S^{n-i}) \to C^\infty(S^i \times S^{n-i})$ by $\mathcal{J}'_i(f)(A,B) = f(A,B)\psi(wB^{-1}w\varepsilon A^{-1}\varepsilon^t)$. Here ε is the matrix with $n-i$ rows and i columns whose first row is the row $(0,0,...,0,1)$ and all other rows are zero.*
- *We define $\mathcal{T}_i : C^\infty(S^i \times S^{n-i}) \to C^\infty(S^{n-i} \times S^i)$ by $\mathcal{T}_i(f)(A,B) = f(B,A)$.*
- *We denote by $\mathfrak{J}_{i,n-i} := \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(S^i \times H^{n-i}) \cap \mathcal{F}^{-1}_{H^{n-i},\psi}(\mathcal{T}_i^{-1}(\mathcal{J}_i'^{-1}(\mathcal{S}^{\Delta_{n-i},\mathbb{R}^\times}(S^{n-i} \times H^i))))$*
- *We define the partial Jacquet transform $\mathcal{J}_i : \mathfrak{J}_{i,n-i} \to \mathcal{S}^{\Delta_{n-i},\mathbb{R}^\times}(S^{n-i} \times H^i))$ by*
$$\mathcal{J}_i := \mathcal{F}_{H^i,\psi} \circ \mathcal{T}_i \circ \mathcal{J}'_i \circ \mathcal{F}_{H^{n-i},\psi}|_{\mathfrak{J}_{i,n-i}}.$$
- *Denote by $\overline{A}$ the set of diagonal matrices in H.*
- *We denote $\mathcal{F}_n : \mathcal{S}^{\Delta_{n-1}}(\overline{A}) \to \mathcal{S}^{\Delta_{n-1}}(\overline{A})$ the Fourier transform w.r.t. the last co-ordinate.*
- *We define*
$$\mathcal{J}_n^{(i)'} : \mathcal{S}^{\Delta_{n-1}}(\overline{A}) \to \mathbb{C}^\infty(A)$$
by
$$\mathcal{J}_n^{(i)'}(f)(a_1,...,a_n) = f(a_1,...,a_{i-1},a_n,a_i..,a_{n-1})\psi(1/a_na_{n-1}).$$
- *We define $\mathcal{J}_n^{(i)} : \mathcal{S}^{\Delta_{n-1}}(\overline{A}) \to \mathbb{C}^\infty(A)$ by $\mathcal{J}_n^{(i)} = \mathcal{J}_n^{(i)'} \circ \mathcal{F}_n$ for $i < n$*
- *We define inductively a sequence of subspaces $\mathfrak{J}_n^{[i]} \subset \mathbb{C}^\infty(A)$ and operators $\mathcal{J}_n^{[i]} : \mathfrak{J}_n^{[i]} \to \mathbb{C}^\infty(A)$ in the following way $\mathfrak{J}_n^{[1]} = \mathcal{S}^{\Delta_{n-1}}(\overline{A})$, $\mathcal{J}_n^{[1]} = \mathcal{F}_n$, $\mathfrak{J}_n^{[i]} = \mathcal{S}^{\Delta_{n-1}}(\overline{A}) \cap (\mathcal{J}_n^{(i)})^{-1}(\mathfrak{J}_n^{[i-1]})$ and $\mathcal{J}_n^{[i]} = \mathcal{J}_n^{[i-1]} \circ \mathcal{J}_n^{(n+1-i)}$.*
- *We define the Jacquet space $\mathfrak{J} := \mathfrak{J}_n$ to be $\mathfrak{J}_n^{[n]}$ and the Jacquet transform $\mathcal{J} := \mathcal{J}_n : \mathfrak{J} \to C^\infty(A)$ to be $\mathcal{J}_n^{[n]}$.*

4.4. **The partial inversion formula.**
In this subsection we prove an analog of Proposition 8 of [Jac03a], namely

Proposition 4.4.1.
(i) $\widetilde{\Omega}_i(\mathcal{S}(H)) \subset \mathfrak{J}_{i,n-i}$
(ii) $\mathcal{J}_i \circ \widetilde{\Omega}_i^\psi|_{\mathcal{S}(H)} = c(D,\psi)^{n(n-i)}\widetilde{\Omega}_{n-i}^{\bar{\psi}} \circ \mathcal{F}_H$

This proposition is equivalent to the following one

Proposition 4.4.2.
$$\mathcal{J}'_i \circ \mathcal{F}_{H^{n-i},\psi} \circ \widetilde{\Omega}_i^\psi|_{\mathcal{S}(H)} = c(D,\psi)^{n(n-i)}\mathcal{T}_i^{-1} \circ (\mathcal{F}_{H^i,\psi})^{-1} \circ \widetilde{\Omega}_{n-i}^{\bar{\psi}} \circ \mathcal{F}.$$

For its proof we will need some auxiliary results.

Lemma 4.4.3. *Let $f \in \mathcal{S}(H)$ be a Schwartz function. Then*
$$\widetilde{\Omega}_i^\psi(f)(A,B) = \eta(\det(A))^{n-i}|\det(A)|^{-(n-i)}\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B + X^tA^{-1}X \end{pmatrix}\right]\psi[\mathrm{Tr}(\varepsilon A^{-1}X)+\mathrm{Tr}(\bar{X}^tA^{-1}\varepsilon^t)]dX$$

The proof is straightforward.

Corollary 4.4.4. *Let $f \in \mathcal{S}(H)$ be a Schwartz function. Then*
$$\mathcal{F}_{H^{n-i},\psi} \circ \widetilde{\Omega}_i^\psi(f)(A, w_{n-i}Cw_{n-i}) = \eta(\det(A))^{n-i}|\det(A)|^{-(n-i)}$$
$$\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B \end{pmatrix}\right]\psi[\mathrm{Tr}(\varepsilon A^{-1}X) + \mathrm{Tr}(\bar{X}^tA^{-1}\varepsilon^t) + \mathrm{Tr}(CX^tA^{-1}X) - Tr(CB)]dXdB$$

Notation 4.4.5.
(i) Let $\xi_{A,B} \in \mathcal{S}^*(H)$ be the distribution defined by
$$\xi_{A,B}(f) = \mathcal{J}'_i \circ \mathcal{F}_{H^{n-i},\psi} \circ \widetilde{\Omega}_i^\psi(f)(A,B).$$

(ii) Let $\zeta_{A,B} \in \mathcal{S}^*(H)$ be the distribution defined by

$$\zeta_{A,B}(f) = \mathcal{T}_i^{-1} \circ (\mathcal{F}_{H^i,\psi})^{-1} \circ \widetilde{\Omega}_{n-i}^{\bar{\psi}}(f)(A,B).$$

Proof of Proposition 4.4.2. We have to show that

$$\xi_{A,B} = c(D,\psi)^{n(n-i)} \mathcal{F}(\zeta_{A,B})$$

Let $f \in \mathcal{S}(H)$ be a Schwartz function. Denote $m := n-i$. By Corollary 4.4.4

$$\xi_{A,C}(f) = \eta(\det(A))^{n-i} |\det(A)|^{-(n-i)} \psi(w_{n-i}C^{-1}w_{n-i}\varepsilon A^{-1}\varepsilon^t)$$
$$\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B\end{pmatrix}\right] \psi[\mathrm{Tr}(\varepsilon A^{-1}X) + \mathrm{Tr}(\bar{X}^t A^{-1}\varepsilon^t) + \mathrm{Tr}(w_{n-i}Cw_{n-i}X^tA^{-1}X) - Tr(w_{n-i}Cw_{n-i}B)]dXdB$$

and

$$\zeta_{A,C}(f) = \eta(\det(C))^i |\det(C)|^{-i} \times$$
$$\int f\left[\begin{pmatrix} C & X \\ \bar{X}^t & B\end{pmatrix}\right] \psi[-\mathrm{Tr}(\varepsilon C^{-1}X + \bar{X}^tC^{-1}\varepsilon^t + w_iAw_iX^tC^{-1}X - w_iAw_iB)]dXdB.$$

Therefore

$$ad(w_n)(\zeta_{A,C})(f) = \eta(\det(C))^i |\det(C)|^{-i} \times$$
$$\int f\left[\begin{pmatrix} B & X \\ \bar{X}^t & w_mCw_m\end{pmatrix}\right] \psi[-\mathrm{Tr}(\varepsilon C^{-1}w_m\bar{X}^tw_m + w_mXw_mC^{-1}\varepsilon^t + AXw_mC^{-1}w_m\bar{X}^t - AB)]dXdB.$$

Thus

$$\mathcal{F}'_{H_i \times H_m}(ad(w_n)(\zeta_{A,C}))(f) = \eta(\det(C))^i |\det(C)|^{-i} \times$$
$$\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B\end{pmatrix}\right] \psi[-\mathrm{Tr}(\varepsilon C^{-1}w_m\bar{X}^tw_m + w_mXw_mC^{-1}\varepsilon^t + AXw_mC^{-1}w_m\bar{X}^t + w_mCw_mB)]dXdB.$$

Therefore by Corollary 4.2.4

$$\mathcal{F}'_{H_i \times H_m{}^{\perp'}}(\mathcal{F}'_{H_i \times H_m}(ad(w_n)(\zeta_{A,C})))(f) = c(D,\psi)^{n(n-i)}\xi_{A,C}(f).$$

□

4.5. **Proof of the inversion formula.**

The inversion formula (Theorem 3.2.4) is equivalent to the following theorem.

