Relative Frobenius Formula

Avraham Aizenbud, Nir Avni and Yoav Krauz

December 21, 2015

Abstract

For a finite group G, Frobenius found a formula for the values of the function $\sum_{\operatorname{Irr} G}(\dim \pi)^{-s}$ for even integers s, where $\operatorname{Irr} G$ is the set of irreducible representations of G. We generalize this formula to the relative case: for a subgroup H, we find a formula for the values of the function $\sum_{\operatorname{Irr} G}(\dim \pi)^{-s}(\dim \pi^H)^{-t}$. We apply our results to compute the E-polynomials of Fock–Goncharov spaces and to relate the Gelfand property to the geometry of generalized Fock–Goncharov spaces.

Contents

1 Frobenius' formula

Let S be a compact surface and let G be a finite group. A fundamental formula of Frobenius relates the number of homomorphisms from the fundamental group of S to G and the dimensions of the irreducible representations of G:

Theorem 1.1. *Let S be a compact surface of genus k and let G be a finite group. Then,*

$$|G|^{2k-1} \sum_{\pi \in \operatorname{Irr} G} (\dim \pi)^{2-2k} = |\operatorname{Hom}(\pi_1(S), G)| = \left| \left\{ (x_1, y_1, \ldots, x_k, y_k) \in G^{2k} \mid [x_1, y_1] \cdots [x_k, y_k] = 1 \right\} \right|,$$

where $\operatorname{Irr} G$ is the set of (isomorphism classes of) irreducible representations of G.

For example, $k = 0$ gives $\sum_{\pi \in \operatorname{Irr} G} (\dim \pi)^2 = |G|$, whereas from $k = 1$ we get

$$|\operatorname{Irr} G| = \frac{1}{|G|} \cdot \left| \left\{ (x, y) \in G^2 \mid xy = yx \right\} \right| = \sum_{x \in G} \frac{|C_G(x)|}{|G|} = \sum_{x \in G} \frac{1}{|x^G|} = |G//G|.$$

Theorem 1.1 also has versions for compact Lie groups and for pro-finite groups (see [Wit91, AA]).

Theorem 1.1 is the case $g = 1$ of the following theorem:

Theorem 1.2. *Let G be a finite group and let $g \in G$. Then,*

$$|G|^{2k-1} \sum_{\pi \in \operatorname{Irr} G} (\dim \pi)^{1-2k} \chi_\pi(g) = \left| \left\{ (x_1, y_1, \ldots, x_k, y_k) \in G^{2k} \mid [x_1, y_1] \cdots [x_k, y_k] = g \right\} \right|.$$

In this paper, we generalize Frobenius' formula to the relative case, i.e., we replace the representation theory of a group G by the harmonic analysis on some G-space X. We apply our result for Gelfand pairs and the Hodge theory of Fock–Goncharov spaces.

2 Relative representation theory

Relative representation theory is motivated by the following example:

Example 2.1. *Let H be a (finite) group, and consider H as a $H \times H$-set via the action*

$$(h_1, h_2) \cdot h := h_1 h h_2^{-1}.$$

Consider the space $\mathbb{C}[H]$ of complex-valued functions on H as a representation of $H \times H$. We have

$$\mathbb{C}[H] = \bigoplus_{\pi \in \operatorname{Irr} H} \pi \otimes \pi^*.$$

This example shows that understanding the $H \times H$-representation $\mathbb{C}[H]$ "is the same" as understanding the representation theory of H. One can reformulate many concepts of the representation theory of H in terms of the $H \times H$-representation $\mathbb{C}[H]$. Relative representation theory (also known as abstract harmonic analysis) deals with those concepts considered in a wider generality: a group G acting on a set X and the representation of G on $\mathbb{C}[X]$.

Two important examples of representation theoretical concepts that have relative counterparts are Schur's Lemma, whose relative counterpart is the Gelfand property (see Definition 4.1 below) and the notion of a character, whose relative counterpart is the notion of spherical (or relative) character (see Definition B.1 below).

