De-Rham theorem and Shapiro lemma for Schwartz functions on Nash manifolds

Avraham Aizenbud and Dmitry Gourevitch *

Abstract

In this paper we continue our work on Schwartz functions and generalized Schwartz functions on Nash (i.e. smooth semi-algebraic) manifolds. Our first goal is to prove analogs of de-Rham theorem for de-Rham complexes with coefficients in Schwartz functions and generalized Schwartz functions. Using that we compute the cohomologies of the Lie algebra $\mathfrak{g}$ of an algebraic group G with coefficients in the space of generalized Schwartz sections of G-equivariant bundle over a G- transitive variety M. We do it under some assumptions on topological properties of G and M. This computation for the classical case is known as Shapiro lemma.

Contents

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1 Introduction

We will use the notions of Schwartz sections and generalized Schwartz sections of Nash (i.e. smooth semi-algebraic) bundles over Nash manifolds introduced in [AG]. These will be reviewed in section 2.

We use the following notation. For a Nash manifold M we denote by $\mathcal{S}(M)$ the space of Schwartz functions on M and by $\mathcal{G}(M)$ the space of generalized Schwartz functions on M. For a Nash vector bundle $E \to M$ we denote by $\mathcal{S}_M^E$ the cosheaf of Schwartz sections of E and by $\mathcal{G}_M^E$ the sheaf of generalized Schwartz sections of E. We also denote the global Schwartz sections of E by $\mathcal{S}(M, E)$ and global generalized Schwartz sections of E by $\mathcal{G}(M, E)$.

Let M be a Nash manifold. We can define the de-Rham complex with coefficients in Schwartz functions

$$DR_{\mathcal{S}}(M) : 0 \to \mathcal{S}(M, \Omega_M^0) \to ... \to \mathcal{S}(M, \Omega_M^n) \to 0$$

We will prove that its cohomologies are isomorphic to the compact support cohomologies of M. Similarly we will define de-Rham complex with coefficients in generalized Schwartz functions

$$DR_{\mathcal{G}}(M) : 0 \to \mathcal{G}(M, \Omega_M^0) \to ... \to \mathcal{G}(M, \Omega_M^n) \to 0$$

and prove that its cohomologies are isomorphic to the cohomologies of M.

Moreover, we will prove relative versions of these statements. Let $F \to M$ be a locally trivial fibration. We will define Nash vector bundles $H^i(F \to M)$ and $H_c^i(F \to M)$ over M such that their fibers will be equal to the cohomologies of the fibers of F and the compact support cohomologies of the fibers of F in correspondance. We will define relative de-Rham complex of $F \to M$ with coefficients in Schwartz functions. We will denote it by $DR_{\mathcal{S}}(F \to M)$ and prove that its cohomologies are canonically isomorphic to the spaces of global Schwartz sections of the bundles $H_c^i(F \to M)$.

Similarly we will define relative de-Rham complex of $F \to M$ with coefficients in generalized Schwartz functions and denote it by $DR_{\mathcal{G}}(F \to M)$. We will prove that its cohomologies are canonically isomorphic to the spaces of global generalized Schwartz sections of the bundles $H^i(F \to M)$. In particular, if the fiber of F is contractible then the higher cohomologies of the relative de-Rham complex with coefficients in generalized Schwartz functions vanish and the zero cohomology is $\mathcal{G}(M)$. Using this result we will prove the following analog of Shapiro lemma.

Theorem 1.0.1 *Let G be a contractible linear algebraic group. Let $H < G$ be a contractible subgroup and let $M = G/H$. Let ρ be a finite dimensional representation of H. Let $E \to M$ be the G-equivariant bundle corresponding to ρ. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{g}$ be the Lie algebra of G. Let V be the space of generalized Schwartz sections of E over M. It carries a natural action of G.*

Then the cohomologies of $\mathfrak{g}$ with coefficients in V are isomorphic to the cohomologies of $\mathfrak{h}$ with coefficients in ρ.

We will need Nash analogs of some known notions and theorems from algebraic topology that we have not found in the literature. They are written in section 2.4.

We have chosen to work in the generality of Nash manifolds for several reasons. The generality of smooth manifolds is too wide since one cannot define Schwartz functions over them. As was explained in [AG], the space of Schwartz functions plays an important role in representation theory because it behaves well under "devisage" and is invariant under Fourier transform.

The generality of real algebraic manifolds is too narrow for us for two reasons. First, any Nash manifold is locally Nash diffeomorphic to $\mathbb{R}^n$ (see Theorem 2.3.26), which is not so in the category of real algebraic manifolds. Second, Nash manifolds appear naturally in representation theory. Namely, an orbit of an algebraic action of a real algebraic group on a real algebraic variety does not have to be an algebraic variety but always has a natural structure of a Nash manifold. For more details, see [AG, section 3.3.1].

1.1 Structure of the paper

In section 2 we give the necessary preliminaries on Nash manifolds and Schwartz functions and distributions over them.

In subsection 2.1 we introduce basic notions of semi-algebraic geometry from [BCR], and [Shi]. In particular we formulate the Tarski-Seidenberg principle of quantifier elimination.

In subsection 2.2 we introduce the notion of restricted topological space (from [DK]) and sheaf theory over it. These notions will be necessary to introduce non-affine Nash manifolds and to formulate the relative de-Rham theorem.

In subsection 2.3 we give basic preliminaries on Nash manifolds from [BCR], [Shi] and [AG].

In subsection 2.4 we repeat known notions and theorems from algebraic topology for the Nash case. In particular we formulate Theorem 2.4.3 which says that the restricted topology is equivalent as a Grothendieck topology to the smooth topology on the category of Nash manifolds.

In subsection 2.5 we give the definitions of Schwartz functions and Schwartz distributions on Nash manifolds from [AG].

In subsection 2.6 we remind some classical facts on nuclear Fréchet spaces and prove that the space of Schwartz functions on a Nash manifold is nuclear.

In section 3 we formulate and prove de-Rham theorem for Schwartz functions on Nash manifolds. Also, we prove its relative version. We need this relative version in the proof of Shapiro Lemma.

In section 4 we formulate and prove a version of Shapiro lemma for Schwartz functions on Nash manifolds.

In section 5 we discuss possible extensions and applications of this work.

In appendix A we prove Theorem 2.4.3 that we discussed above.

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2 Preliminaries

During the whole paper we mean by smooth infinitely differentiable.

2.1 Semi-algebraic sets and Tarski-Seidenberg principle

In this subsection we will give some preliminaries on semi-algebraic geometry from [BCR] and [Shi].

Definition 2.1.1 *A subset $A \subset \mathbb{R}^n$ is called a **semi-algebraic set** if it can be presented as a finite union of sets defined by a finite number of polynomial equalities and inequalities. In other words, if there exist finitely many polynomials $f_{ij}, g_{ik} \in R[x_1, ..., x_n]$ such that*

$$A = \bigcup_{i=1}^{r} \{x \in \mathbb{R}^n | f_{i1}(x) > 0, ..., f_{is_i}(x) > 0, g_{i1}(x) = 0, ..., g_{it_i}(x) = 0\}.$$

Lemma 2.1.2 *The collection of semi-algebraic sets is closed with respect to finite unions, finite intersections and complements.*

Example 2.1.3 *The semi-algebraic subsets of $\mathbb{R}$ are unions of finite number of intervals.*

Definition 2.1.4 *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semi-algebraic sets. A mapping $\nu : A \to B$ is called **semi-algebraic** iff its graph is a semi-algebraic subset of $\mathbb{R}^{m+n}$.*

Proposition 2.1.5 *Let ν be a bijective semi-algebraic mapping. Then the inverse mapping ν^{-1} is also semi-algebraic.*

Proof. The graph of ν is obtained from the graph of ν^{-1} by switching the coordinates. □

One of the main tools in the theory of semi-algebraic spaces is the Tarski-Seidenberg principle of quantifier elimination. Here we will formulate and use a special case of it. We start from the geometric formulation.

Theorem 2.1.6 *Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset and $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the standard projection. Then the image $p(A)$ is a semi-algebraic subset of $\mathbb{R}^{n-1}$.*

By induction and a standard graph argument we get the following corollary.

Corollary 2.1.7 *An image of a semi-algebraic subset of $\mathbb{R}^n$ under a semi-algebraic map is semi-algebraic.*

Sometimes it is more convenient to use the logical formulation of the Tarski-Seidenberg principle. Informally it says that any set that can be described in semi-algebraic language is semi-algebraic. We will now give the logical formulation and immediately after that define the logical notion used in it.

Theorem 2.1.8 *(Tarski-Seidenberg principle) Let Φ be a formula of the language $L(\mathbb{R})$ of ordered fields with parameters in $\mathbb{R}$. Then there exists a quantifier - free formula Ψ of $L(\mathbb{R})$ with the same free variables $x_1, \dots, x_n$ as Φ such that $\forall x \in \mathbb{R}^n, \Phi(x) \Leftrightarrow \Psi(x)$.*

For the proof see Proposition 2.2.4 on page 28 of [BCR].