Theorem 4.5.1.
(i) $\widetilde{\Omega}(\mathcal{S}(H)) \subset \mathfrak{J}$.
(ii) $\mathcal{J} \circ \widetilde{\Omega}^{\psi}|_{\mathcal{S}(H)} = c(D,\psi)^{n(n-1)/2}\widetilde{\Omega}^{\bar{\psi}} \circ \mathcal{F}_H$.

The proof is by induction. We will need the following straightforward lemma.

Lemma 4.5.2. *The induction hypotheses implies that*
(i) $\widetilde{\Omega}^{1,n-1}(\mathcal{S}^{\Delta_1}(S^1 \times H^{n-1})) \subset \mathfrak{J}_n^{[n-1]}$
(ii)

$$\widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \mathcal{F}_{H^{n-1},\psi} = c(D,\psi)^{(n-1)(n-2)/2} \mathcal{J}_n^{[n-1]} \widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{S}^{\Delta_1}(S^1 \times H^{n-1})}$$

Proof of Theorem 4.5.1. First let us prove (i). It is easy to see that

(1) $$\widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{S}^{\Delta_1}(S^1 \times H^{n-1})} \circ \mathcal{T}_{n-1} \circ \mathcal{J}'_{n-1}|_{\mathcal{F}_{H^1,\psi}(\mathfrak{J}_{n-1,1})} = \mathcal{J}_n^{(i)'} \circ \widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{F}_{H^1,\psi}(\mathfrak{J}_{n-1,1})}$$

This implies that

(2) $$\widetilde{\Omega}^{1,n-1}|_{\mathcal{S}^{\Delta_1}(S^1 \times H^{n-1})} \circ \mathcal{T}_{n-1} \circ \mathcal{J}'_{n-1} \circ \mathcal{F}_{H^1,\psi} \circ \widetilde{\Omega}_{n-1}|_{\mathcal{S}(H)} = \mathcal{J}_n^{(i)'} \circ \mathcal{F}_n \circ \widetilde{\Omega}^{1,n-1,\psi} \circ \widetilde{\Omega}_{n-1}|_{\mathcal{S}(H)}$$

By Proposition 3.2.1 this implies

(3) $$\widetilde{\Omega}^{1,n-1}|_{\mathcal{S}^{\Delta_1}(S^1 \times H^{n-1})} \circ \mathcal{T}_{n-1} \circ \mathcal{J}'_{n-1} \circ \mathcal{F}_{H^1,\psi} \circ \widetilde{\Omega}_{n-1}|_{\mathcal{S}(H)} = \mathcal{J}_n^{(i)'} \circ \mathcal{F}_n \circ \widetilde{\Omega}|_{\mathcal{S}(H)}$$

This together with Lemma 4.5.2 implies (i).

Now let us prove (ii). By Propositions 3.2.1 and 4.4.1 we have

$$\widetilde{\Omega}^{\bar{\psi}} \circ \mathcal{F}_H = \widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \widetilde{\Omega}^{\bar{\psi}}_1 \circ \mathcal{F}_H = c(D,\psi)^{(n-1)} \widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \mathcal{J}_{n-1} \circ \widetilde{\Omega}^{\psi}_{n-1}|_{\mathcal{S}(H)} = \\ = c(D,\psi)^{(n-1)} \widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \mathcal{F}_{H^{n-1},\psi} \circ \mathcal{T}_{n-1} \circ \mathcal{J}'_{n-1} \circ \mathcal{F}_{H^1,\psi} \circ \widetilde{\Omega}^{\psi}_{n-1}|_{\mathcal{S}(H)} \tag{4}$$

(ii) follows now from (3), (4), and Lemma 4.5.2. ☐

5. Proof of the Key Lemma

We will use the following notation and lemma.

Notation 5.0.1. *Denote*

$$Z' := \{x \in Z | x_{ij} = 0 \text{ for } i+j < n+1 \text{ and } x_{i,n+1-i} = x_{j,n+1-j} \in \mathbb{R} \text{ for any } 1 \le i,j \le n\}.$$

Denote also $U' := H - Z'$.

Notation 5.0.2. *We call a matrix* $x \in H$ *relevant if* $\chi|_{N_x} \equiv 1$*, and irrelevant otherwise.*

Lemma 5.0.3 ([Jac03a], §3, §5)**.** *Every relevant orbit in* $H^n(D)$ *has a unique representative of the form*

$$\begin{pmatrix} a_1 w_{m_1} & 0 & ... & 0 \\ 0 & a_2 w_{m_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_n w_{m_n} \end{pmatrix} \tag{5}$$

where $m_1 + ... + m_j = n$*,* $a_1, ..., a_j \in \mathbb{R}$*, and if* $\det(g) = 0$ *then* $\Delta_{n-1}(g) \neq 0$*.*

For the sake of completeness we will repeat the proof here.

Proof. Step 1. Proof for $S^n(\mathbb{R} \oplus \mathbb{R})$
Let W_n denote the group of permutation matrices. By Bruhat decomposition, every orbit has a unique representative of the form wa with $w \in W_n$ and $a \in A^n$. If this element is relevant, then for every pair of positive roots (α_1, α_2) such that $w\alpha_2 = -\alpha_1$, and for $u_i \in N_{\alpha_i}(\mathbb{R})$ (where N_{α_i} denotes the one-dimensional subgroup of N corresponding to α_i) we have

$$u_1^t wau_2 = wa \Rightarrow \chi(u_1, u_2) = 0. \tag{6}$$

This condition implies that α_1 is simple if and only if α_2 is simple. Thus w and its inverse have the property that if they change a simple root to a negative one, then they change it to the opposite of a simple root. Let S be the set of simple roots α such that $w\alpha$ is negative. Then S is also the set of simple roots α such that $w^{-1}\alpha$ is negative and $wS = S$. Let M be the standard Levi subgroup determined by S. Thus S is the set of simple roots of M for the torus A, w is the longest element of the Weyl group of M, and $w^2 = 1$. This being so, if α_2 is simple, then condition (6) implies $\alpha_2(a) = 1$. Thus a is in the center of M. Hence wa is of the form (5).

Step 2. Proof for $S^n(\mathbb{C})$.
Every orbit has a unique representative of the form wa with $w \in W_n$, and diagonal $a \in GL_n(\mathbb{C})$ (for proof see e.g. [Spr85, Lemma 4.1(i)], for the involution $g \mapsto w_n \overline{g}^{-t} w_n$, where $w_n \in W_n$ denotes the longest element). Since $wa \in S$, we have $w = w^t$ and hence $w^2 = 1$ and $waw = \overline{a}$.

Suppose that α is a simple root such that $w\alpha = -\beta$ where β is positive. For $u_\alpha \in N_\alpha$, define

$$u_\beta := w\overline{a}^{-1} \overline{u}_\alpha^{-t} \overline{a} w \in N_\beta.$$

Then

$$\overline{u}_\beta^t wau_\alpha = wa = \overline{u}_\alpha^t wau_\beta.$$

There exists an element $u_{\alpha+\beta} \in N_{\alpha+\beta}$ (i.e. $u_{\alpha+\beta} = 1$ if $\alpha+\beta$ is not a root) such that $u := u_{\alpha+\beta} u_\alpha u_\beta$ satisfies $\overline{u}^t wau = wa$. If wa is relevant, this relation implies $\chi(u_\alpha u_\beta) = 1$.

Thus β is simple. Since $w^2 = 1$, we see that, as before, there is a standard Levi subgroup M such that w is the longest element in its Weyl group, and $a \in Z(M) \cap A^n$.