3 Relative version of Frobenius' formula

We prove the following theorem in §6:

Theorem 3.1. *Let G be a finite group acting on a finite set X, let $g \in G$, and let $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 1}$. Then:*

$$\sum_{\pi \in irrG} \frac{\dim(\mathrm{Hom}_G(\pi, \mathbb{C}[X]))^m}{\dim \pi^{m+2k-1}} \chi_\pi(g) = \frac{1}{\#G^{m+2k-1}} \cdot$$

$$\cdot \#\{p_1, \dots p_m \in X, h_1, \dots h_m, a_1, \dots a_k, b_1, \dots b_k \in G | h_i \in G_{p_i}, \prod_{i=1}^{m} h_i \cdot \prod_{i=1}^{k} [a_i, b_i] = g\} =$$

$$= \frac{1}{\#G^{m+2k-1}} \sum_{h_2, \dots h_m, a_1, \dots a_k, b_1, \dots b_k \in G} \#X^{g^{-1} \cdot h_2 \cdots h_m \cdot [a_1, b_1] \cdots [a_k, b_k]} \prod_{i=2}^{m} \#X^{h_i},$$

where $[a, b] := aba^{-1}b^{-1}$ is the commutator of a and b.

In Appendix B we reformulate this theorem in terms of spherical characters.

4 A criterion for Gelfand pairs

Recall the definition of Gelfand pairs:

Definition 4.1. *Let G be a finite group.*

1. *Assume that G acts on a finite set X. We say that X is multiplicity free if, for any $\pi \in \mathrm{Irr}(G)$, we have* $\dim \mathrm{Hom}_G(\pi, \mathbb{C}[X]) \leq 1$.

2. *Let $H < G$. We say that (G, H) is a Gelfand pair if G/H is a multiplicity free G-set.*

Theorem 3.1 gives us the following criterion for Gelfand pairs:

Corollary 4.2. *Let $H \subset G$ be a pair of groups, and let $X = G/H$. Then the pair (G, H) is a Gelfand pair if and only if*

$$\sum_{g,h\in G} \#X^{[g,h]} = \sum_{g,h\in G} \#X^g \cdot \#X^h \cdot \#X^{gh}.$$

In fact, Theorem 3.1 implies also the following more general statement:

Corollary 4.3. *Let $H \subset G$ be a pair of groups and let $X = G/H$. For every $k, m \in \mathbb{Z}_{\geq 0}$ denote:*

$$f(k,m) := \sum_{h_1,\dots h_m,a_1,\dots a_k,b_1,\dots b_k\in G} \#X^{h_1\cdots h_m\cdot[a_1,b_1]\cdots[a_k,b_k]} \prod_{i=1}^{m} \#X^{h_i}.$$

Then, the following are equivalent:

- *The pair (G, H) is a Gelfand pair.*
- *For every $k, m \in \mathbb{Z}_{\geq 0}$ and $0 < l \leq k$, we have $f(k-l, m) = f(k, m+2l)$.*
- *For some $k, m \in \mathbb{Z}_{\geq 0}$ and $0 < l \leq k$, we have $f(k-l, m) = f(k, m+2l)$.*

5 Fock–Goncharov spaces

Theorem 3.1 can also be interpreted as a counting formula for (generalized) Fock–Goncharov spaces, which we proceed to define. The setting for this section is as follows: let $\overline{S}$ be a compact surface, let $p_1, \dots, p_m \in \overline{S}$, $m \geq 1$, be distinct points, and denote $S = \overline{S} \smallsetminus \{p_1, \dots, p_m\}$. Such S is called a surface of finite type. Choose a base point $s \in S$ and, for each $i = 1, \dots, m$, choose a representative $\tau_i \in \pi_1(S, s)$ from the conjugacy class corresponding to a circle around p_i.

Definition 5.1. *Let G be a group acting on a set X. An X-framed representation $\pi_1(S, s) \to G$ is a tuple $(\rho, x_1, \dots, x_m)$, where $\rho : \pi_1(S, s) \to G$ is a homomorphism, and $x_i \in X$ satisfy $\rho(\tau_i)x_i = x_i$. The collection of all X-framed representations is denoted by $\widehat{\mathcal{X}}_{S,s,(\tau_i),G,X}$.*

If s' and τ_i' are different choices of a point and loops, then there is a bijection (depending on a choice of a path from s to s') between $\widehat{\mathcal{X}}_{S,s,(\tau_i),G,X}$ and $\widehat{\mathcal{X}}_{S,s',(\tau_i'),G,X}$. When no confusion arises, we will omit s and τ_i from the notation.