Definition 2.1.9 *A **formula of the language of ordered fields with parameters in** $\mathbb{R}$ is a formula written with a finite number of conjunctions, disjunctions, negations and universal and existential quantifiers ($\forall$ and $\exists$) on variables, starting from atomic formulas which are formulas of the kind $f(x_1, \dots, x_n) = 0$ or $g(x_1, \dots, x_n) > 0$, where f and g are polynomials with coefficients in $\mathbb{R}$. The free variables of a formula are those variables of the polynomials which are not quantified. We denote the language of such formulas by $L(\mathbb{R})$.*

Notation 2.1.10 *Let Φ be a formula of $L(\mathbb{R})$ with free variables $x_1, \dots, x_n$. It defines the set of all points $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ that satisfy Φ. We denote this set by S_Φ. In short,*

$$S_\Phi := \{x \in \mathbb{R}^n | \Phi(x)\}.$$

Corollary 2.1.11 *Let Φ be a formula of $L(\mathbb{R})$. Then S_Φ is a semi-algebraic set.*

Proof. Let Ψ be a quantifier-free formula equivalent to Φ. The set S_Ψ is semi-algebraic since it is a finite union of sets defined by polynomial equalities and inequalities. Hence S_Φ is also semi-algebraic since $S_\Phi = S_\Psi$. □

Proposition 2.1.12 *The logical formulation of the Tarski-Seidenberg principle implies the geometric one.*

Proof. Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset, and $pr : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the standard projection. Then there exists a formula $\Phi \in L(\mathbb{R})$ such that $A = S_\Phi$. Then $pr(A) = S_\Psi$ where

$$\Psi(y) = \text{“}\exists x \in \mathbb{R}^n \, (\mathrm{pr}(x) = y \wedge \Phi(x))\text{”}.$$

Since $\Psi \in L(\mathbb{R})$, the proposition follows now from the previous corollary.

Remark 2.1.13 *In fact, it is not difficult to deduce the logical formulation from the geometric one.*

Let us now demonstrate how to use the logical formulation of the Tarski-Seidenberg principle.

Corollary 2.1.14 *The closure of a semi-algebraic set is semi-algebraic.*

Proof. Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset, and let $\overline{A}$ be its closure. Then $\overline{A} = S_\Psi$ where

$$\Psi(x) = \text{“}\forall \varepsilon > 0 \, \exists y \in A \, |x-y|^2 < \varepsilon\text{”}.$$

Clearly, $\Psi \in L(\mathbb{R})$ and hence $\overline{A}$ is semi-algebraic. □

Corollary 2.1.15 *Images and preimages of semi-algebraic sets under semi-algebraic mappings are semi-algebraic.*

Corollary 2.1.16
(i) The composition of semi-algebraic mappings is semi-algebraic.
(ii) The $\mathbb{R}$-valued semi-algebraic functions on a semi-algebraic set A form a ring, and any nowhere vanishing semi-algebraic function is invertible in this ring.

We will also use the following theorem from [BCR] (Proposition 2.4.5).

Theorem 2.1.17 *Any semi-algebraic set in $\mathbb{R}^n$ has a finite number of connected components.*

2.2 Sheaf theory over restricted topological spaces

The usual notion of topology does not fit semi-algebraic geometry. Therefore we will need a different notion of topology called restricted topology, that was introduced in [DK].

Definition 2.2.1 *A* ***restricted topological space*** *M is a set M equipped with a family $\overset{\circ}{\mathfrak{S}}(M)$ of subsets of M, called the open subsets that includes M and the empty set and is closed with respect to finite unions and finite intersections.*

Remark 2.2.2 *When we work on a restricted topological space and we say that some property is satisfied locally we mean locally in the restricted topology, i.e. that there exists a finite open cover $X = \bigcup_{i=1}^n U_i$ such that the property is satisfied for any of the U_i.*

Remark 2.2.3 *In general, there is no closure in restricted topology since infinite intersection of closed sets does not have to be closed.*

Remark 2.2.4 *A restricted topological space M can be considered as a site in the sense of Grothendieck. The category of the site has as objects the open sets of M and as morphisms the inclusion maps. The covers $(U_i \to U)_{i \in I}$ are the finite systems of inclusions with $\bigcup_{i=1}^{n} U_i = U$. This gives us the notions of sheaf and cosheaf on M. We will repeat the definitions of these notions in simpler terms.*

Definition 2.2.5 *A* ***pre-sheaf*** *F on a restricted topological space M is a contravarinant functor from the category $Top(M)$ which has open sets as its objects and inclusions as morphisms to the category of abelian groups, vector spaces etc.*

In other words, it is an assignment $U \mapsto F(U)$ that assigns for every open U an abelian group (or a vector spaces or a topological vector space, etc.), and for every inclusion of open sets $V \subset U$ - a restriction morphism $res_{U,V} : F(U) \to F(V)$ that satisfy $res_{U,U} = Id$ and for $W \subset V \subset U$, $res_{V,W} \circ res_{U,V} = res_{U,W}$. A ***morphism of pre-sheaves*** *$\phi : F \to G$ is a collection of morphisms $\phi_U : F(U) \to G(U)$ for any open set U that commute with the restrictions.*

Definition 2.2.6 *A **sheaf** F on a restricted topological space M is a pre-sheaf fulfilling the usual sheaf conditions, except that now only finite open covers are admitted. The conditions are: for any open set U and any finite cover U_i of M by open subsets, the sequence*

$$0 \to F(U) \stackrel{res_1}{\to} \prod_{i=1}^{n} F(U_i) \stackrel{res_2}{\to} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} F(U_i \cap U_j)$$

is exact.
The map res_1 above is defined by $res_1(\xi) = \prod_{i=1}^{n} res_{U,U_i}(\xi)$ and the map res_2 by

$$res_2(\prod_{i=1}^{n} \xi_i) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} res_{U_i,U_i \cap U_j}(\xi_i) - res_{U_j,U_i \cap U_j}(\xi_j)$$

.

Definition 2.2.7 *A **pre-cosheaf** F on a restricted topological space M is a covarinant functor from the category $Top(M)$ to the category of abelian groups, vector spaces etc.*

In other words, it is the assignment $U \mapsto F(U)$ for every open U with abelian groups, vector spaces etc. as values, and for every inclusion of open sets $V \subset U$ - an extension morphism $ext_{V,U} : F(V) \to F(U)$ that satisfy: $ext_{U,U} = Id$ and for $W \subset V \subset U$, $ext_{V,U} \circ ext_{W,V} = ext_{W,U}$. A morphism of pre-cosheaves $\phi : F \to G$ is a collection of morphisms $\phi_U : F(U) \to G(U)$ for any open set U that commute with the extensions.

Definition 2.2.8 *A **cosheaf** F on a restricted topological space M is a pre-cosheaf on M fulfilling the conditions dual to the usual sheaf conditions, and with only finite open covers allowed. This means: for any open set U and any finite cover U_i of M by open subsets, the sequence*

$$\bigoplus_{i=1}^{n-1} \bigoplus_{j=i+1}^{n} F(U_i \cap U_j) \to \bigoplus_{i=1}^{n} F(U_i) \to F(U) \to 0$$

is exact.
Here, the first map is defined by

$$\bigoplus_{i=1}^{n-1} \bigoplus_{j=i+1}^{n} \xi_{ij} \mapsto \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} ext_{U_i \cap U_j, U_i}(\xi_{ij}) - ext_{U_i \cap U_j, U_j}(\xi_{ij})$$

and the second one by

$$\bigoplus_{i=1}^{n} \xi_i \mapsto \sum_{i=1}^{n} ext_{U_i,U}(\xi_i).$$

Remark 2.2.9 *As in the usual case, we have the functors of sheafification and cosheafification, which assign to every pre-sheaf (pre-cosheaf) a canonical sheaf (cosheaf). They are defined as left adjoint (right adjoint) functors to the forgetful functor from sheaves (cosheaves) to pre-sheaves (pre-cosheaves). Note that in the construction of cosheafification quotient objects are needed. So cosheafification always exists for sheaves with values in abelian categories. Pre-cosheaves of Fréchet spaces whose extension maps have closed image also have cosheafification.*

Definition 2.2.10 *Let M be a restricted topological space, and F be a sheaf on M. Let $Z \subset M$ be a closed subset. A global section of F is said to be **supported in** Z if its restriction to the complement of Z is zero.*

Remark 2.2.11 *Unfortunately, if we try to define support of a section, it will not be a closed set in general, since infinite intersection of closed sets in the restricted topology does not have to be closed.*

Remark 2.2.12 *Till the end of this section we will consider only sheaves and cosheaves of linear spaces over* $\mathbb{R}$.

Definition 2.2.13 *Let* M *be a restricted topological space and* V *be a linear space over* $\mathbb{R}$. *A function* $f : M \to V$ *is called* ***locally constant*** *if there exists a* *finite* *cover* $M = \bigcup_{i=1}^{k} U_i$ *s.t.* $\forall i. f|_{U_i} = const.$

Remark 2.2.14 *Till the end of this section we will consider only those restricted topological spaces in which any open set is a finite disjoint union of its open connected subsets. In such spaces a locally constant function is a function which is constant on every connected component.*

Using this notion, we define constant sheaf in the usual way, i.e.

Definition 2.2.15 *Let* M *be a restricted topological space. Let* V *be a linear space over* $\mathbb{R}$. *We define* ***constant sheaf over*** M ***with coefficients in*** V *by* $V_M(U) := \{f : U \to V | f$ *is locally constant on* V *in the induced restricted topology* $\}$ *for any open* $U \subset M$.

Definition 2.2.16 *Let* M *be a restricted topological space and* F *be a sheaf (cosheaf) over it. We define a conjugate cosheaf (sheaf) by* $F^*(U) := F(U)^*$.

Definition 2.2.17 *Let* M *be a restricted topological space. Let* V *be a finite dimensional linear space over* $\mathbb{R}$. *We define* ***constant cosheaf over*** M ***with coefficients in*** V *by* $V'_M := (V^*_M)^*$.