Step 3. Proof for $H^n(D) - S^n(D)$.
Let $s \in H^n(D)$ with $\det(s) = 0$ be relevant. Then $s = u^t w b$ with $u \in N(D)$, $w \in W_n$ and b upper triangular. If a column of s of index $i < n$ would be zero, then the row with index i would also be zero, and hence s would be irrelevant. Hence $b_{1,1} \neq 0$ and acting on s by $N(D)$ we can bring b to the form $b = \begin{pmatrix} b' & 0 \\ 0 & 0 \end{pmatrix}$, where b' is diagonal and invertible. In particular, the last row of b is zero. We may replace s by $wb\overline{u}^{-1}$. The last row of $b\overline{u}^{-1}$ is again zero. Since the rows of $wb\overline{u}^{-1}$ with index less than n cannot be zero, w must have the form $w = \begin{pmatrix} w' & 0 \\ 0 & 0 \end{pmatrix}$. The theorem follows now from the 2 previous cases. □

Since Z and Z' are N-invariant we obtain

Corollary 5.0.4. *Every relevant $x \in Z$ lies in Z'.*

Using Corollary 2.2.16 we obtain

Corollary 5.0.5. *Recall that we consider the action of N on $\mathcal{S}(H)$ to be the standard action twisted by χ. Then $\mathcal{S}(U') = \mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U')$.*

Lemma 5.0.6. $Z' \not\supseteq Z'^{\perp}$.

Proof. For $n > 2$ this is obvious since $\dim Z' < \frac{n^2}{2} = \frac{\dim H}{2}$.
For $n = 2$, $\dim Z' = \frac{n^2}{2} = \frac{\dim H}{2}$. Hence it is enough to show that $Z' \neq (Z')^{\perp}$. Now

$$B\left(\begin{pmatrix} 0 & a \\ a & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ c & d \end{pmatrix}\right) = 2ac,$$

which is not identically 0. □

Corollary 5.0.7. $\mathcal{S}(H) = \mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U'))$.

Proof. Follows from the previous lemma and Theorem 2.2.13. □

Proof of the Key Lemma (Lemma 3.2.5). By Corollaries 5.0.5 and 5.0.7,

$$\begin{aligned}\mathcal{S}(H) = \mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U')) &= \mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U')) = \\ &= \mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}\mathcal{F}(\mathcal{S}(U')) = \\ &= \mathcal{S}(U) + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}(\mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U'))) \subset \mathcal{S}(U) + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}(\mathcal{S}(H)).\end{aligned}$$

The opposite inclusion is obvious. □

6. Non-regular Kloostermann integrals

In this section we define Kloostermann integrals over relevant non-regular orbits. We prove that if two functions match then their non-regular Kloostermann integrals also equal, up to a matching factor. We also prove that if all regular Kloostermann integrals of a function vanish then all Kloostermann integrals of this function vanish. In the non-Archimedean case this was done in [Jac03b] and the proofs we give here are very similar.

Recall that $g \in H^n(D)$ is called relevant if the character χ is trivial on the stabilizer $N(D)_g$ of g. For every relevant $g \in H^n(D)$ and every $\Psi \in \mathcal{S}^{\det,\mathbb{R}^\times}(S^n(D))$ we define

$$\Omega_D^{n,\psi}(\Psi, g) := \int_{N/N_g} \Psi(\overline{u}^t a u)\chi(u)du.$$

Recall the description of relevant orbits given in Lemma 5.0.3: every relevant orbit in $H^n(D)$ has a unique element of the form

$$g = \begin{pmatrix} a_1 w_{m_1} & 0 & \dots & 0 \\ 0 & a_2 w_{m_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n w_{m_n} \end{pmatrix}, \tag{7}$$

where $m_1 + ... + m_j = n$, $a_1, ..., a_j \in \mathbb{R}$, and if $\det(g) = 0$ then $\Delta_{n-1}(g) \neq 0$. In particular, $H^n(\mathbb{C})$ and $H^n(\mathbb{R} \oplus \mathbb{R})$ have the same set of representatives of regular orbits.

Notation 6.0.1. *We extend the definition of the transfer factor γ to all g of the form (7) by*

$$\text{For } g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in S^i(\mathbb{C}) \times H^{n-i}(\mathbb{C}) \quad \gamma(g) = \gamma(x)\gamma(y)\,\mathrm{sign}(\det(x))^i \tag{8}$$

$$\gamma(aw_n) = \gamma(-a^{-1}w_{n-1}, \overline{\psi})c(\mathbb{C}, \psi)^{n(n-1)/2}\,\mathrm{sign}(\det(-a^{-1}w_{n-1})) \tag{9}$$

Remark 6.0.2. *Since $c(\mathbb{C}, \psi)^2 = -1$ and $c(\mathbb{C}, \psi)c(\mathbb{C}, \overline{\psi}) = 1$, we have $\gamma(aw_{n+8}) = \gamma(aw_n)$ and for $1 \leq n \leq 8$, $\gamma(aw_n)$ is determined by the sequence*

$$1,\ c(\mathbb{C}, \psi)\,\mathrm{sign}(-a),\ \mathrm{sign}(a),\ 1,\ -1,\ c(\mathbb{C}, \psi)\,\mathrm{sign}(-a),\ \mathrm{sign}(-a),\ 1.$$

In particular $\gamma(g)$ is always a fourth root of unity.

Theorem 6.0.3. *Let $\Phi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{R} \oplus \mathbb{R}))$ and $\Psi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{C}))$. Suppose that*

$$\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi) = \gamma\Omega^{n,\psi}_{\mathbb{C}}(\Psi).$$

Then for any g of the form (7) we have

$$\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi, g) = \gamma(g, \psi)\Omega^{n,\psi}_{\mathbb{C}}(\Psi, g).$$

For proof see §§6.2.

By substituting 0 in place of Φ or Ψ we obtain the folowing corollary

Corollary 6.0.4 (Density)**.** *Let $\Phi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(D))$. Suppose that $\Omega_D(\Phi) = 0$. Then $\Omega_D(\Phi, g) = 0$ for any relevant $g \in D$.*

For the proof of Theorem 6.0.3 we will need the following lemma, which is a more elementary version of the inversion formula.

Lemma 6.0.5. *Let $n > 1$. For any $\Phi \in \mathcal{S}(H^n(D))$, define the function f_Φ on $\mathbb{R}^\times$ by $f_\Phi(a) := \Omega_D(\Phi, aw_n)$. Then $f_\Phi \in \mathcal{S}(\mathbb{R}^\times)$ and*

$$f_\Phi(a) = |a|^{-n^2+1} \int \Omega^{n,\overline{\psi}}_D(\mathcal{F}(\Phi), \begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & b \end{pmatrix})db.$$

6.1. **Proof of Lemma 6.0.5.**

Notation 6.1.1.

- *We denote $V := \{\{a_{i,j}\} \in H | a_{i,j} = 0 \text{ if } i + j \leq n + 1\} \subset H$.*
- *Note that $V^\perp = \{\{a_{i,j}\} \in H | a_{i,j} = 0 \text{ if } i + j < n + 1\} \subset H$.*
- *We denote $e := \{e_{i,j}\} \in H$. where $e_{i,j} = \delta_{i+j,n}$.*

The following two lemmas follow from change of variables.

Lemma 6.1.2. *We have*

$$f_\Phi(a) = |a|^{(n-n^2)/2} \int_{v \in V} \Phi(aw_n + v)\psi(< a^{-1}e, v >)dv$$

Lemma 6.1.3. *We have*

$$\int \Omega^{n,\psi}_D(\Phi, \begin{pmatrix} aw_{n-1} & 0 \\ 0 & b \end{pmatrix})db = |a|^{-(n+n^2)/2+1} \int_{v \in V^\perp} \Phi(ae + v)\psi(< a^{-1}w, v >)dv.$$

Lemma 6.1.4. *The function f_Φ is in $\mathcal{S}(\mathbb{R}^\times)$.*

Proof. Let $W = \mathrm{Span}(w_n) \oplus V$. Let $\Xi = \Phi|_W \in \mathcal{S}(W)$. Let $\hat{\Xi}_V \in \mathcal{S}(\mathrm{Span}(w_n) \oplus V^*)$ be the partial Fourier transform of Ξ w.r.t. V. For any $a \in \mathbb{R}^\times$ let $\phi(a) \in V^*$ be the functional defined by $\phi(a)(v) =< ae, v >$. Consider the closed embedding $\varphi : \mathbb{R}^\times \to \mathrm{Span}(w_n) \oplus V^*)$ defined by $\varphi(a) = (a, \phi(a^{-1}))$. Now by Lemma 6.1.2, $f_\Phi = \hat{\Xi}_V \circ \varphi \in \mathcal{S}(\mathbb{R}^\times)$. ☐

Proof of Lemma 6.0.5. It is left to prove that

$$f_\Phi(a) = |a|^{-n^2+1} \int \Omega_D^{n,\overline{\psi}}(\mathcal{F}(\Phi), \begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & b \end{pmatrix})db.$$

Let $\delta_{ae+V} \in \mathcal{S}(H)$ and $\delta_{aw_n+V^\perp} \in \mathcal{S}(H)$ be the Haar measures on $ae+V$ and $aw_n+V^\perp$ correspondingly. Let $f_a, g_a \in C^\infty(H)$ be defined by $f_a(x) = \psi(< ae, x >)$ and $g_a(x) = \psi(< aw_n, x >)$. By Lemmas 6.1.2 and 6.1.3 the assertion follows from the fact that

$$\delta_{ae+V} g_{-a^{-1}} = \mathcal{F}^*(\delta_{-a^{-1}w_n+V^\perp} f_a).$$

□

6.2. **Proof of Theorem 6.0.3.**

We prove the theorem by induction on n. From now on we suppose that it holds for every $r < n$.