If $\mathbf{G}$ is group scheme acting on a scheme $\mathbf{X}$, then the functor sending a scheme T to $\widehat{\mathcal{X}}_{S,\mathbf{G}(T),\mathbf{X}(T)}$ is representable by a scheme that we denote by $\widehat{\mathcal{X}}_{S,\mathbf{G},\mathbf{X}}$.

Definition 5.2. *Let $\mathbf{G}$ be a group scheme acting on a scheme $\mathbf{X}$. Then, $\mathbf{G}$ acts on $\widehat{\mathcal{X}}_{S,\mathbf{G},\mathbf{X}}$, and we denote the quotient stack by $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$. Similarly, if a group G acts on a set X, we denote the quotient groupoid $G\backslash\widehat{\mathcal{X}}_{S,G,X}$ by $\mathcal{X}_{S,G,X}$.*

Remark 5.3.

- *If $\mathbf{X}$ is the flag variety of a reductive group $\mathbf{G}$, then the stack $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$ was defined in [FG06]. The authors of [FG06] defined the notion of a framed $\mathbf{G}$ local system and showed that $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$ is the moduli stack of framed $\mathbf{G}$ local systems on S (see [FG06, §2]). The notion of a framed $\mathbf{G}$ local system extends to general $\mathbf{G}$ and $\mathbf{X}$, and the same proof shows that $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$ is the moduli space of framed $(\mathbf{G},\mathbf{X})$-local systems.*
- *If $\mathbf{G}$ is connected, then, by Lang's Theorem, $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}(\mathbb{F}_p) \cong \mathcal{X}_{S,\mathbf{G}(\mathbb{F}_p),\mathbf{X}(\mathbb{F}_p)}$.*

In terms of the definitions above, Theorem 3.1 implies:

Theorem 5.4. *Let G be a finite group acting on a finite set X. Then*

$$\#\widehat{\mathcal{X}}_{S,G,X} = (\#G)^{1-\chi(S)} \sum_{\pi\in\operatorname{Irr} G} \frac{\dim(\operatorname{Hom}_G(\pi,\mathbb{C}[X]))^{\#(\overline{S}\smallsetminus S)}}{\dim\pi^{-\chi(S)}},$$

and

$$vol(\mathcal{X}_{S,G,X}) := \sum_{x \text{ is an isomorphism class of } \mathcal{X}_{S,G,X}} \frac{1}{\#Aut(x)} = (\#G)^{-\chi(S)} \sum_{\pi\in\operatorname{Irr} G} \frac{\dim(\operatorname{Hom}_G(\pi,\mathbb{C}[X]))^{\#(\overline{S}\smallsetminus S)}}{\dim\pi^{-\chi(S)}}.$$

Corollary 5.5. *Let G be a finite group acting on a finite set X. The following are equivalent:*

- *X is a multiplicity free G-space.*
- *For any two non-compact surfaces of finite type S_1, S_2 such that $\chi(S_1)=\chi(S_2)$, we have $vol(\mathcal{X}_{S_1,G,X}) = vol(\mathcal{X}_{S_2,G,X})$.*
- *There are two non homeomorphic non-compact surfaces of finite type S_1, S_2 such that $\chi(S_1)=\chi(S_2)$ and $vol(\mathcal{X}_{S_1,G,X}) = vol(\mathcal{X}_{S_2,G,X})$.*

Definition 5.6. *We say that a set T of prime powers is dense if, for any finite Galois extension $E/\mathbb{Q}$ and for any conjugacy class $\gamma \subset Gal(E/\mathbb{Q})$, there exists $p^n \in T$ such that p is unramified in E and $\gamma = Fr_p^n$.*

Remark 5.7.