Definition 2.2.18 *A sheaf(cosheaf)* F *over a restricted topological space* M *is called* ***locally constant*** *if there exists a* *finite* *cover* $M = \bigcup_{i=1}^{k} U_i$ *such that for any* i, $F|_{U_i}$ *is isomorphic to a constant sheaf(cosheaf) on* U_i.

Definition 2.2.19 *We define internal* $\mathcal{H}om$ *in the categories of sheaves and cosheaves over restricted topological space the same way as it is done in the usual case, i.e.* $\mathcal{H}om(F, G)(U) := Hom(F|_U, G|_U)$.

Definition 2.2.20 *Let* F *be a sheaf over a restricted topological space* M. *We define its dual sheaf* $D(F)$ *by* $D(F) := \mathcal{H}om(F, \mathbb{R}_M)$.

Definition 2.2.21 *Let* F *be a cosheaf over a restricted topological space* M. *We define its dual cosheaf* $D(F)$ *by* $D(F) := \mathcal{H}om(F, \mathbb{R}'_M)$.

Notation 2.2.22 *To every sheaf(cosheaf)* F *over a restricted topological space* M *we associate a cosheaf (sheaf)* F' *by* $F' := D(F)^*$.

Remark 2.2.23 *The constant sheaf (cosheaf) is evidently a sheaf (cosheaf) of algebras, and any sheaf (cosheaf) has a canonical structure of a sheaf (cosheaf) of modules over the constant sheaf (cosheaf).*

Definition 2.2.24 *Let* F *and* G *be sheaves (cosheaves). We define* $F \otimes G$ *to be the sheafification (cosheafification) of the presheaf (precosheaf)* $U \mapsto F(U) \underset{\mathbb{R}_M(U)}{\otimes} G(U)$.

2.3 Nash manifolds

In this section we define the category of Nash manifolds, following [BCR], [Shi] and [AG].

Definition 2.3.1 *A **Nash map** from an open semi-algebraic subset U of $\mathbb{R}^n$ to an open semi-algebraic subset $V \subset \mathbb{R}^m$ is a smooth (i.e. infinitely differentiable) semi-algebraic function. The ring of $\mathbb{R}$-valued Nash functions on U is denoted by $\mathcal{N}(U)$. A **Nash diffeomorphism** is a Nash bijection whose inverse map is also Nash.*

As we are going to do semi-algebraic differential geometry, we will need a semi-algebraic version of implicit function theorem.

Theorem 2.3.2 (Implicit Function Theorem.) *Let $(x^0, y^0) \in \mathbb{R}^{n+p}$, and let $f_1, ..., f_p$ be semi-algebraic smooth functions on an open neighborhood of (x^0, y^0), such that $f_j(x^0, y^0) = 0$ for $j = 1, .., p$ and the matrix $[\frac{\partial f_j}{\partial y_i}(x^0, y^0)]$ is invertible. Then there exist open semi-algebraic neighborhoods U (resp. V) of x^0 (resp. y^0) in $\mathbb{R}^n$ (resp. $\mathbb{R}^p$) and a Nash mapping ϕ, such that $\phi(x^0) = y^0$ and $f_1(x, y) = ... = f_p(x, y) = 0 \Leftrightarrow y = \phi(x)$ for every $(x, y) \in U \times V$.*

The proof is written on page 57 of [BCR] (corollary 2.9.8).

Definition 2.3.3 *A **Nash submanifold** of $\mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$ which is a smooth submanifold .*

By the implicit function theorem it is easy to see that this definition is equivalent to the following one, given in [BCR]:

Definition 2.3.4 *A semi-algebraic subset M of $\mathbb{R}^n$ is said to be a **Nash submanifold of** $\mathbb{R}^n$ **of dimension** d if, for every point x of M, there exists a Nash diffeomorphism ϕ from an open semi-algebraic neighborhood Ω of the origin in $\mathbb{R}^n$ onto an open semi-algebraic neighborhood Ω' of x in $\mathbb{R}^n$ such that $\phi(0) = x$ and $\phi(\mathbb{R}^d \times \{0\} \cap \Omega) = M \cap \Omega'$.*

Definition 2.3.5 *A **Nash map** from a Nash submanifold M of $\mathbb{R}^m$ to a Nash submanifold N of $\mathbb{R}^n$ is a semi-algebraic smooth map.*

Remark 2.3.6 *Any open semi-algebraic subset of a Nash submanifold of $\mathbb{R}^n$ is also a Nash submanifold of $\mathbb{R}^n$.*

Theorem 2.3.7 *Let $M \subset \mathbb{R}^n$ be a Nash submanifold. Then it has the same dimension as its Zarisky closure.*

For proof see section 2.8 in [BCR].

Unfortunately, open semi-algebraic sets in $\mathbb{R}^n$ do not form a topology, since their infinite unions are not always semi-algebraic. This is why we need restricted topology.

Definition 2.3.8 *A $\mathbb{R}$-**space** is a pair $(M, \mathcal{O}_M)$ where M is a restricted topological space and $\mathcal{O}_M$ a sheaf of $\mathbb{R}$-algebras over M which is a subsheaf of the sheaf $\mathbb{R}[M]$ of real-valued functions on M.*

*A **morphism between** $\mathbb{R}$-**spaces** $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ is a continuous map $f : M \to N$, such that the induced morphism of sheaves $f^* : f^*(\mathbb{R}[N]) \to \mathbb{R}[M]$ maps $\mathcal{O}_N$ to $\mathcal{O}_M$.*

Example 2.3.9 *Take for M a Nash submanifold of $\mathbb{R}^n$, and for $\mathring{\mathfrak{S}}(M)$ the family of all open subsets of M which are semi-algebraic in $\mathbb{R}^n$. For any open (semi-algebraic) subset U of M we take as $\mathcal{O}_M(U)$ the algebra $\mathcal{N}(U)$ of Nash functions $U \to \mathbb{R}$.*

Definition 2.3.10 *An **affine Nash manifold** is an $\mathbb{R}$-space which is isomorphic to an $\mathbb{R}$-space of a closed Nash submanifold of $\mathbb{R}^n$. A morphism between two affine Nash manifolds is a morphism of $\mathbb{R}$-spaces between them.*

Example 2.3.11 *Any real nonsingular affine algebraic variety has a natural structure of an affine Nash manifold.*

Remark 2.3.12 *Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be Nash submanifolds. Then a Nash map between them is the same as a morphism of affine Nash manifolds between them.*

Let $f : M \to N$ be a Nash map. Since an inverse of a semi-algebraic map is semi-algebraic, f is a diffeomorphism if and only if it is an isomorphism of affine Nash manifolds. Therefore we will call such f a Nash diffeomorphism.

In [Shi] there is another but equivalent definition of affine Nash manifold.

Definition 2.3.13 *An **affine** C^∞ **Nash manifold** is an $\mathbb{R}$-space over $\mathbb{R}$ which is isomorphic to an $\mathbb{R}$-space of a Nash submanifold of $\mathbb{R}^n$.*

The equivalence of the definitions follows from the following theorem.

Theorem 2.3.14 *Any affine C^∞ Nash manifold is Nash diffeomorphic to a union of finite number of connected components of a real nonsingular affine algebraic variety.*

This theorem is an immediate corollary of theorem 8.4.6 in [BCR] and Theorem 2.1.17.

Remark 2.3.15 *[Shi] usually uses the notion of affine C^ω Nash manifold instead of affine C^∞ Nash manifold. The two notions are equivalent by the theorem of Malgrange (see [Mal] or Corollary I.5.7 in [Shi]) and hence equivalent to what we call just affine Nash manifold.*

Definition 2.3.16 *A **Nash manifold** is an $\mathbb{R}$-space $(M, \mathcal{N}_M)$ which has a finite cover (M_i) by open sets M_i such that the $\mathbb{R}$-spaces $(M_i, \mathcal{N}_M|_{M_i})$ are isomorphic to $\mathbb{R}$-spaces of affine Nash manifolds.*

*A **morphism between Nash manifolds** is a morphism of $\mathbb{R}$-spaces between them. Such morphisms are called Nash maps, and isomorphisms are called Nash diffeomorphisms.*

Remark 2.3.17 *By Proposition 2.1.17, any Nash manifold is a union of a finite number of connected components.*

Definition 2.3.18 *A Nash manifold is called **separated** if its restricted topological space satisfies the standard Hausdorff separation axiom.*

Remark 2.3.19 *Any Nash manifold has a natural structure of a smooth manifold, and any separated Nash manifold is separated as a smooth manifold.*

Remark 2.3.20 *There is a theorem by B.Malgrange (see [Mal]) saying that any Nash manifold has a natural structure of a real analytic manifold and any Nash map between Nash manifolds is analytic. The proof is also written on page 44 in [Shi] (corollary I.5.7).*

Example 2.3.21 *Any real nonsingular algebraic variety has a natural structure of a Nash manifold.*

Proposition 2.3.22 *Any Nash submanifold of the projective space $\mathbb{P}^n$ is affine.*

Proof.
It is enough to show that $\mathbb{P}^n$ is affine. This is written on page 72 of [BCR] (theorem 3.4.4) $\square$

Remark 2.3.23 *So, quasiprojective Nash manifold is the same as affine Nash manifold.*

Notation 2.3.24 *By **open semi-algebraic subset** of a Nash manifold we mean its open subset in the restricted topology.*

The following theorem is a version of Hironaka's theorem for Nash manifolds.