Lemma 6.2.1. *It is enough to prove Theorem 6.0.3 for the case $\Phi \in \mathcal{S}(H^n(\mathbb{R}\oplus\mathbb{R}))$ and $\Psi \in \mathcal{S}(H^n(\mathbb{C}))$.*

Proof. Suppose that there exist $\Phi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{R}\oplus\mathbb{R}))$ and $\Psi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{C}))$ that form a counterexample for Theorem 6.0.3. We have to show that then there exist $\Phi' \in \mathcal{S}(H^n(\mathbb{R}\oplus\mathbb{R}))$ and $\Psi' \in \mathcal{S}(H^n(\mathbb{C}))$ that also form a counterexample.

We have $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi) = \gamma\Omega^{n,\psi}_{\mathbb{C}}(\Psi)$ but $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi, g) \neq \gamma(g,\psi)\Omega^{n,\psi}_{\mathbb{C}}(\Psi, g)$ for some g. Let $f \in C_c^\infty(\mathbb{R})$ such that $f(\det(g)) = 1$. Let $f' := f \circ \det$, and define $\Phi' := f'\Phi$ and $\Psi' := f'\Psi$. Note that Φ' and Ψ' are Schwartz functions and form a counterexample since determinant is invariant under the action of N. □

Lemma 6.2.2. *Let $\Phi \in \mathcal{S}(H^n(\mathbb{R}\oplus\mathbb{R}))$ and $\Psi \in \mathcal{S}(H^n(\mathbb{C}))$ such that $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi) = \gamma\Omega^{n,\psi}_{\mathbb{C}}(\Psi)$. Let $g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $x \in S^i(D)$ and $y \in H^{n-i}(D)$. Then $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi, g) = \gamma(g,\psi)\Omega^{n,\psi}_{\mathbb{C}}(\Psi, g)$.*

This lemma follows from the induction hypotheses using intermediate Kloostermann integrals, i.e. integration over $N_i^n(D)$ (cf. §§3.2.1).

Lemma 6.2.3. *Let $\Phi \in \mathcal{S}(H^n(\mathbb{R}\oplus\mathbb{R}))$ and $\Psi \in \mathcal{S}(H^n(\mathbb{C}))$ such that $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi) = \gamma\Omega^{n,\psi}_{\mathbb{C}}(\Psi)$. Let $g = aw_n$ where $a \in \mathbb{R}^\times$. Then $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi, g) = \gamma(g,\psi)\Omega^{n,\psi}_{\mathbb{C}}(\Psi, g)$.*

This lemma follows from the previous one using Lemma 6.0.5.

The theorem follows now from the last 3 Lemmas.

Appendix A. Schwartz functions on Nash manifolds

In this appendix we give some complementary facts about Nash manifolds and Schwartz functions on them and prove Property 2.2.7 and Theorems 2.2.15 and 2.2.13 from the preliminaries.

Theorem A.0.1 (Local triviality of Nash manifolds)**.** *Any Nash manifold can be covered by finite number of open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

For proof see [Shi87, Theorem I.5.12].

Theorem A.0.2. *[Nash tubular neighborhood] Let M be a Nash manifold and $Z \subset M$ be closed Nash submanifold. Then there exists an finite cover $Z = \cup Z_i$ by open Nash submanifolds of Z, and open embeddings $N^M_{Z_i} \hookrightarrow M$ that are identical on the zero section.*

This follows from e.g. [AG08, Corollary 3.6.3].

Notation A.0.3. *We fix a system of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ in the following way:*

$$\mathfrak{N}_k(f) := \max_{\{\alpha\in\mathbb{Z}^n_{\geq 0} \mid |\alpha|\leq k\}} \max_{\{\beta\in\mathbb{Z}^n_{\geq 0} \mid |\beta|\leq k\}} \sup_{x\in\mathbb{R}^n} |x^\alpha \frac{\partial^{|\beta|}}{(\partial x)^\beta} f|.$$

Notation A.0.4. *For any Nash vector bundle E over X we denote by $\mathcal{S}(X,E)$ the space of Schwartz sections of E.*

The properties of Schwartz functions on Nash manifolds listed in the preliminaries hold also for Schwartz sections of Nash bundles.

Remark A.0.5. *One can put the notion of push of Schwartz functions in a more invariant setting. Let $\phi : X \to Y$ be a morphism of Nash manifolds. Let E be a bundle on Y. Let E' be a bundle on X defined by $E' := \phi^*(E \otimes D_Y^{-1}) \otimes D_X$, where D_X and D_Y denote the bundles of densities on X and Y. Then we have a well defined map $\phi_* : \mathcal{S}(X, E') \to \mathcal{S}(Y, E)$.*

A.1. **Analog of Dixmier-Malliavin theorem.**
In this subsection we prove Property 2.2.7. Let us remind its formulation.

Theorem A.1.1. *Let $\phi : M \to N$ be a Nash map of Nash manifolds. Then multiplication defines an onto map $\mathcal{S}(M) \otimes \mathcal{S}(N) \twoheadrightarrow \mathcal{S}(M)$.*

First let us remind the formulation of the classical Dixmier-Malliavin theorem.

Theorem A.1.2 (see [DM78])**.** *Let a Lie group G acct continuously on a Fréchet space E. Then $C_c^\infty(G)E = E^\infty$, where E^∞ is the subspace of smooth vectors in E and $C_c^\infty(G)$ acts on E by integrating the action of G.*

Corollary A.1.3. *Let $L \subset V$ be finite dimensional linear spaces, and let L act on V by translations. Then $\mathcal{S}(L) * \mathcal{S}(V) = \mathcal{S}(V)$, where $*$ means convolution.*

Proof of Theorem A.1.1. Step 1. The case $N = \mathbb{R}^n$, $M = \mathbb{R}^{n+k}$, ϕ is the projection.
Follows from Corollary A.1.3 after applying Fourier transform.
Step 2. The case $N = \mathbb{R}^n$, $M = \mathbb{R}^k$, ϕ - general.
Identify N with the graph of ϕ in $N \times M$. The assertion follows now from the previous step using Property 2.2.5.
Step 3. The general case.
Follows from the previous step using Property 2.2.4 and Theorem A.0.1. □

A.2. **Coinvariants in Schwartz functions.**

Definition A.2.1. *Let a Nash group G act on a Nash manifold X. A* **tempered G-equivariant bundle** *E over X is a Nash bundle E with an equivariant structure $\phi : a^*(E) \to p^*(E)$ (here $a : G \times X \to X$ is the action map and $p : G \times X \to X$ is the projection) such that ϕ corresponds to a tempered section of the bundle* $\mathrm{Hom}(a^*(E), p^*(E))$ *(for the definition of tempered section see e.g.* [AG08]*), and for any element α in the Lie algebra of G the derivation map $a(\alpha) : C^\infty(X, E) \to C^\infty(X, E)$ preserves the sub-space of Nash sections of E.*

In this subsection we prove the following generalization of Theorem 2.2.15.

Theorem A.2.2. *Let a connected algebraic group G act on a real algebraic manifold X. Let Z be a G-invariant Zariski closed subset of X. Let $\mathfrak{g}$ be the Lie algebra of G. Let E be a tempered G-equivariant bundle over X. Suppose that for any $z \in Z$ and $k \in \mathbb{Z}_{\geq 0}$ we have*

$$(E|_z \otimes \mathrm{Sym}^k(CN_{z,Gz}^X) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z}))_{\mathfrak{g}_z} = 0.$$

Then

$$(\mathcal{S}(X, E)/\mathcal{S}(X - Z, E))_{\mathfrak{g}} = 0.$$

For the proof of this theorem we will need some auxiliary results.

Lemma A.2.3. *Let V be a representation of a Lie algebra $\mathfrak{g}$. Let F be a finite $\mathfrak{g}$-invariant filtration of V. Suppose $gr_F(V)_{\mathfrak{g}} = 0$. Then $V_{\mathfrak{g}} = 0$.*

The proof is evident by induction on the length of the filtration.