- *The Chebotarev Density Theorem says that the set of all primes is dense.*
- *The Grothendieck trace formula implies that if X_1, X_2 are two schemes such that $X_1(\mathbb{F}_q) = X_1(\mathbb{F}_q)$ when q ranges over a dense set of prime powers, then $X_1(\mathbb{F}_{p^n}) = X_1(\mathbb{F}_{p^n})$ for almost all primes p and for all natural numbers n.*

The last corollary and [Kat08] implies:

Corollary 5.8. *Let $\mathbf{G}$ be a group scheme over $\mathbb{Z}$ acting on a scheme $\mathbf{X}$. The following are equivalent:*

- *There is a dense set T of prime powers such that, for any $q \in T$, the set $\mathbf{X}(\mathbb{F}_q)$ is a multiplicity free $\mathbf{G}(\mathbb{F}_q)$ space.*
- *For all but finitely many primes p and for all n, the set $\mathbf{X}(\mathbb{F}_{p^n})$ is a multiplicity free $\mathbf{G}(\mathbb{F}_{p^n})$ space.*

Moreover, if these conditions hold then, for any two non-compact surfaces S_1, S_2 such that $\chi(S_1) = \chi(S_2)$, the varieties $\widehat{\mathcal{X}}_{S_1,\mathbf{G},\mathbf{X}}$ and $\widehat{\mathcal{X}}_{S_2,\mathbf{G},\mathbf{X}}$ have the same E-polynomial[1].

We will now apply Theorem 5.4 for the case of $\mathbf{GL}_n$ acting on its flag variety $\mathbf{Fl}_n$. Recall that, if $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of n and λ^* is the conjugate partition, then

$$h_\lambda(i,j) = \lambda_i - j + \lambda_j^* - i + 1$$

is the length of the hook in the Young diagram corresponding to λ passing through the box (i,j). We prove the following:

Theorem 5.9.

- $$vol\left(\mathcal{X}_{S,\mathbf{GL}_n,\mathbf{Fl}_n}(\mathbb{F}_q)\right) = (n!)^{\#\overline{S} \smallsetminus S} \sum_{\lambda \text{ is a partition of } n} q^{\sum_k (k-1)\lambda_k \chi(S)} \prod_{i,j\,:\, j \leq \lambda_i} \frac{(q^{h_\lambda(i,j)} - 1)^{-\chi(S)}}{h_\lambda(i,j)^{\#\overline{S} \smallsetminus S}}.$$

[1] For the definition of the E-polynomial see e.g. [Kat08]

- *The E polynomial of* $\widehat{\mathcal{X}}_{S,\mathbf{GL}_n,\mathbf{Fl}_n}$ *is*

$$(n!)^{\#\overline{S}\smallsetminus S}\prod_{k=1}^{n}(x^ny^n-x^ky^k)\sum_{\lambda}(xy)^{\sum_k(k-1)\lambda_k\chi(S)}\prod_{i,j\,:\,j\le\lambda_i}\frac{((xy)^{h_\lambda(i,j)}-1)^{-\chi(S)}}{h_\lambda(i,j)^{\#\overline{S}\smallsetminus S}}.$$

For the proof, we collect the following facts:

Proposition 5.10 ([Jam84]). *For every partition λ of n, there exists a unique irreducible representation R_λ of $\mathbf{GL}_n(\mathbb{F}_q)$ satisfying:*

- *R_λ appears in the permutation representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_\lambda(\mathbb{F}_q)]$, where $\mathbf{P}_\lambda$ is the standard parabolic corresponding to λ (see [Jam84, Chapter 11]).*
- *R_λ does not appear in the permutation representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_\mu(\mathbb{F}_q)]$, for $\mu<\lambda$ (see [Jam84, Chapter 15]).*
- $$\dim R_\lambda=q^{\sum_k(k-1)\lambda_k}\frac{\#\,\mathrm{GL}_n(\mathbb{F}_q)}{\prod_{i,j:j\le\lambda_i}(q^{h_\lambda(i,j)}-1)}.$$

  (see [Jam84, Page 2]).