Theorem 2.3.25 (Hironaka) *Let M be an affine Nash manifold. Then there exists a compact affine nonsingular algebraic variety N and a closed algebraic subvariety Z of N, which is empty if M is compact, such that Z has only normal crossings in N and M is Nash diffeomorphic to a union of connected components of $N - Z$.*

The proof is written on page 49 of [Shi] (Corollary I.5.11). This is a consequence of Hironaka desingularization Theorem [Hir].
It implies the following interesting theorem.

Theorem 2.3.26 (Local triviality of Nash manifolds) *Any Nash manifold can be covered by finite number of open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

The proof is written on page 50 of [Shi] (theorem I.5.12)

2.4 Nash algebraic topology

In this section we repeat known notions and theorems from algebraic topology for the Nash case. Part of them can be found in [BCR] and [Shi].

Definition 2.4.1 *Let $f : M \to N$ be a Nash map of Nash manifolds. It is called a **Nash locally trivial fibration** with fiber Z if Z is a Nash manifold and there exist a finite cover $N = \bigcup U_i$ of N by open (semi-algebraic) sets and Nash diffeomorphisms ϕ_i of $f^{-1}(U_i)$ with $U_i \times Z$ such that the composition $f \circ \phi_i^{-1}$ is the natural projection.*

Now we will give another definition of Nash locally trivial fibration.

Definition 2.4.2 *Let $f : M \to N$ be a Nash map of Nash manifolds. It is called a **Nash locally trivial fibration** if there exist a Nash manifold N' and surjective submersive Nash map $g : N' \to N$ such that the base change $h : N' \underset{N}{\times} M \to N'$ is trivializable, i.e. there exists a Nash manifold Z and a Nash diffeomorphism $k : N' \underset{N}{\times} M \to M \times Z$ such that $\pi \circ k = h$ where $\pi : M \times Z \to M$ is the standard projection.*

In order to prove that this definition is equivalent to the previous one, it is enough to prove the following theorem.

Theorem 2.4.3 *Let M and N be Nash manifolds and $\nu : M \to N$ be a surjective submersive Nash map. Then locally (in the restricted topology) it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that ν has a Nash section on each U_i.*

For proof see Appendix A.

Definition 2.4.4 *A **Nash vector bundle** over a Nash manifold M is a linear space object in the category of locally trivial fibrations over M. In other words, it is an $\mathbb{R}$-vector bundle such that the total space, the projection, the fiber and the trivializations are Nash.*

Remark 2.4.5 *In some books, for example [BCR], such vector bundles are called pre-Nash vector bundles. They are called Nash if they can be embedded to a trivial bundle.*

Remark 2.4.6 *Direct sum and tensor product of Nash vector bundles have canonical structure of Nash vector bundles.*

Definition 2.4.7 *Let M be a Nash manifold. To any locally constant sheaf $\mathcal{F}$ on M there corresponds a canonical bundle $B(\mathcal{F})$ on M. Let us give an explicit construction.*

Choose a cover $M = \bigcup_{i=1}^{k} U_i$ such that $\mathcal{F}|_{U_i}$ is isomorphic to the constant sheaf on U_i with fiber V_i, where V_i are some linear spaces. Define $N = \bigsqcup U_i \times V_i$. We define equivalence relation: Let $u_1 \in U_i, u_2 \in U_j, v_1 \in V_i, v_2 \in V_j$ we say that $(u_1, v_1) \sim (u_2, v_2)$ if u_1 and u_2 are the same point in M and $res_{U_i, U_i \cap U_j}(v_1) = res_{U_j, U_i \cap U_j}(v_2)$. We define $B(F) = N/\sim$ with the obvious structure of Nash bundle over M. It is easy to see that the definition does not depend on the choice of the cover and the trivializations.

Definition 2.4.8 *Let $\pi : F \to M$ be a Nash locally trivial fibration. Consider the constant sheaf in the usual topology $\mathbb{R}_F^{us}$ on F. Let π_* denote the push functor from the category of sheaves on F to the category of sheaves on M.*

Let $R^i\pi_*$ denote the i-th right derived functor of π_*. Consider $R^i\pi_*(\mathbb{R}_M^{us})$ and restrict it to the restricted topology. We get a locally constant sheaf in the restricted topology on M. We denote it by $\mathcal{H}^i(F \to M)$.

Definition 2.4.9 *Let $\pi : F \to M$ be a Nash locally trivial fibration. Consider the constant sheaf in the usual topology $\mathbb{R}_F^{us}$ on F. Let $\pi_!$ denote the functor of push with compact support from the category of sheaves on F to the category of sheaves on M.*

Consider $R^i\pi_!(\mathbb{R}_F^{us})$ and restrict it to restricted topology. We get a locally constant sheaf in the restricted topology on M. We denote it by $\mathcal{H}_c^i(F \to M)$.

Definition 2.4.10 *Let $\pi : F \to M$ be a Nash locally trivial fibration. Consider the locally constant sheaf of orientations $\mathcal{O}rient_F^{us}$ on F in the usual topology. Consider the sheaf of relative orientations $\mathcal{O}rient_F^{us} \otimes \pi^*(\mathcal{O}rient_M^{us})$. Consider $R^i\pi_!(\mathcal{O}rient_F^{us} \otimes \pi^*(\mathcal{O}rient_M^{us}))$ and restrict it to the restricted topology. We get a locally constant sheaf in the restricted topology on M. We denote it by $\mathcal{T}\mathcal{H}_c^i(F \to M)$. For the definition of the sheaf of orientations see e.g. [AG, subsection A.1.1].*

Notation 2.4.11 *We denote $H^i(F \to M) := B(\mathcal{H}^i(F \to M))$, $H_c^i(F \to M) := B(\mathcal{H}_c^i(F \to M))$, $TH_c^i(F \to M) := B(\mathcal{T}\mathcal{H}_c^i(F \to M)))$.*

Proposition 2.4.12 *Tangent, normal and conormal bundles, the bundle of differential k-forms, the bundle of orientations, the bundle of densities, etc. have canonical structure of Nash bundles.*

For proof see e.g. Theorems 3.4.3 and 3.4.4 in [AG].

Notation 2.4.13 *Let M be a Nash manifold. We denote by $Orient_M$ the bundle of orientations on M and by D_M the bundle of densities on M.*

Notation 2.4.14 *Let $E \to M$ be a Nash bundle. We denote $\widetilde{E} := E^* \otimes D_M$.*

Using Hironaka theorem (2.3.25) we will prove the following useful result.

Theorem 2.4.15 *Let M be a Nash manifold. Then $H^i(M)$,$H_c^i(M)$,$H_c^i(M, \mathcal{O}rient_M^{us})$ are finite dimensional.*

For this we will need the following lemma.

Lemma 2.4.16 *Let M be a smooth manifold. Let $N \subset M$ be a closed submanifold and denote $U = M - N$. Let $\mathcal{F}$ be a locally constant sheaf on M such that $H^i_c(M, \mathcal{F})$ and $H^i_c(N, \mathcal{F}|_N)$ are finite dimensional, where by $\mathcal{F}|_N$ we mean restriction as a local system. Then $H^i_c(U, \mathcal{F}|_U)$ is also finite dimensional.*

Proof of the lemma. Let $\phi : N \to M$ and $\psi : U \to M$ be the standard imbeddings. Note that $\phi_!$ and $\psi_!$ are exact and $\phi_* = \phi_!$ so $H^i_c(U, \mathcal{F}|_U) \cong H^i_c(M, \psi_!(\mathcal{F}|_U))$ and $H^i_c(N, \mathcal{F}|_N) \cong H^i_c(M, \phi_!(\mathcal{F}|_N))$. So from the short exact sequence $0 \to \psi_!(\mathcal{F}|_U) \to \mathcal{F} \to \phi_!(\mathcal{F}|_N) \to 0$ of sheaves on M we see that $H^i_c(U, \mathcal{F}|_U)$ is also finite dimensional. □

Proof of the theorem. Intersection of affine open Nash submnaifolds is affine, hence by Mayer - Vietories long exact sequence (see e.g. [BT], section I.2) it is enough to prove the theorem for affine Nash manifolds. Note that $H^i_c(M) = H^i_c(M, \mathbb{R}^{us}_M)$ where $\mathbb{R}^{us}_M$ is the constant sheaf on M.

By Hironaka theorem (Theorem 2.3.25) there exists a compact affine nonsingular algebraic variety K and a closed algebraic subvariety Z of K, such that Z has only normal crossings in K and M is Nash diffeomorphic to a union of connected components of $K - Z$.

Let $Z = Z_n \supset ... \supset Z_0 = \emptyset$ be a stratification of Z such that $Z_k - Z_{k-1}$ is a Nash manifold. We will prove by induction on k that $H^*_c(K - Z_k)$ are finite dimensional. The basis is known and the induction step follows form the lemma. Hence $H^*_c(K - Z)$ are finite dimensional and hence $H^*_c(M)$ are finite dimensional.

Similarly $H^*_c(M, \mathcal{O}rient^{us}_M)$ are finite dimensional. By Poincaré duality $H^i(M) \cong H^{dimM-i}_c(M, \mathcal{O}rient^{us}_M)^*$ and hence is finite dimensional. □

2.5 Schwartz functions on Nash manifolds

In this section we will review some preliminaries on Schwartz functions on Nash manifolds defined in [AG].

The Fréchet space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions on $\mathbb{R}^n$ was defined by Laurant Schwartz to be the space of all smooth functions that decay faster than $1/|x|^n$ for all n.

In [AG] we have defined for any Nash manifold M the Fréchet space $\mathcal{S}(M)$ of Schwartz functions on it.

As Schwartz functions cannot be restricted to open subsets, but can be continued by 0 from open subsets, they form a cosheaf rather than a sheaf.