Lemma A.2.4. *Let V be a representation of a finite dimensional Lie algebra $\mathfrak{g}$. Let F_i be a countable decreasing $\mathfrak{g}$-invariant filtration of V. Suppose $\bigcap F^i(V) = 0$, $F^0(V) = V$ and that the canonical map $V \to \lim\limits_{\leftarrow}(V/F^i(V))$ is an isomorphism. Suppose also that $gr_F^i(V)_{\mathfrak{g}} = 0$. Then $V_{\mathfrak{g}} = 0$.*

This lemma is standard and we included its prove for the sake of completeness.

Proof. We have to prove that the map $\mathfrak{g}\otimes V \to V$ is onto. Let $v \in V$. We will construct in an inductive way a sequence of vectors $w_i \in \mathfrak{g}\otimes V/F^i(V)$ s.t. their image under the action map $\mathfrak{g}\otimes V/F^i(V) \to V/F^i(V)$ coincides with the image of v under the quotient map $V \to V/F^i(V)$. Define $w_0 = 0$. Suppose we have already defined w_n and we have to define w_{n+1}. Let w'_{n+1} be an arbitrary lifting of w_n to $\mathfrak{g}\otimes V/F^{n+1}(V)$. Let v'_{n+1} be the image of w'_{n+1} under the action map $\mathfrak{g}\otimes V/F^{n+1}(V) \to V/F^{n+1}(V)$ and let v_{n+1} be the image of v under the quotient map $V \to V/F^{n+1}(V)$. Let $dv = v_{n+1} - v'_{n+1}$. Clearly dv lies in $F^n(V)/F^{n+1}(V)$. Let dw be its lifting to $\mathfrak{g}\otimes(F^n(V)/F^{n+1}(V))$. Denote $w_{n+1} = w'_{n+1} + dw$.

Since $\mathfrak{g}$ is finite dimensional, the canonical map $\mathfrak{g}\otimes V \to \lim\limits_{\leftarrow} \mathfrak{g}\otimes(V/F^i(V))$ is an isomorphism. Therefore there exists a unique $w \in \mathfrak{g}\otimes V$ s.t. its image in $\mathfrak{g}\otimes(V/F^i(V))$ is w_i. Thus the image of w under the map $\mathfrak{g}\otimes V \to V$ is v. □

Notation A.2.5. *Let Z be a locally closed semi-algebraic subset of a Nash manifold X. Let E be a Nash bundle over X. Denote*

$$\mathcal{S}_X(Z,E) := \mathcal{S}(X - (\overline{Z} - Z))/\mathcal{S}(X - \overline{Z}, E).$$

Lemma A.2.6. *Let X be a Nash manifold and $Z \subset X$ be a locally closed semi-algebraic subset. Let E be a Nash bundle over X. Let Z_i be a finite stratification of Z by locally closed semi-algebraic subsets. Then $\mathcal{S}_X(Z,E)$ has a canonical filtration s.t.*

$$gr_i(\mathcal{S}_X(Z,E)) \cong \mathcal{S}_X(Z_i,E)).$$

Proof. It follows immediately from property 2.2.3. □

Lemma A.2.7. *Let X be a Nash manifold and $Z \subset X$ be Nash submanifold. Then $\mathcal{S}_X(Z)$ has a canonical countable decreasing filtration satisfying $\bigcap(\mathcal{S}_X(Z))^i = 0$ s.t. $gr_i(\mathcal{S}_X(Z,E)) \cong \mathcal{S}(Z, \mathrm{Sym}^i(CN_Z^X)\otimes E)$.*

Proof. It follows from the proof of Corollary 5.5.4. in [AG08]. □

Lemma A.2.8 (E. Borel)**.** *Let X be a Nash manifold and $Z \subset X$ be Nash submanifold. Then the natural map*

$$\mathcal{S}_X(Z,E) \to \lim_{\leftarrow}(\mathcal{S}_X(Z,E))/\mathcal{S}_X(Z,E))^i)$$

is an isomorpihsm.

Proof. Step 1. Reduction to the case when X is a total space of a bundle over Z.
It follows immediately from Theorem A.0.2.

Step 2. Reduction to the case when $Z = \mathbb{R}^n$ is standardly embedded inside $X = \mathbb{R}^{n+k}$.
It follows immediately from Theorem A.0.1 and Property 2.2.4.

Step 3. Proof for the case when $Z = \mathbb{R}^n$ standardly embedded inside $X = \mathbb{R}^{n+k}$.
It is the same as the proof of the classical Borel Lemma. □

Definition A.2.9. *We call an action of a Nash group G on a Nash manifold X factorisable if the map $\phi_{G,X} : G \times X \to X \times X$ defined by $(g,x) \mapsto (gx,x)$ has a Nash image and is a submersion onto it.*

Theorem A.2.10 (Chevalley)**.** *Let a real algebraic group act on a real algebraic variety X. Then there exists a finite G-invariant smooth stratification X_i of X s.t. the action of G on X_i is factorisable.*

Proof. By the classical Chevalley Theorem there exists a Zariski open subset $U \subset G \times X$ s.t. the map $\phi_{G,X}|_U$ is a submersion to its smooth image. Let $X_0 \subset X$ be the projection of U to X. It is easy to see that $\phi_{G,X}|_{G\times X_0}$ is a submersion to its smooth image. The theorem now follows by Noetherian induction. □

Theorem A.2.11. *Let a Nash group G act factorisably on a Nash manifold X and E be a tempered G-equivariant bundle over X. Suppose that for any $x \in X$ we have*

$$((E|_x \otimes ((\Delta_G)|_{G_x}/\Delta_{G_x})))_{\mathfrak{g}_x} = 0.$$

Then

$$(\mathcal{S}(X,E))_{\mathfrak{g}} = 0.$$

For the proof see section A.2.1 below.

Now we ready to prove Theorem A.2.2.

Proof of Theorem A.2.2.
Step 1. Reduction to the case that the action of G on Z is factorisable.
It follows from Theorem A.2.10 and Lemmas A.2.6 and A.2.3.

Step 2. Reduction to the case that the action of G on Z is factorisable and $Z = X$.
It follows from Theorems A.2.8 and A.2.4.

Step 3. Proof for the case that the action of G on Z is factorisable and $Z = X$.
It follows from Theorem A.2.11. □

A.2.1. *Proof of Theorem A.2.11.*

Notation A.2.12. *Let $\phi : X \to Y$ be a map of (Nash) manifolds.*
(i) Denote $D_Y^X := D_\phi := \phi^*(D_Y^*) \otimes D_X$.
(ii) Let $E \to Y$ be a (Nash) bundle. Denote $\phi^?(E) = \phi^*(E) \otimes D_Y^X$.

Remark A.2.13. *Note that*
(i) If ϕ is a submersion then for all $y \in Y$ we have $D_Y^X|_{\phi^{-1}(y)} \cong D_{\phi^{-1}(y)}$.
(ii) If ϕ is a submersion then by Remark A.0.5 we have a well defined map $\phi_* : \mathcal{S}^*(X, \phi^?(E)) \to \mathcal{S}^*(Y, E)$.
(iii) If a Lie group G acts on a smooth manifold X and E is a G-equivariant vector bundle (i.e. we have a map $p^*(E) \to a^*(E)$, where $p : G \times X \to X$ is the projection and $a : G \times X \to X$ is the action) then we also have a natural map $p^?(E) \to a^?(E)$. If G, X and E are Nash and the actions of G on X and E are Nash then the map $p^?(E) \to a^?(E)$ is Nash. If the action of G on E is tempered then the map $p^?(E) \to a^?(E)$ corresponds to a tempered section of $\operatorname{Hom}(p^?(E), a^?(E))$.

Notation A.2.14. *Let G be a Nash group. We denote*

$$\mathcal{S}(G, D_G)_0 := \{f \in \mathcal{S}(G, D_G) | \int_G f = 0\}.$$

Lemma A.2.15. *Let G be a connected Nash group and $\mathfrak{g}$ be its Lie algebra. Then $\mathfrak{g}\mathcal{S}(G, D_G) = \mathcal{S}(G, D_G)_0$.*

Proof. The inclusion $\mathfrak{g}\mathcal{S}(G, D_G) \subset \mathcal{S}(G, D_G)_0$ is evident. The theorem follows now from the fact that $\dim \mathcal{S}(G, D_G)_{\mathfrak{g}} = 1$, which is proved in the same way as Proposition 4.0.11 in [AG]. □

Notation A.2.16. *Let G be a Nash group, X be a Nash manifold and E be a Nash bundle over X. Let $p : G \times X \to X$ be the projection. Denote by $\mathcal{S}(G \times X, p^?(E))_{0,X}$ the kernel of the map $p_* : \mathcal{S}(G \times X, p^?(E)) \to \mathcal{S}(X, E)$. In cases when there is no ambiguity we will denote it just by $\mathcal{S}(G \times X, p^?(E))_0$.*