Let $\mathbf{B}\subset\mathbf{GL}_n$ be the standard Borel. Taking $T_\lambda=R_\lambda^{\mathbf{B}(\mathbb{F}_q)}$, we get

Corollary 5.11. *For every partition λ of n, we have*

- *T_λ appears in the representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_\lambda(\mathbb{F}_q)]^{\mathbf{B}(\mathbb{F}_q)}$.*
- *T_λ does not appear in the representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_\mu(\mathbb{F}_q)]^{\mathbf{B}(\mathbb{F}_q)}$, for $\mu<\lambda$.*

The following is classical:

Proposition 5.12. *For every partition λ of n, there exists a unique irreducible representation π_λ of S_n satisfying:*

- *π_λ appears in the permutation representation $\mathbb{C}[S_n/S_\lambda]$, where $S_{(\lambda_1,\dots,\lambda_m)}=S_{\lambda_1}\times\cdots\times S_{\lambda_m}\subset S_n$.*
- *π_λ does not appear in the permutation representation $\mathbb{C}[S_n/S_\mu]$, for $\mu<\lambda$.*
- $\dim\pi_\lambda=\frac{n!}{\prod_{i,j:i\le\lambda_j}h_\lambda(i,j)}$.

Proof of Theorem 5.9. Since $\dim\mathrm{Hom}(R_\lambda, \mathbb{C}[\mathbf{Fl}_n]) = \dim T_\lambda$, it is enough to show that $\dim T_\lambda = \dim \pi_\lambda$, for every λ. Recall that the Hecke algebra $H^{S_n}(t)$ corresponding to the Coxeter group S_n is a (polynomial) one parameter family of algebras whose underlying vector space is $\mathbb{C}[S_n]$; we denote the product in $H^{S_n}(t)$ by $*_t$. Recall that the product $*_1$ is the convolution on $\mathbb{C}[S_n]$ and that, if t is a prime power, then the product $*_t$ corresponds to the convolution in $\mathbb{C}[\mathbf{B}(\mathbb{F}_t)\backslash\mathbf{GL}_n(\mathbb{F}_t)/\mathbf{B}(\mathbb{F}_t)]$ under the identification $\mathbb{C}[\mathbf{B}(\mathbb{F}_t)\backslash\mathbf{GL}_n(\mathbb{F}_t)/\mathbf{B}(\mathbb{F}_t)] \cong \mathbb{C}[S_n]$ given by the Bruhat decomposition. Let $M_\lambda(t) \subset H^{S_n}(t)$ be the subspace of S_λ-(right)-invariant elements of $\mathbb{C}[S_n]$. For every prime power t, $M_\lambda(t)$ is an ideal, and, hence, the same is true for every t. Using the interpolation of the natural inner product, we get that, for $t \in \mathbb{R}_{\geq 1}$, the algebra $H^{S_n}(t)$ is semisimple, and, hence, there is an (analytic) trivialization of $H^{S_n}(t)$ over $\mathbb{R}_{\geq 1}$. Since there are only finitely many isomorphism types of representations of a given dimension, we get that $M_\lambda(t)$ can also be trivialized over $\mathbb{R}_{\geq 1}$. Corollary 5.11 and Proposition 5.12 imply that, under the algebra isomorphism $\mathbb{C}[S_n] \to \mathbb{C}[\mathbf{B}(\mathbb{F}_q)\backslash\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{B}(\mathbb{F}_q)]$, the modules T_λ and π_λ are isomorphic, and hence have the same dimension. □

6 Proof of Theorem 3.1

The case $k = 0$, $m = 1$ of theorem 3.1 is easy:

Lemma 6.1. *Let G be a finite group acting on a finite set X. Then:*

$$\sum_{\pi \in \mathrm{Irr}\, G} \dim(\mathrm{Hom}_G(\pi, \mathbb{C}[X])) \cdot \chi_\pi(g) = \chi_{\mathbb{C}[X]}(g) = \# X^g. \tag{1}$$

In order to deduce the general case we need a basic fact about convolution of characters. Recall that for two functions $f, g \in \mathbb{C}[G]$, the convolution is defined by

$$(f * g)(h) = \sum_{u \in G} f(u) g(u^{-1}h).$$

Lemma 6.2. *For any $\pi, \tau \in \mathrm{Irr}\, G$ we have:*

$$\chi_\pi * \chi_\tau = \frac{\delta_{\pi,\tau} \# G}{\dim(\pi)} \chi_\pi.$$

Now we ready to prove the main theorem.