We have defined for any Nash bundle E over M the cosheaf $\mathcal{S}^E_M$ over M (in the restricted topology) of Schwartz sections of E. These cosheaves satisfy: $\mathcal{S}^E_M(U) = \mathcal{S}(U, E|_U) = \{\xi \in \mathcal{S}(M, E) | \xi$ vanishes with all its derivatives on $M - U\}$. We have also defined the sheaf $\mathcal{G}^E_M$ of generalized Schwartz sections of E by $\mathcal{G}^E_M(U) = (\mathcal{S}^{\widetilde{E}}_M(U))^*$. This sheaf is flabby.

The fact that $\mathcal{S}^E_M$ satisfies the cosheaf axioms follows from the following version of partition of unity:

Theorem 2.5.1 (Partition of unity) *Let M be a Nash manifold, and $(U_i)_{i=1}^n$ - finite open cover by affine Nash submanifolds. Then there exist smooth functions $\alpha_1, ..., \alpha_n$ such that $supp(\alpha_i) \subset U_i$, $\sum_{i=1}^n \alpha_i = 1$ and for any $g \in \mathcal{S}(M)$, $\alpha_i g \in \mathcal{S}(U_i)$.*

For proof see [AG], section 5.2.
We will use the following proposition which follows trivially from the definition of the sheaves of Schwartz sections and generalized Schwartz sections given in [AG], section 5.

Proposition 2.5.2 *Let $\mathcal{F}$ be a locally constant sheaf over a Nash manifold M. Then $\mathcal{S}^{B(\mathcal{F})}_M \cong \mathcal{S}_M \otimes \mathcal{F}'$ and $\mathcal{G}^{B(\mathcal{F})}_M \cong \mathcal{G}_M \otimes \mathcal{F}$. Moreover, if E is a Nash vector bundle over M then $\mathcal{S}^{B(\mathcal{F}) \otimes E}_M \cong \mathcal{S}^E_M \otimes \mathcal{F}'$ and $\mathcal{G}^{B(\mathcal{F}) \otimes E}_M \cong \mathcal{G}^E_M \otimes \mathcal{F}$. Recall that $B(\mathcal{F})$ is the bundle corresponding to $\mathcal{F}$ and $\mathcal{F}'$ is the cosheaf corresponding to $\mathcal{F}$ as they were defined in sections 2.4 and 2.2.*

To conclude, we will list the important statements from [AG]:

1 *Compatibility: For open semi-algebraic subset* $U \subset M$, $\mathcal{S}_M^E(U) = \mathcal{S}(U, E|_U)$.

2 $\mathcal{S}(\mathbb{R}^n)$ = *Classical Schwartz functions on* $\mathbb{R}^n$.

3 *For compact* M, $\mathcal{S}(M,E)$ = *smooth global sections of* E.

4 $\mathcal{G}_M^E = (\mathcal{S}_M^{\widetilde{E}})^*$, *where* $\widetilde{E} = E^* \otimes D_M$ *and* D_M *is the bundle of densities on* M .

5 *Let* $Z \subset M$ *be a Nash closed submanifold. Then restriction maps* $\mathcal{S}(M,E)$ *onto* $\mathcal{S}(Z, E|_Z)$.

6 *Let* $U \subset M$ *be a semi-algebraic open subset. Then*

$$\mathcal{S}_M^E(U) \cong \{\xi \in \mathcal{S}(M,E) | \quad \xi \text{ is 0 on } M - U \text{ with all derivatives}\}.$$

7 *Let* $Z \subset M$ *be a Nash closed submanifold. Consider* $V = \{\xi \in \mathcal{G}(M,E) | \xi$ *is supported in* $Z\}$. *It has canonical filtration* V_i *such that its factors are canonically isomorphic to* $\mathcal{G}(Z, E|_Z \otimes Sym^i(CN_Z^M) \otimes D_M^*|_Z \otimes D_Z)$ *where* CN_Z^M *is the conormal bundle of* Z *in* M *and* Sym^i *means i-th symmetric power.*

2.6 Nuclear Fréchet spaces

Definition 2.6.1 *We call a complex of topological vector spaces* ***admissible*** *if all its differentials have closed images.*

We will need the following classical facts from the theory of nuclear Fréchet spaces.

- Let V be a nuclear Fréchet space and W be a closed subspace. Then both W and V/W are nuclear Fréchet spaces.
- Let
$$\mathcal{C} : 0 \to C_1 \to ... \to C_n \to 0$$
be an admissible complex of nuclear Fréchet spaces. Then the complex $\mathcal{C}^*$ is also admissible and $H^i(\mathcal{C}^*) \cong H^i(\mathcal{C}_i)^*$.
- Let V be a nuclear Fréchet space. Then the complex $\mathcal{C}\widehat{\otimes}V$ is an admissible complex of nuclear Fréchet spaces and $H^i(\mathcal{C} \otimes V) \cong H^i(\mathcal{C}) \otimes V$.
- $\mathcal{S}(\mathbb{R}^n)$ is a nuclear Fréchet space.
- $\mathcal{S}(\mathbb{R}^{n+m}) = \mathcal{S}(\mathbb{R}^n)\widehat{\otimes}\mathcal{S}(\mathbb{R}^m)$.

A good exposition on nuclear Fréchet spaces can be found in Appendix A in [CHM].

Corollary 2.6.2 *Let* M *be a Nash manifold and* E *be a Nash bundle over it. Then* $\mathcal{S}(M,E)$ *is a nuclear Fréchet space.*

Proof. By definition of $\mathcal{S}(M,E)$ and by Theorem 2.3.26, $\mathcal{S}(M,E)$ is a quotient of direct sum of several copies of $\mathcal{S}(\mathbb{R}^n)$. □

Corollary 2.6.3 *Let* M_i, $i = 1,2$ *be Nash manifolds and* E_i *be Nash bundles over* M_i. *Then*

$$\mathcal{S}(M_1 \times M_2, E_1 \boxtimes E_2) = \mathcal{S}(M_1, E_1)\widehat{\otimes}\mathcal{S}(M_2, E_2),$$

where $E_1 \boxtimes E_2$ *denotes the exterior product.*

Lemma 2.6.4 *Let $\phi : V \to W$ be a morphism of Fréchet spaces. Suppose that $Im\phi$ has finite codimension. Then $Im\phi$ is closed in W.*

Proof. Clearly, it is enough to show for the case when ϕ is an embedding. In this case let us choose a finite dimensional subspace $L \subset W$ such that $W = L \oplus Im\phi$ as abstract linear spaces. Consider the map $\psi : L \oplus V \to W$. By Banach open map theorem (see theorem 2.11 in [Rud]) ψ is an isomorphism of topological vector spaces which implies that $Im\phi$ is closed. □

Corollary 2.6.5 *Let C be an complex of Fréchet spaces. Suppose that C has finite - dimensional cohomologies. Then C is admissible.*

3 De-Rham theorem for Schwartz functions on Nash manifolds

3.1 De-Rham theorem for Schwartz functions on Nash manifolds

Theorem 3.1.1 *Let M be an affine Nash manifold. Consider the de-Rham complex of M with compactly supported coefficients*

$$DR_c(M) : 0 \to C_c^\infty(M, \Omega_M^0) \to ... \to C_c^\infty(M, \Omega_M^n) \to 0$$

and the natural map $i : DR_c(M) \to DR_{\mathcal{S}}(M)$. Then i is a quasiisomorphism, i.e. it induces an isomorphism on the cohomologies.

Proof. Let $N \supset M$ be the compactification of M given by Hironaka theorem, i.e. N is a compact Nash manifold, $N = M \dot\cup D \dot\cup U$ where M and U are open and $D = \bigcup_{i=1}^{k} D_i$ where $D_i \subset N$ is a closed Nash submanifold of codimension 1 and all the intersections are normal, i.e. every $y \in N$ has a neighborhood V with a diffeomorphism $\phi : V \to \mathbb{R}^n$ such that $\phi(D_i \cap V)$ is either a coordinate hyperplane or empty. Denote $Z = N - M$.

N has a structure of compact smooth manifold. We build two complexes $\mathcal{DR}_1$ and $\mathcal{DR}_2$ of sheaves on N in the classical topology by $\mathcal{DR}_1^k(W) := \{\omega \in C^\infty(W, \Omega^k) | \omega$ vanishes in a neighborhood of $Z\}$ and $\mathcal{DR}_2^k(W) = \{\omega \in C^\infty(W, \Omega^k) | \omega$ vanishes on Z with all its derivatives $\}$. As the differential we take the standard de-Rham differential.

Note that we have a natural embedding of complexes $\mathcal{I} : \mathcal{DR}_1 \to \mathcal{DR}_2$. Note also that $\mathcal{DR}_1(M) \cong DR_c(M)$ and $\mathcal{DR}_2(M) \cong DR_{\mathcal{S}}(M)$. The theorem follows from the facts that $\mathcal{DR}_{1,2}^i$ are Γ - acyclic sheaves and that $\mathcal{I}$ is a quasiisomorphism. Let us prove these two facts now.

$\mathcal{DR}_{1,2}^i$ are fine (i.e. have partition of unity), which follows from the classical partition of unity. So, by theorem 5.25 from [War] they are acyclic.