Lemma A.2.17. *Let G be a Nash group, X be a Nash manifold and E be a Nash bundle over X. Let $p : G \times X \to X$ be the projection. Then*

$$\mathcal{S}(G \times X, p^?(E))_0 \cong \mathcal{S}(G, D_G)_0 \widehat{\otimes} \mathcal{S}(X, E).$$

Proof. The sequence

$$0 \to \mathcal{S}(G, D_G)_0 \to \mathcal{S}(G, D_G) \to \mathbb{C} \to 0$$

is exact. Therefore by Proposition 2.3.2 the sequence

$$0 \to \mathcal{S}(G, D_G)_0 \widehat{\otimes} \mathcal{S}(X, E) \to \mathcal{S}(G, D_G) \widehat{\otimes} \mathcal{S}(X, E) \to \mathcal{S}(X, E) \to 0$$

is also exact. Thus it is enough to show that the map $\mathcal{S}(G, D_G) \widehat{\otimes} \mathcal{S}(X, E) \to \mathcal{S}(X, E)$ corresponds to the map $p_* : \mathcal{S}(G \times X, p^?(E)) \to \mathcal{S}(X, E)$ under the identification $\mathcal{S}(G, D_G) \widehat{\otimes} \mathcal{S}(X, E) \cong \mathcal{S}(G \times X, p^?(E))$. Since those maps are continuous it is enough to check that they are the same on the image of $\mathcal{S}(G, D_G) \otimes \mathcal{S}(X, E)$, which is evident. □

Corollary A.2.18. *Let G be a Nash group, X be a Nash manifold and E be a Nash bundle over X. Let $\mathfrak{g}$ be the Lie algebra of G. Let $p : G \times X \to X$ be the projection. Let G act on $\mathcal{S}(G \times X, p^?(E))$ by acting on the G coordinate. Then $\mathfrak{g}\mathcal{S}(G \times X, p^?(E)) = \mathcal{S}(G \times X, p^?(E))_0$.*

Proof. The corollary follows from the last two lemmas using Proposition 2.3.2. □

Corollary A.2.19. *Let G be a connected Nash group and $\mathfrak{g}$ be its Lie algebra. Let G act on a Nash manifold X and let E be a tempered G-equivariant bundle over X. Let $p : G \times X \to X$ be the projection. Let $a : G \times X \to X$ be the action map.*

Then $\mathfrak{g}\mathcal{S}(X, E)$ is the image $a_*(\mathcal{S}(G \times X, a^?(E))_0)$ where $\mathcal{S}(G \times X, a^?(E))_0$ denotes the image of $\mathcal{S}(G \times X, p^?(E))_0$ under the identification $\mathcal{S}(G \times X, p^?(E)) \cong \mathcal{S}(G \times X, a^?(E))$.

Proof. Let G act on $\mathcal{S}(G\times X, p^?(E))$ by acting on the G coordinate. The identification $\mathcal{S}(G\times X, p^?(E)) \cong \mathcal{S}(G\times X, a^?(E))$ gives us an action of G on $\mathcal{S}(G\times X, a^?(E))$. It is easy to see that $a_* : \mathcal{S}(G\times X, a^?(E)) \to \mathcal{S}(X, E)$ is a morphism of G-representations. By property 2.2.6 a_* is surjective. Therefore $(\mathfrak{g}\mathcal{S}(X, E)) = a_*((\mathfrak{g}\mathcal{S}(G \times X, a^?(E)))$. The assertion follows now by the previous corollary. □

Definition A.2.20. *A **Nash family of groups** over a Nash manifold X is a surjective submersion $G \to X$, a Nash map $m : G \times_X G \to G$ and a Nash section $e : X \to G$ s.t. for any $x \in X$ the map $m|_{G|_x \times G|_x}$ gives a group structure on the fiber $G|_x$ and $e(x)$ is the unit of this group.*

Definition A.2.21. *A **Nash family of Lie algebras** over a Nash manifold X is a Nash bundle $\mathfrak{g} \to X$, a Nash section m of the bundle $\mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ s.t. for any $x \in X$ the map $m(x) : \mathfrak{g}|_x \otimes \mathfrak{g}|_x \to \mathfrak{g}|_x$ gives a Lie algebra structure on the fiber $\mathfrak{g}|_x$.*

Definition A.2.22. *A Nash family of Lie algebras of a Nash family of groups G over a Nash Manifold X is the bundle $e^*(N^G_{e(X)})$ equipped with the natural structure of a Nash family of Lie algebras. We will denote it by $\mathrm{Lie}(G)$.*

Notation A.2.23. *Let G be a Nash family of groups over a Nash manifold X. Let E be a bundle over X. Let $p : G \to X$ be the projection. Denote by $\mathcal{S}(G, p^?(E))_{0,G}$ the kernel of the map $p_* : \mathcal{S}(G, p^?(E))) \to \mathcal{S}(X, E)$. If there is no ambiguity we will denote it by $\mathcal{S}(G, p^?(E))_0$.*

Lemma A.2.24. *Let G be a Nash family of groups over a Nash manifold X and $\mathfrak{g}$ be its family of Lie algebras. Then the image of the natural map $\mathcal{S}(X, \mathfrak{g}) \otimes \mathcal{S}(G, D_G) \to \mathcal{S}(G, D_G)$ is included in $\mathcal{S}(G, D_G)_0$.*

Proof. It follows immediately from the case when X is one point and E is $\mathbb{C}$ which follows from Lemma A.2.15. □

Definition A.2.25. *A Nash family of representations of a Nash family of Lie algebras $\mathfrak{g}$ over a Nash manifold X is a bundle E over X and a Nash section a of the bundle $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ s.t. for any $x \in X$ the map $a(x) : \mathfrak{g}|_x \otimes E|_x \to E|_x$ gives a structure of a representation of $\mathfrak{g}|_x$ on the fiber $E|_x$.*

Definition A.2.26. *Let G be a Nash family of groups over a Nash manifold X. Let $\mathfrak{g}$ be its family of Lie algebras. Let $p : G \to X$ be the projection. A tempered (finite dimensional) family of representations of G is a pair (E, a) where E is a bundle over X and a is a tempered section of the bundle $\mathrm{End}(p^*E)$ s.t. for any $x \in X$ the section $a|_{G|_x}$ gives a structure of a representation of $G|_x$ on the fiber $E|_x$ and s.t. the differential of a considered as a section of $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ gives a structure of a Nash family of representations of $\mathfrak{g}$ on E.*

Lemma A.2.24 gives us the following corollary.

Corollary A.2.27. *Let G be a Nash family of groups over a Nash manifold X and $\mathfrak{g}$ be its Lie algebra. Let (E, a) be a tempered (finite dimensional) family of representations of G. Let ϕ denote the composition $\mathcal{S}(G, p^?(E)) \xrightarrow{a} \mathcal{S}(G, p^?(E)) \xrightarrow{p_*} \mathcal{S}(X, E)$. Then the image of the natural map $\mathcal{S}(X, \mathfrak{g})\otimes\mathcal{S}(X, E) \to \mathcal{S}(X, E)$ is included in $\phi(\mathcal{S}(G, p * (E) \otimes D_G)|_0)$.*

Lemma A.2.28. *Let $\mathfrak{g}$ be a Nash family of Lie algebras over a Nash manifold X. Let E be a Nash family of its representations. Consider $\mathcal{S}(X, \mathfrak{g})$ as a Lie algebra and $\mathcal{S}(X, E)$ as its representation. Suppose that for any $x \in X$ we have $(E|_x)_{\mathfrak{g}|_x} = 0$. Then $(\mathcal{S}(X, E))_{\mathcal{S}(X,\mathfrak{g})} = 0$.*

Proof. For any $x \in X$ denote by a_x the map $\mathfrak{g}|_x \otimes E|_x \to E|_x$. By property 2.2.4 we may assume that E and $\mathfrak{g}$ are trivial bundles with fibers V and W. Fix a basis for V and W and the corresponding basis for $W \otimes V$. Let $\mathfrak{S}$ be the collection of coordinate subspaces of $W \otimes V$ of dimension $\dim V$. For any $L \in \mathfrak{S}$ denote $U_L = \{x \in X | a_x(L) = V\}$. Clearly $X = \bigcup U_L$. Thus by property 2.2.4 we may assume that $X = U_L$ for some L. For this case the lemma is evident. □

Corollary A.2.29. *Let G be a Nash family of groups over a Nash manifold X and $\mathfrak{g}$ be its Lie algebra. Let (E, a) be a tempered (finite dimensional) family of representations of G. Let ϕ denote the composition*

$$\mathcal{S}(G, p^?(E)) \xrightarrow{a} \mathcal{S}(G, p^?(E)) \xrightarrow{p_*} \mathcal{S}(X, E).$$