Proof of theorem 3.1. Applying Lemma 6.2, the assertion follows by convolving (1) with itself m times and with the formula in Theorem 1.2. □

7 Acknowledgments

We thank Inna Entova Aizenbud for a helpful conversation. A.A. was partially supported by ISF grant 687/13 and a Minerva foundation grant. N.A. was partially supported by NSF grant DMS-1303205. A.A. and N.A. were partially supported by BSF grant 2012247. N.A. thanks the Weizmann Institute for hospitality.

A An alternative proof of the Frobenius formula

Lemma 6.1 gives an alternative proof of the Frobenius formula (Theorem 1.1).

Let G be a finite group acting on a finite set X. For a representation π of G, define a function on $X \times X$ by

$$\chi_\pi^X(x,y) = \frac{1}{\#G} \sum_{h\,:\,hx=y} \chi_\pi(h). \tag{2}$$

Lemma A.1. *Consider the 2-sided action of $G \times G$ on G. Let π be a representation of G. Then*

$$\chi^G_{\pi\otimes\pi^*}(1,g) = \frac{1}{\#G\dim\pi}\chi_\pi(g).$$

Proof.

$$\chi^G_{\pi\otimes\pi^*}(1,g) = \frac{1}{\#G} \sum_{h_1,h_2\,:\,h_1h_2^{-1}=g} \chi_\pi(h_1)\chi_\pi(h_1^{-1}) = \frac{(\chi_\pi * \chi_\pi)(g)}{\#G} = \frac{1}{\#G\dim\pi}\chi_\pi(g),$$

where the last equality is by Lemma 6.2 □

Proof of Theorem 1.1. the case $k=1$ follows from the Lemma 6.1 and lemma A.1. The general case follows by taking convolution power of the case $k=1$ and using Lemma 6.2. □

B The spherical character

The relative counterpart of the notion of the character of a representation is given in the following definition:

Definition B.1. *Let G be a finite group acting on a finite set X. Let π be a representation of G.*

1. *Let $\phi : \pi \to \mathbb{C}[X]$ and $\psi : \pi^* \to \mathbb{C}[X]$ be morphisms of representations. Denote by ϕ^t and ψ^t the dual maps. We define the spherical character $\chi_\pi^{\phi\otimes\psi} \in \mathbb{C}[X \times X]$ by*

$$\chi_\pi^{\phi\otimes\psi}(x, y) = \langle \phi^t(\delta_x), \psi^t(\delta_y)\rangle,$$

where $\delta_x \in \mathbb{C}[X] = \mathbb{C}[X]^*$ is the Kronecker delta function supported at x.

2. *This definition extends (by linearity) to the case when $\phi \otimes \psi$ is replaced by any element of $Hom(\pi, \mathbb{C}[X]) \otimes Hom(\pi^*, \mathbb{C}[X]) = End(Hom(\pi, \mathbb{C}[X]))$.*

Lemma B.2.

$$\chi_\pi^X := \chi_\pi^{Id_{Hom(\pi,\mathbb{C}[X])}}.$$

Proof. For $x \in X$, let $L_\pi^x : \mathrm{Hom}_G(\pi, \mathbb{C}[X]) \to \pi^*$ be the linear map defined by

$$\phi \in \mathrm{Hom}_G(\pi, \mathbb{C}[X]) \mapsto (u \in \pi \mapsto \phi(u)(x)).$$

Note that $\mathrm{Hom}_G(\pi, \mathbb{C}[X])$, $\mathrm{Hom}_G(\pi^*, \mathbb{C}[X])$ are naturally dual to each other by the pairing

$$\langle \phi, \psi\rangle := \sum_{x\in X} \langle L_\pi^x \phi, L_{\pi^*}^x \psi\rangle \qquad (\phi \in \mathrm{Hom}_G(\pi, \mathbb{C}[X]);\ \psi \in \mathrm{Hom}_G(\pi^*, \mathbb{C}[X]))$$

therefore we shall identify $\mathrm{Hom}_G(\pi^*, \mathbb{C}[X])$ with $\mathrm{Hom}_G(\pi, \mathbb{C}[X])^*$.