The statement that $\mathcal{I}$ is a quasiisomorphism is a local statement, so we will verify that $\mathcal{I} : \mathcal{DR}_1(W) \to \mathcal{DR}_2(W)$ is a quasiisomorphism for small enough W. Since all the intersections in D are normal, it is enough to check it for the case $W \cong \mathbb{R}^n$ and $D \cap W$ is a union of coordinate hyperplanes. In this case, the proof is technical and all its ideas are taken from classical proof of Poincaré lemma. We will give it now only for completeness and we recommend the reader to skip to the end of the proof.
$(N - D) \cap W$ splits to a union of connected components of the form $R_{>0}^k \times \mathbb{R}^l$. Hence complexes $\mathcal{DR}_{1,2}(W)$ split to direct sum of the complexes corresponding to the connected components. Therefore, it is enough to check this statement in the following two cases:
Case 1 $W = R_{\geqslant 0}^k \times \mathbb{R}^l$, $U \cap W = R_{>0}^k \times \mathbb{R}^l$, $M \cap W = \emptyset$
Case 2 $W = R_{\geqslant 0}^k \times \mathbb{R}^l$, $U \cap W = \emptyset$, $M \cap W = R_{>0}^k \times \mathbb{R}^l$.
Case 1 is trivial, as $\mathcal{DR}_1(W) = \mathcal{DR}_2(W) = 0$ in this case.
Case 2: If $k = 0$ then $\mathcal{I} = Id$. Otherwise we will show that the cohomologies of both complexes vanish. Clearly $H_{1,2}^0 = 0$ since the only constant function which vanishes on D is 0. Now let $\omega \in \mathcal{DR}_{1,2}^m(W)$ be a closed form. We can write ω in coordinates $dx_1, ..., dx_{k+l}$: $\omega = \omega_1 \wedge dx_1 + \omega_2$ where neither ω_1 nor ω_2

contain dx_1. Let f_j be the coefficients of ω_1 and define $g_j(x_1, ..., x_{k+l}) = \int\limits_0^{x_1} f_j(t, x_2, ..., x_{k+l})dt$ and let λ be the form with coefficients g_j. It is easy to check that $d\lambda = \omega$ and $\lambda \in \mathcal{DR}_{1,2}^{m-1}(W)$. □

Remark 3.1.2 *Our proof of the previous theorem heavily uses the Hironaka theorem. This explains why our proof works only in the generality of affine Nash manifolds. However, we expect that one can give another proof which will not use the Hironaka theorem and will work in the full generality of Nash manifolds. We elaborate on this in section 5.*

Theorem 3.1.3 *Let M be an affine Nash manifold. Consider the de-Rham complex of M with coefficients in classical generalized functions, i.e. functionals on compactly supported densities.*

$$DR_{-\infty}(M) : 0 \to C^{-\infty}(M, \Omega_M^0) \to ... \to C^{-\infty}(M, \Omega_M^n) \to 0$$

and the natural map $i : DR_{\mathcal{G}}(M) \to DR_{-\infty}(M)$. Then i is a quasiisomorphism.

Proof. Let N, D, D_i, U and Z be the same as in the proof of Theorem 3.1.1. We again build two complexes $\mathcal{DR}_1$ and $\mathcal{DR}_2$ of sheaves on N in the classical topology by $\mathcal{DR}_1^k(W) :=$ k-forms on $W - Z$ with generalized coefficients and $\mathcal{DR}_2^k(W) :=$ k-forms with generalized coefficients on W modulo k-forms with generalized coefficients on W supported in $Z \cap W$. We have an embedding $\mathcal{I} : \mathcal{DR}_2 \hookrightarrow \mathcal{DR}_1$. Again, by classical partition of unity the sheaves are fine and hence acyclic, so it is enough to prove that $\mathcal{I}$ is a quasiisomorphism. Again, we check it locally and the only interesting case is $W = R_{\geq 0}^k \times \mathbb{R}^l$, $U \cap W = \emptyset$, $M \cap W = R_{>0}^k \times \mathbb{R}^l$ where $k > 0$. Define a map $\phi : \mathbb{R} \to \mathcal{DR}(W)_{1,2}^0$ by setting $\phi(c)$ to be the constant generalized function c. It gives us extensions $\widetilde{\mathcal{DR}_{1,2}}(W)$ of complexes $\mathcal{DR}_{1,2}(W)$ and $\widetilde{\mathcal{I}}$ of $\mathcal{I}$. It is enough to prove that $\widetilde{\mathcal{I}}$ is a quasiisomorphism. For this we will prove that both complexes are acyclic. Fix standard orientation on N. Now our complexes become dual to

$$C_1 : 0 \leftarrow \mathbb{R} \leftarrow C_c^\infty(W \cap M, \Omega_{W\cap M}^n) \leftarrow ... \leftarrow C_c^\infty(W \cap M, \Omega_{W\cap M}^0) \leftarrow 0$$

and

$$C_2 : 0 \leftarrow \mathbb{R} \leftarrow C_c^\infty(W, W \cap D, \Omega_W^n) \leftarrow ... \leftarrow C_c^\infty(W, W \cap D, \Omega_W^0) \leftarrow 0$$

where $C_c^\infty(W, W \cap D, \Omega_W^n)$ are compactly supported forms which vanish with all their derivatives on $W \cap D$. We will prove that $C_{1,2}$ are homotopically equivalent to zero and this will give us that $\widetilde{\mathcal{DR}_{1,2}}(W)$ are also homotopically equivalent to zero and hence are acyclic. The complex C_1 is isomorphic to the following complex

$$C_1' : 0 \leftarrow \mathbb{R} \leftarrow C_c^\infty(\mathbb{R}^n, \Omega_{\mathbb{R}^n}^n) \leftarrow ... \leftarrow C_c^\infty(\mathbb{R}^n, \Omega_{\mathbb{R}^n}^0) \leftarrow 0.$$

In section I.4 of [BT] (Poincaré lemma for compactly supported cohomologies) it is proven that C_1 is homotopy equivalent to zero. In the same way we can prove that C_2 is homotopically equivalent to zero. □

The following theorem is classical.

Theorem 3.1.4 *Let M be a smooth manifold. Consider the de-Rham complex of M with coefficients in classical generalized functions $DR_{-\infty}(M)$, the de-Rham complex of M with coefficients in smooth functions $DR(M)$ and the natural map $i : DR(M) \to DR_{-\infty}(M)$. Then i is a quasiisomorphism.*

Proof. Let $\mathcal{DR}_{-\infty}$ and $\mathcal{DR}$ be the de-Rham complex of M with coefficients in the sheaves of classical generalized functions and smooth functions correspondingly. The sheaves in these complexes are acyclic hence it is enough to show that the natural map $\mathcal{I} : \mathcal{DR} \to \mathcal{DR}_{-\infty}$ is a quasiisomorphism. This is proven by a local computation similar to the one in the proof of the last theorem. □

Definition 3.1.5 *Let M be a Nash manifold. We define the* ***twisted bundle of k-differential forms on M by*** $T\Omega_M^k := \Omega_M^k \otimes Orient_M$ *and correspondingly the* ***twisted de-Rham complexes***

$$TDR_{-\infty}(M),\ TDR_{\mathcal{G}}(M),\ TDR(M),\ TDR_{\mathcal{S}}(M),\ TDR_c(M).$$

Remark 3.1.6 *Note that* $T\Omega_M^{n-k} \cong \widetilde{\Omega_M^k}$. *This gives us a natural pairing between* $\mathcal{S}(M, T\Omega_M^{n-k})$ *and* $\mathcal{G}(M, \Omega_M^k)$.

Remark 3.1.7 *The theorems 3.1.1, 3.1.3 and 3.1.4 hold true also for the twisted de-Rham complexes and the proofs are the same.*

The bottom line of this section is the following version of de-Rham theorem

Theorem 3.1.8 *Let M be an affine Nash manifold of dimension n. Then*
$H^i(DR_{\mathcal{G}}(M)) \cong H^i(M)$
$H^i(DR_{\mathcal{S}}(M)) \cong H_c^i(M)$
$H^i(TDR_{\mathcal{S}}(M)) \cong H_c^i(M, \mathcal{O}rient_M^{us})$
and the pairing between $\mathcal{G}(M, \Omega_M^i)$ *and* $\mathcal{S}(M, T\Omega_M^{n-i})$ *gives an isomorphism between* $H^i(DR_{\mathcal{G}}(M))$ *and* $(H^{n-i}(TDR_{\mathcal{S}}(M)))^*$.

Proof. The theorem is a direct corollary from theorems 3.1.1 3.1.3 3.1.4 for the standard and the twisted cases and from classical Poincaré duality. □

Corollary 3.1.9 *The complexes* $DR_{\mathcal{G}}(M), DR_{\mathcal{S}}(M)$ *and* $TDR_{\mathcal{S}}(M)$ *are admissible.*

Proof. The complexes $DR_{\mathcal{S}}(M)$ and $TDR_{\mathcal{S}}(M)$ are admissible by the theorem and corollary 2.6.5. The complex $DR_{\mathcal{G}}(M)$ is admissible since $DR_{\mathcal{G}}(M) = (TDR_{\mathcal{S}}(M))^*$ and the dual of an admissible complex of nuclear Fréchet spaces is admissible. □

3.2 Relative de-Rham theorem for Nash locally trivial fibration

Definition 3.2.1 *Let* $F \xrightarrow{\pi} M$ *be a locally trivial fibration. Let* $E \to M$ *be a Nash bundle. We define* $T_{F \to M} \subset T_F$ *by* $T_{F \to M} = ker(d\pi)$. *We denote*

$$\Omega_{F \to M}^{i,E} := ((T_{F \to M})^*)^{\wedge i} \otimes \pi^* E,\ Orient_{F \to M} = Orient_F \otimes \pi^*(Orient_M)$$

$$\text{and } T\Omega_{F \to M}^{i,E} := \Omega_{F \to M}^{i,E} \otimes Orient_{F \to M}.$$

Now we can define ***the relative de-Rham complexes***

$$DR_{\mathcal{G}}^E(F \to M),\ DR_{\mathcal{S}}^E(F \to M),\ TDR_{\mathcal{G}}^E(F \to M),\ TDR_{\mathcal{S}}^E(F \to M).$$

If E is trivial we will omit it.