Suppose that for any $x \in X$ we have $(E|_x)_{\mathfrak{g}_x} = 0$. Then

$$\phi(\mathcal{S}(G, p^?(E))|_0) = \mathcal{S}(X, E).$$

Definition A.2.30. *We call a set G equipped with a map $m : G \times G \times G \to G$* ***a torsor*** *if there exists a group structure on G s.t. $m(x, y, z) = z((z^{-1}x)(z^{-1}y))$. One may say that a torsor is a group without choice of identity element.*

Definition A.2.31. *A Nash family of torsors over a Nash manifold X is a surjective submersion $G \to X$ and a Nash map $m : G \times_X G \times_X G \to G$ s.t. for any $x \in X$ the map $m|_{G|_x \times G|_x \times G|_x}$ gives a torsor structure on the fiber $G|_x$.*

Definition A.2.32. *Let G be a Nash family of torsors over a Nash manifold X. Let $p : G \to X$ be the projection. Consider* $\operatorname{Ker} dp$ *as a subbundle of TG. It has a natural structure of a family of Lie algebras over G. We will call this family the family of Lie algebras of G.*

Remark A.2.33. *One could define the family of Lie algebras of G to be a family of Lie algebras over X. This definition would be more adequate, but it is technically harder to phrase it. We did not do it since it is unnecessary for our purposes.*

Definition A.2.34. *A* **representation of a torsor** *G is a pair (V, W) of vector spaces and a morphism of torsors $G \to \operatorname{Iso}(V, W)$.*

Definition A.2.35. *Let G be a Nash family of torsors over a Nash manifold X. Let $\mathfrak{g}$ be its family of Lie algebras. Let $p : G \to X$ by the projection. A* **tempered (finite dimensional) family of representations of** *G is a triple (E, L, a), where E and L are (Nash) bundles over X and a is a tempered section of the bundle $Hom(p^*E, p^*L)$ s.t. for any $x \in X$ the section $a|_{G|_x}$ gives a structure of a representation of $G|_x$ on the fibers $E|_x$ and $L|_x$ and s.t. the differential of a considered as a section of* $\operatorname{Hom}(\mathfrak{g} \otimes p^*L, p^*L)$ *gives a structure of a Nash family of representations of $\mathfrak{g}$ on p^*L.*

Corollary A.2.29 gives us the following corollary.

Corollary A.2.36. *Let G be a Nash family of torsors over a Nash manifold X and $\mathfrak{g}$ be its Lie algebra. Let (E, L, a) be a tempered (finite dimensional) family of representations of G. Let ϕ denote the composition*

$$\mathcal{S}(G, p^?(E)) \xrightarrow{a} \mathcal{S}(G, p^?(L)) \xrightarrow{p_*} \mathcal{S}(X, L).$$

Suppose that for any $x \in G$ we have $(L|_{p(x)})_{\mathfrak{g}_x} = 0$. Then

$$\phi(\mathcal{S}(G, p^?(E))_0) = \mathcal{S}(X, L).$$

Proof. It follows from Corollary A.2.29 using property 2.2.4 and [AG, Theorem 2.4.16]. □

Now we are ready to prove Theorem A.2.11.

Proof of Theorem A.2.11. By Lemma A.2.19 it is enough to show that

$$a_*(\mathcal{S}(G \times X, a^?(E))_0) = \mathcal{S}(X, E).$$

Let Y be the image of the map $b : G \times X \to X \times X$ defined by $b(g, x) = (x, gx)$. Let $E_i = p_i^?(E)$ for $i = 1, 2$. Here $p_i : Y \to X$ is the projection to the i's coordinate. Note that $G \times X$ has a natural structure of a family of torsors over Y and the G-equivariant structure on E gives a family of representations (ψ, E_1, E_2) of the family of torsors $b : G \times X \to Y$. It is enough to show that

$$b_*(\mathcal{S}(G \times X, a^?(E)))_0) = \mathcal{S}(Y, E_2).$$

Recall that $\mathcal{S}(G \times X, a^?(E))_0$ is the image of $\mathcal{S}(G \times X, p^?(E))_{0,X}$ under the identification $\phi : \mathcal{S}(G \times X, a^?(E)) \to \mathcal{S}(G \times X, p^?(E)))$. Note that $\mathcal{S}(G \times X, p^?(E))_{0,X}$ includes $\mathcal{S}(G \times X, p^?(E))_{0,Y}$. Therefore it is enough to show that the image of $\mathcal{S}(G \times X, b^?(E_1))_{0,Y}$ under the composition

$$\mathcal{S}(G \times X, b^?(E_1)) \to \mathcal{S}(G \times X, b^?(E_2)) \to \mathcal{S}(Y, E_2)$$

is $\mathcal{S}(Y, E_2)$. This follows by Corollary A.2.36 from the fact that for every $y \in Y$ we have $((E_2)|_y)_{\mathfrak{g}_{p_2(y)}} = 0$. This fact is a reformulation of the fact that

$$((E|_{p_2(y)} \otimes ((\Delta_G)|_{G_{p_2(y)}}/\Delta_{G_{p_2(y)}})))_{\mathfrak{g}_{p_2(y)}} = 0,$$

which is part of the assumptions of the theorem. □

A.3. **Dual uncertainty principle.**

Notation A.3.1. *Let V be a finite dimensional real vector space. Let ψ be a non-trivial additive character of $\mathbb{R}$. Let μ be a Haar measure on V. Let $f \in \mathcal{S}(V)$ be a function. We denote by $\widehat{f} \in \mathcal{S}(V^*)$ the Fourier transform of f defined by μ and ψ.*

In this subsection we prove the following generalization of Theorem 2.2.13.

Theorem A.3.2. *Let V be a linear space, $L \subset V$ and $L' \subset V^*$ be subspaces. Suppose that $(L')_\perp \subsetneq L$. Then*

$$\mathcal{S}(V - L) + \mathcal{S}(\widehat{V^* - L'}) = \mathcal{S}(V).$$

The following lemma is obvious.

Lemma A.3.3. *There exists $f \in \mathcal{S}(\mathbb{R})$ such that f vanishes at 0 with all its derivatives and $\mathcal{F}(f)(0) = 1$.*

Corollary A.3.4. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}$ be enhanced with the obvious quadratic form. Let $g \in \mathcal{S}(L)$. Then there exists $f \in \mathcal{S}(V)$ such that $f \in \mathcal{S}(V - L)$ and $\mathcal{F}(f)|_L = g$.*

Corollary A.3.5. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form. Let $g \in \mathcal{S}(L)$. Let i be a natural number. Then there exists $f \in \mathcal{S}(V)$ such that $f \in \mathcal{S}(V - L)$, $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$.*

Corollary A.3.6. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form. Let $g \in \mathcal{S}(L)$. Then for all i and ε there exists $f \in \mathcal{S}(V)$ such that $\mathfrak{N}_{i-1}(f) < \varepsilon$, $f \in \mathcal{S}(V - L)$, $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$.*

Proof. Let $f \in \mathcal{S}(V)$ be s.t. $f \in \mathcal{S}(V - L)$, $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$.

Let $f^t \in \mathcal{S}(V)$ defined by $f^t(x + \alpha e_1) = t^{i+2} f(x + t\alpha e_1)$. It is easy to see that $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$. Also it is easy to see that $\lim_{t \to 0} \mathfrak{N}_{i-1}(f^t) = 0$. This implies the assertion. □

Corollary A.3.7. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form. Let $\{g_i\}_{i=0}^\infty \in \mathcal{S}(L)$. Then there exists $f \in \mathcal{S}(V)$ such that f vanishes on L with all its derivatives and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g_i$.*

Proof. Define 3 sequences of functions $f_i, h_i \in S(V), g_i' \in S(L)$ recursively in the following way: $f_0 = 0$. $g_i' = g_i - \frac{\partial^i \mathcal{F}(f_{i-1})}{(\partial e)^i}|_L$. Let $h_i \in \mathcal{S}(V)$ s.t. $h_i \in \mathcal{S}(V - L)$, $\mathfrak{N}_{i-1}(h_i) < 1/2^i$, $\frac{\partial^i \mathcal{F}(h_i)}{(\partial e)^i}|_L = g_i'$ and $\frac{\partial^i \mathcal{F}(h_i)}{(\partial e)^j}|_L = 0$ for any $j < i$. Define $f_i = f_{i-1} + h_i$.

Clearly $f := \lim_{i \to \infty} f_i$ exists and satisfies the requirements. □

Corollary A.3.8. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form.*

Then $\mathcal{S}(V - L) + \mathcal{F}(\mathcal{S}(V - L)) = \mathcal{S}(V)$.