Let $\phi \in \mathrm{Hom}_G(\pi, \mathbb{C}[X])$, $\psi \in \mathrm{Hom}_G(\pi^*, \mathbb{C}[X])$. Then by definition,

$$\chi_\pi^{\phi\otimes\psi}(x, y) = \langle \phi^t(\delta_x), \psi^t(\delta_y)\rangle = \langle L_\pi^x \phi, L_{\pi^*}^y \psi\rangle = \langle \left(L_{\pi^*}^y\right)^t L_\pi^x \phi, \psi\rangle,$$

so $\chi_\pi^{Id_{Hom(\pi,\mathbb{C}[X])}}(x, y) = \mathrm{tr}\left(\left(L_{\pi^*}^y\right)^t L_\pi^x\right)$.

It is easy to see that $\left(L_\pi^x\right)^t : \pi \to \mathrm{Hom}_G(\pi^*, \mathbb{C}[X])$ can be computed by

$$\forall u \in \pi, f \in \pi^* : \left(\left(L_\pi^x\right)^t u\right)(f) = \frac{1}{\#G} \sum_{h\in G} f(\pi(h)u)\, \delta_{hx}.$$

Now, $\chi_\pi^{Id_{Hom(\pi,\mathbb{C}[X])}} = \mathrm{tr}\left(\left(L_{\pi^*}^y\right)^t L_\pi^x\right) = \mathrm{tr}\left(L_\pi^x \left(L_{\pi^*}^y\right)^t\right)$. Note that $L_\pi^x \left(L_{\pi^*}^y\right)^t$ is the linear mapping $\pi^* \to \pi^*$ defined by

$$\forall f \in \pi^* : \left(L_\pi^x \left(L_{\pi^*}^y\right)^t\right) f = \left(u \in \pi \mapsto \frac{1}{\#G} \sum_{h\in G} \langle u, (\pi^*(h)f)\rangle \delta_{hy,x}\right) = \frac{1}{\#G} \sum_{h \text{ s.t. } hy=x} \pi^*(h) f$$

so

$$\chi_\pi^{Id_{Hom(\pi,\mathbb{C}[X])}} = \mathrm{tr}\left(L_\pi^x \left(L_{\pi^*}^y\right)^t\right) = \frac{1}{\#G} \sum_{h \text{ s.t. } hy=x} \chi_{\pi^*}(h) = \frac{1}{\#G} \sum_{h \text{ s.t. } hx=y} \chi_\pi(h) = \chi_\pi^X(x, y)$$

□

We reformulate Theorem 3.1 in terms of the spherical character:

Theorem B.3. *Let G be a finite group that acts on a finite set X. Then:*

$$\sum_{\pi \in \operatorname{Irr} G} \frac{\dim(\operatorname{Hom}_G(\pi, \mathbb{C}[X]))^m}{\dim \pi^{m+2k-1}} \chi_\pi^X(x_1, x_2) = \frac{1}{\# G^{m+2k}} \cdot$$

$$\cdot \#\{p_1, \dots p_m \in X, h_1, \dots h_m, a_1, \dots a_k, b_1, \dots b_k \in G | h_i \in G_{p_i}, \prod_{i=1}^{m} h_i \cdot \prod_{i=1}^{k} [a_i, b_i] \cdot x_1 = x_2\}.$$

References

[AA] Aizenbud, A.; Avni, N.; *Representation growth and rational singularities of the moduli space of local systems.* Arxiv 1307.0371, to appear in Inventiones Mathematicae.

[FG06] V. Fock, A. Goncharov, *Moduli spaces of local systems and higher Teichmuller theory.* Publications Mathematiques de l'Institut des Hautes Etudes Scientifiques 103/1, (2006).

[HR08] T. Hausel, F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties.* Invent. Math. 174/3, (2008).

[Jam84] G.D. James, *Representations of general linear groups,* London Math. Soc. Lec. Notes Series, 94 (1984).

[Kat08] N. Katz, *Appendix of [HR08]: E-polynomials, zeta-equivalence, and polynomial-count varieties.* Invent. Math. 174/3, (2008).

[Wit91] Witten, Edward *On quantum gauge theories in two dimensions.* Comm. Math. Phys. 141 (1991), no. 1, 153–209.