The goal of this section is to prove the following theorem.

Theorem 3.2.2 *Let $p : F \to M$ be a Nash locally trivial fibration. Then*

$$H^k(DR_{\mathcal{S}}^E(F \to M)) \cong \mathcal{S}(M, H_c^k(F \to M) \otimes E). \quad (1)$$

$$H^k(TDR_{\mathcal{S}}^E(F \to M)) \cong \mathcal{S}(M, TH_c^k(F \to M) \otimes E). \quad (2)$$

$$H^k(DR_{\mathcal{G}}^E(F \to M)) \cong \mathcal{G}(M, H^k(F \to M) \otimes E). \quad (3)$$

Proof
(1) Step 1. Proof for the case $M = \mathbb{R}^n$, the fibration $F \to M$ is trivial and E is trivial.

It follows from Theorem 3.1.8 using subsection 2.6.

Step 2. Proof in the general case.

Let $C_i \subset \mathcal{S}(F, \Omega^{i,E}_{F\to M})$ be the subspace of closed forms. We have to construct a continuous onto map $\phi_i : C_i \twoheadrightarrow \mathcal{S}(M, H^i(F \to M) \otimes E)$ whose kernel is the space of exact forms. Fix a cover $M = \bigcup_{k=1}^m U_k$ such that U_k are Nash diffeomorphic to $\mathbb{R}^n$ and $F|_{U_k}$ and $E|_{U_k}$ are trivial. Fix a partition of unity $1 = \sum \alpha_i$ such that and for any $g \in \mathcal{S}(F)$, $\alpha_i g \in \mathcal{S}(p^{-1}(U_i))$. Note that for any $\omega \in \mathcal{S}(F, \Omega^{i,E}_{F\to M})$ we have $\alpha_i\omega \in \mathcal{S}(p^{-1}(U_i), \Omega^{i,E}_{F\to M})$. By the previous step,

$$H^i(DR^{E|_{U_k}}_{F|_{U_k}\to U_k}) \cong \mathcal{S}(U_k, H^i(F|_{U_k} \to U_k) \otimes E|_{U_k}).$$

For any form $\nu \in \mathcal{S}(p^{-1}(U_k), \Omega^{i,E|_{U_k}}_{F|_{U_k}\to U_k})$ we consider the class $[\nu]$ as an element

$$[\nu] \in \mathcal{S}(U_k, H^i(F|_{U_k} \to U_k) \otimes E|_{U_k}) \subset \mathcal{S}(M, H^i(F \to M) \otimes E).$$

Now let $\omega \in C_i$. Define

$$\phi_i(\omega) := \sum_{k=1}^m [\alpha_i \omega].$$

It is easy to see that ϕ satisfies the requirements and does not depend on the choice of U_k and α_k.
(2) Is proven in the same way as (1).
(3) follows from (2) using subsection 2.6.

4 Shapiro lemma

In this section we formulate and prove a version of Shapiro lemma for generalized Schwartz sections of Nash equivariant bundles.

Definition 4.0.1 *Let $\mathfrak{g}$ be a Lie algebra of dimension n. Let ρ be its representation. Define $H^i(\mathfrak{g}, \rho)$ to be the cohomologies of the complex:*

$$C(\mathfrak{g}, \rho) : 0 \to \rho \to \mathfrak{g}^* \otimes \rho \to (\mathfrak{g}^*)^{\wedge 2} \otimes \rho \to ... \to (\mathfrak{g}^*)^{\wedge n} \otimes \rho \to 0$$

with the differential defined by

$$d\omega(x_1, ..., x_{n+1}) = \sum_{i=1}^{n+1} (-1)^i \rho(x_i)\omega(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}) + \\ + \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_1, ..., x_{i-1}, x_{i+1}, ..., x_{j-1}, x_{j+1}, ..., x_{n+1})$$

where we interpret $(\mathfrak{g}^*)^{\wedge k} \otimes \rho$ as anti-symmetric ρ-valued k-forms on $\mathfrak{g}$.

Remark 4.0.2 *$H^i(\mathfrak{g}, \rho)$ is the i-th derived functor of the functor $\rho \mapsto \rho^{\mathfrak{g}}$.*

Definition 4.0.3 *A **Nash group** is a group object in the category of Nash manifolds, i.e. a Nash manifold G together with a point $e \in G$ and Nash maps $m : G \times G \to G$ and $inv : G \to G$ which satisfy the standard group axioms.*

*A **Nash G- manifold** is a Nash manifold M together with a Nash map $a : G \times M \to M$ satisfying $a(gh, x) = a(g, a(h, x))$.*

*A **Nash G - equivariant bundle** is a Nash vector bundle E over a Nash G-manifold M together with an isomorphism of Nash bundles $pr^*(E) \simeq a^*(E)$ where $pr : G \times M \to M$ is the standard projection.*

Definition 4.0.4 *Let G be a Nash group and M be a Nash G manifold. We define* ***the quotient space*** *$G \setminus M$ to be the following $\mathbb{R}$-space. As a set, it is the set theoretical quotient. A subset $U \subset G \setminus M$ is open iff $\pi^{-1}(U)$ is open, where π is the standard projection $M \to G \setminus M$. The sheaf of regular functions is defined by $\mathcal{O}(U) = \{f | f \circ \pi \in \mathcal{N}(\pi^{-1}(U))\}$.*

Definition 4.0.5 *A Nash action of a Nash group G on a Nash manifold M is called* ***strictly simple*** *if it is simple (i.e. all stabilizers are trivial) and $G \setminus M$ is a separated Nash manifold.*

Proposition 4.0.6 *Let G be a Nash group and M be a Nash G manifold. Suppose that the action is strictly simple. Then the projection $\pi : M \to G \setminus M$ is a Nash locally trivial fibration.*

Proof. From differential topology we know that π is a submersion. Consider the base change $M \underset{G\setminus M}{\times} M \to M$. It is Nash diffeomorphic to the trivial projection $M \times G \to M$. □

Corollary 4.0.7 *Let G be a Nash group and M be a Nash G manifold with strictly simple action. Let N be any G manifold. Then the diagonal action on $M \times N$ is strictly simple.*

Proof. If the fibration $M \to G \setminus M$ is trivial, the statement is clear. It is locally trivial by the proposition, and the statement is local on $G \setminus M$. □

Remark 4.0.8 *Let G be a Nash group, M be a Nash G-manifold and $E \to M$ be a Nash G-equivariant bundle. Then the spaces $\mathcal{S}(M, E)$ and $\mathcal{G}(M, E)$ have natural structure of G-representations. Moreover, they are smooth G-representations and hence they have a natural structure of $\mathfrak{g}$-representations where $\mathfrak{g}$ is the Lie algebra of G.*

Now we give a recipe how to compute the cohomologies of such representations.

Theorem 4.0.9 *Let G be a Nash group. Let M be a Nash G-manifold and $E \to M$ be a Nash G-equivariant bundle. Let N be a strictly simple Nash G-manifold. Suppose that N and G are cohomologically trivial (i.e. all their cohomologies except H^0 vanish and $H^0 = \mathbb{R}$) and affine. Denote $F = M \times N$. Note that the bundle $E \boxtimes \Omega_N^i$ has Nash G-equivariant structure given by diagonal action. Hence the relative de-Rham complex $DR_{\mathcal{G}}^E(F \to M)$ is a complex of representations of $\mathfrak{g}$. Then $H^i(\mathfrak{g}, \mathcal{G}(M, E)) = H^i((DR_{\mathcal{G}}^E(F \to M))^{\mathfrak{g}})$.*

For this theorem we will need the following lemma.

Lemma 4.0.10 *Let G be a Nash group. Let F be a strictly simple Nash G-manifold. Denote $M := G \setminus F$ let $E \to M$ be a Nash bundle. Then the relative de-Rham complex $DR_{\mathcal{G}}^E(F \to M)$ is isomorphic to the complex $C(\mathfrak{g}, \mathcal{G}(F, \pi^* E))$, where $\pi : F \to M$ is the standard projection.*

Proof. By partition of unity it is enough to prove for the case that the fibration $\pi : F \to M$ is trivial. In this case we can imbed $\mathfrak{g}$ into the space of Nash sections of the bundle $T_{F \to M} \to F$ and its image will generate the space of all Nash sections of $T_{F \to M} \to F$ over $\mathcal{N}(F)$. This gives us an isomorphism between $\mathcal{G}(F) \otimes \mathfrak{g}$ and $\mathcal{G}(F, T_{F \to M})$ and in the same way between $(\mathfrak{g}^*)^{\wedge k} \otimes \mathcal{G}(F, \pi^* E)$ and $\mathcal{G}(F, \Omega_{F \to M}^{k,E})$. It is easy to check that the last isomorphisms form an isomorphism of complexes between $DR_{\mathcal{G}}^E(F \to M)$ and $C(\mathfrak{g}, \mathcal{G}(F, \pi^* E))$.