Proof. Let $f \in \mathcal{S}(V)$. Let $f' \in \mathcal{S}(V - L)$ s.t. $\frac{\partial^i \mathcal{F}(f')}{(\partial e)^i}|_L = \frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L$. Let $f'' = f - f'$. Clearly $f'' \in \mathcal{F}(\mathcal{S}(V - L))$. □

Corollary A.3.9. *Let V be a linear space, $L \subset V$ and $L' \subset V^*$ be subspaces of codimension 1. Suppose that $(L')_\perp \nsubseteq L$. Then*

$$\mathcal{S}(V - L) + \mathcal{S}(\widehat{V^* - L'}) = \mathcal{S}(V).$$

Proof. Choose a non-degenerate quadratic form on V s.t. $L \perp (L')_\perp$. This form gives an identification $V \to V^*$ which maps L to L'. Now the corollary follows from the previous corollary. □

Now we are ready to prove Theorem A.3.2.

Proof of Theorem A.3.2. Let $M \supset L$ be a sub-space in V of codimension 1 s.t. $M^\perp \nsubseteq L'$. Let $M' \supset L'$ be a sub-space in V^* of codimension 1 s.t. $M^\perp \nsubseteq M'$. The theorem follows now from the previous corollary. □

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Chapter 3

Discussion

3.1 Gelfand pairs and Spherical pairs

There are still lot of unsolved problems regarding Gelfand pairs. One can formulate a conjecture that (in case it is true) will enlighten this problem. Namely:

Conjecture 1. *Let (G, H) be a spherical pair (i.e. H has finite number of orbits on the flag variety of G), then for any irreducible representation ρ, the multiplicity* $\dim \mathrm{Hom}_H(\rho, \mathbb{C})$ *is bounded by the multiplicity of a generic principal series representation which is distinguished by H.*

This conjecture will significantly simplify the situation since the multiplicity of a generic principal series representation can be computed in geometric terms. I doubt that this conjecture will be proven only with tools based on the Gelfand-Kazhdan criterion. Another interesting task is to understand what can be said about the multiplicity spaces when they are not 1-dimensional (i.e. the pair is not a Gelfand pair).

3.2 Invariant distributions

In the realm of invariant distributions there are also many open questions. Firstly, I believe one should establish a full analog of the integrability theorem. Secondly, it is interesting to investigate the connection of invariant distributions and the co-invariants in the space of functions. A priori, in the Archimedean case these spaces are not necessarily dual, since there is a topology on the space of Schwartz functions. This connection is important in problems of matching, and in [AG6] we studied it. We proved there that, in some cases, these spaces are dual. We also produced analogs of some tools to work with invariant distributions for the case of co-invariant functions. Those question should be studied in wider generality.

An ultimate problem in the field of invariant distributions could be computing the space of $G-$invariant sections of a bundle E over a variety X. A slightly more sophisticated problem is to compute $G-$co-invariants of the space of Schwartz sections $\mathcal{S}(X, E)$, and more generally, to compute $H_i(G, \mathcal{S}(X, E))$. I believe that the solutions to these problems can be related to certain type of algebraic objects over a categorical quotient $X//G$. The solution to these problems will give a solution of all the matching problems of the type mentioned above. It will also provide a lot of examples of Gelfand pairs (those for which the Gelfand property can be proven using invariant distributions). An interesting generalisation of these problems is obtained by replacing X/G by a general groupoid or algebraic stack.

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תקציר

זוג גלפנד הוא זוג (G, H) כך שההצגה הקוואזי-רגולארית של G על המרחב F(G/H) של הפונקציות על G/H "מכילה" כל הצגה בלתי פריקה של G עם ריבוי מרבי 1.
למחקר אודות זוגות גלפנד שימושים רבים בתורת ההצגות, באנליזה הרמונית ובתבניות אוטומורפיות. הכלי המרכזי להוכחת תכונת גלפנד הוא שיטת גלפנד-קשדן, המבוססת על ניתוח של התפלגויות על G שהן אינווריאנטיות לגבי הפעולה הדו-צדדית של H.
בתזה זו אנו מציגים את הוכחת תכונת גלפנד עבור מגוון זוגות, ובפרט אנו מוכיחים כי הזוגות הבאים הם זוגות גלפנד

- $(GL_{n+k}(F), GL_k(F)\times GL_n(F))$ עבור $F = \mathbb{R}, \mathbb{C}$. המקרה הלא ארכימדי הוכח ב [JR]
- $(GL_n(\mathbb{C}), GL_n(\mathbb{R}))$. המקרה הלא ארכימדי הוכח ב [Fli]
- $(O_{n+k}(\mathbb{C}), O_k(\mathbb{C})\times O_n(\mathbb{C}))$
- $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$
- $(GL_{2n}(F), Sp_{2n}(F))$ הלא ארכימדי הוכח ב [HR]
- הזוג $(GL_{n+1}(F), (GL_n(F)))$ הוא זוג גלפנד חזק
(ז״א שהזוג $(GL_{n+1}(F)\times GL_n(F), \dot{}(GL_n(F)))$ הוא זוג גלפנד)

ההוכחות מבוססות על מגוון כלים שפיתחנו כדי לעבוד עם התפלגויות אינווריאנטיות.
מלבד שיטת גלפנד-קשדן ישנן שיטות נוספות להוכחת תכונת גלפנד, רובן מבוססות על הסקת תכונת גלפנד של זוג אחד מתכונת גלפנד של זוג אחר. נציג כאן שתי דוגמאות לשיטות כאלה.
לתורה של התפלגויות אינווריאנטיות ישנן שימושים נוספים בתורת ההצגות. נדון גם בכמה משימושים אלו.

תוכן התזה

במבוא נדון בתוצאות המוצגות בתזה זו ביתר פירוט. פרק 2 מורכב מהמאמרים הנידונים במבוא שנכתבו במהלך לימודי לדוקטורט. בפרק 3 נבחן אפשרויות להרחבת תוצאות התזה.

תודות

ראשית ברצוני להודות למנחי **יוסף ברנשטיין ולדימיר ברקוביץ'** על שהאירו את דרכי במהלך לימודי הדוקטורט. אני לומד **מיוסף ברנשטיין** מאז היותי בן חמש עשרה. במהלך תקופה זו למדתי ממנו את רוב המתמטיקה שאני יודע, כמו כן הוא חשף בפני את גישתו המיוחדת למתמטיקה. אני מקווה שהצלחתי להפנים לפחות חלק מגישה זו. כשהתחלתי את לימודי במכון ויצמן זכיתי במנחה נוסף, **ולדימיר ברקוביץ'**. הוא דאג לי רבות והציג בפני גישה שונה ומרתקת למתמטיקה.

אני מודה מקרב ליבי **לדימיטרי גורביץ'**, חברי ועמיתי מזה למעלה מעשור. הרוב המכריע של לימודי ומחקרי המתמטיים התבצע בשיתוף פעולה קרוב אתו, בפרט חלק גדול מתיזה זו מבוסס על עבודה משותפת אתו.

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במשך כל השנים שהקדשתי ללימודי המתמטיקה נהניתי משיחות עם אנשים רבים. לא אוכל למנות כאן את כולם, אך ברצוני להודות **לעמר אופן, סמיון אלסקר, לב בוכובסקי, פאול בירן, משה ברוך, מריה גורליק, סטיבן גלברט, יעקב ורשבסקי, איליה טיומקין, אנטוני יוסף, אוקסנה יקימובה, סידהרתא סאהי, דוד סודרי, בִּינְיוּנג סאן, יאניס סקאלארדיס, ליאוניד פלטרוביץ', יובל פליקר, גדי קוזמה, ואדים קוסוי, ברנרד קרוץ, מטיאס קרק, דוד קשדן, לב רדיבילובסקי ואנדריי רזניקוב** על שיחותנו השימושית.

בנוסף תודתי נתונה למחלקה למתמטיקה של מכון ויצמן למדע על שסיפקה אוירה חמה ופורייה ללימודי לקראת הדוקטורט. בפרט אני מודה **לסטיבן גלברט, מריה גורליק, סרגיי יעקובנקו ודימיטרי נוביקוב** על התמיכה הדאגה והעידוד שהעניקו לי. ברצוני גם להודות למזכירת החוג למתמטיקה **ג'יזל מימון** וקודמתה בתפקיד **טרי דבש** על שהעלימו את הביורוקרטיה עבורי.

במהלך לימודי לדוקטורט ביקרתי במרכז האוסדורף למתמטיקה ובמכון מקס פלנק למתמטיקה בבון. אני מודה למוסדות אלה על האווירה ותנאי העבודה שהעניקו לי.

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עבודה זאת התבצעה תחת הנחייתם של

פרופסור יוסף ברנשטיין
ופרופסור ולדימיר ברקוביץ'

לאשתי האהובה

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מוגשת למועצה המדעית של
מכון ויצמן למדע
רחובות, ישראל

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אברהם איזנבוד

זוגות גלפנד והתפלגויות אינווריאנטיות
Gelfand pairs and Invariant Distributions

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July 2010

אב תש"ע