□

Proof of Theorem 4.0.9. From relative de-Rham theorem (3.2.2), we know that the complex $DR_{\mathcal{G}}^E(F \to M)$ is a resolution of $\mathcal{G}(M, E)$ (i.e. all its higher cohomologies vanish and the 0's cohomology is equal to $\mathcal{G}(M, E)$). So it is enough to prove that the representations $\mathcal{G}(F, E \boxtimes \Omega_N^i)$ are $\mathfrak{g}$- acyclic. Denote $Z := G \setminus F$. The fact that the bundle $E \boxtimes \Omega_N^i \to F$ is G- equivariant gives us an action of G on the total space $E \boxtimes \Omega_N^i$. Denote $B := G \setminus (E \boxtimes \Omega_N^i)$. Note that $B \to Z$ is a Nash bundle and $F \to Z$ is a Nash locally trivial fibration. By the lemma, the complex $C(\mathfrak{g}, \mathcal{G}(F, E \boxtimes \Omega_N^i))$ is isomorphic to the relative de-Rham complex $DR_{\mathcal{G}}^{B}(F \to Z)$ and again by relative de-Rham theorem $H^i(DR_{\mathcal{G}}^{B}(F \to Z)) = 0$ for $i > 0$. □

Proposition 4.0.11 *Let G be a connected Nash group and F be a Nash G manifold with strictly simple action. Denote $M := G \setminus F$ and let $E \to M$ be a Nash bundle. Then $(\mathcal{G}(F, \pi^*(E)))^{\mathfrak{g}} \cong \mathcal{G}(M, E)$ where $\pi : F \to M$ is the standard projection.*

Proof. It is a direct corollary of Lemma 4.0.10 and relative de-Rham theorem (3.2.2) □

Corollary 4.0.12 *Let G be a Nash group and M be a transitive Nash G manifold. Let $x \in M$ and denote $H := stab_G(x)$. Consider the diagonal action of G on $M \times G$. Let $E \to M \times G$ be a G equivariant Nash bundle. Then $\mathcal{G}(M \times G, E)^{\mathfrak{g}} \cong \mathcal{G}(\{x\} \times G, E|_{\{x\} \times G})^{\mathfrak{h}}$.*

Proof. By the previous proposition, $\mathcal{G}(M \times G, E)^{\mathfrak{g}} \cong \mathcal{G}(G \setminus (M \times G), E')$, where E' is a bundle over $G \setminus (M \times G)$ such that $E = \pi^* E'$.

On the other hand, by the previous proposition $\mathcal{G}(\{x\} \times G, E|_{\{x\} \times G})^{\mathfrak{h}} \cong \mathcal{G}(H \setminus (\{x\} \times G), E'')$. The corollary follows from the fact that $H \setminus (\{x\} \times G) \cong G \setminus (M \times G)$ and under this identification the bundles are the same. □

Now we can prove the Shapiro lemma.

Theorem 4.0.13 (Shapiro lemma) *Let G be a Nash group and M be a transitive Nash G manifold. Let $x \in M$ and denote $H := stab_G(x)$. Let $E \to M$ be a G equivariant Nash bundle. Let V be the fiber of E in x. Suppose G and H are cohomologically trivial. Then $H^i(\mathfrak{g}, \mathcal{G}(M, E)) \cong H^i(\mathfrak{h}, V)$.*

Proof. From the recipe of computing cohomologies (Theorem 4.0.9) we see that

$$H^i(\mathfrak{g}, \mathcal{G}(M, E)) \cong H^i((DR_{\mathcal{G}}^{E}(M \times G \to M))^{\mathfrak{g}}).$$

Now, by Corollary 4.0.12,

$$\mathcal{G}(M \times G, \Omega^{i,E}_{M \times G \to M})^{\mathfrak{g}} \cong \mathcal{G}(\{x\} \times G, \Omega^{i,E}_{\{x\} \times G \to \{x\}})^{\mathfrak{h}}$$

hence

$$(DR_{\mathcal{G}}^{E}(M \times G \to M))^{\mathfrak{g}} \cong (DR_{\mathcal{G}}^{V}(\{x\} \times G \to \{x\}))^{\mathfrak{h}}$$

and hence

$$H^i((DR_{\mathcal{G}}^{E}(M \times G \to M))^{\mathfrak{g}}) \cong H^i((DR_{\mathcal{G}}^{V}(\{x\} \times G \to \{x\}))^{\mathfrak{h}})$$

and again by the recipe of computing cohomologies (Theorem 4.0.9)

$$H^i((DR_{\mathcal{G}}^{V}(\{x\} \times G \to \{x\}))^{\mathfrak{h}}) \cong H^i(\mathfrak{h}, V).$$

□

To make the theorem complete we need to prove that a quotient of a Nash group by its Nash subgroup is a Nash manifold. We prove it in the case of linear Nash group.

Proposition 4.0.14 *Let $H < G < GL_n$ be Nash groups. Then the action of H on G is strictly simple.*

To prove the proposition we will need the following lemma.

Lemma 4.0.15 *Let $H < G$ be Nash groups and M be a Nash G-manifold. Suppose that the actions of H on G and of G on M are strictly simple. Then the action of H on M is also strictly simple.*

Proof.
Consider the locally trivial fibration $M \to G \setminus M$. If it is trivial, the statement is clear. It is locally trivial and the statement is local. $\square$

Proof of Proposition 4.0.14.
Case 1. $dimH = dimG$.
From the theory of Lie groups we know that in this case H is a union of connected components of G. G has a finite number of connected components by Proposition 2.1.17. Hence G/H is finite.
Case 2. H and G are Zarisky closed in GL_n.
In this case they are linear algebraic groups, and for them this statement is known.
Case 3. G is Zarisky closed in GL_n.
Denote by $\overline{H}$ the Zarisky closure of H. It has the same dimension as H by Theorem 2.3.7. From case 1 the action of H on $\overline{H}$ is strictly simple. From the case 2 the action of $\overline{H}$ on G is strictly simple. Hence by the lemma the action of H on G is strictly simple.
Case 4. General.
From the proof for case 1 we see that G/H is a union of a finite number of connected components of $\overline{G}/H$ which is a Nash manifold by case 3. $\square$

5 Possible extensions and applications

We believe that it is possible to obtain an alternative proof of de-Rham theorem which will be valid also in non-affine Nash case. That proof goes in the following way. First one should prove for $M = \mathbb{R}^n$ in the same way as we did. Then one should prove that the cohomologies of a Nash manifold in classical topology are equal to its cohomologies in the restricted topology and to the cohomologies of its de-Rham complex with generalized Schwartz coefficients. If M has a finite cover by open semi-algebraic subsets Nash diffeomorphic to $\mathbb{R}^n$ such that all their intersections are also Nash diffeomorphic to $\mathbb{R}^n$ then the statement is easy because all these cohomologies are isomorphic to the cohomologies of the Chěch complex of this cover. But in general the intersection of the open sets in the cover can be not Nash diffeomorphic to $\mathbb{R}^n$. However we can always construct a hypercover by open semi-algebraic sets Nash diffeomorphic to $\mathbb{R}^n$. So one should prove that the Chěch cohomologies of this hypercover are isomorphic to the required cohomologies. For the notion of hypercover see [Del].

After one proves de-Rham theorem for general Nash manifolds, the relative de-Rham theorem and Shapiro lemma will follow in the same way as in this paper.

It is possible to prove that for any Nash groups $H < G$, the action of H on G is strictly simple. In fact, for any closed Nash equivalence relation $R \subset M \times M$ we can build a structure of $\mathbb{R}$-space on M/R. It is easy to see that if the projection $pr : R \to M$ is ètale then M/R is a Nash manifold. It is left to prove that M/R is Nash manifold in case of any submersive pr. This problem is analogous to the following known theorem in algebraic geometry. Let M be an algebraic variety. Let $R \subset M \times M$ be a closed algebraic equivalence relation. Suppose that the projection $pr : R \to M$ is smooth. Then M/R is an algebraic space. This theorem is proven using the fact that any surjective smooth map has a section locally in ètale topology. In our case any surjective submersion has a section locally in the restricted topology. So we think that our statement can be proven in the same way.

In the classical case Shapiro lemma has a stronger version which enables to compute the cohomologies of $\mathfrak{g}$ in the case that G and H are not cohomologically trivial. We think that our techniques enable to prove its Schwartz version.

Using Shapiro lemma and [AG] one can estimate $H^i(\mathfrak{g}, \mathcal{G}(M, E))$, where M is a Nash G - manifold with finite number of orbits, and E is G-equivariant Nash bundle over M. These cohomologies are important in representation theory since sometimes the space of homomorphisms between two induced representations is $H^0(G, \mathcal{G}(M, E))$ for certain Nash bundle $E \to M$.

A Proof of Theorem 2.4.3

In this Appendix we prove Theorem 2.4.3. Let us first remind its formulation.

Theorem A.0.1 *Let M and N be Nash manifolds and $\nu : M \to N$ be a surjective submersive Nash map. Then locally (in the restricted topology) it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that ν has a Nash section on each U_i.*

This theorem follows immediately from the following three statements.

Theorem A.0.2 *Any semi-algebraic surjection $f : M \to N$ of semi-algebraic sets has a semi-algebraic section.*

Theorem A.0.3 *Let $f : M \to N$ be a semi-algebraic map of Nash manifolds. Then there exists a finite stratification of M by Nash manifolds $M = \bigcup_{i=1}^{k} M_i$ such that $f|_{M_i}$ is Nash.*

Proposition A.0.4 *Let M and N Nash manifolds and $\nu : M \to N$ be a Nash submersion. Let $L \subset N$ be a Nash submanifold and $s : L \to M$ be a section of ν. Then there exist a finite open Nash cover $L \subset \bigcup_{i=1}^{n} U_i$ and sections $s_i : U_i \to M$ of ν such that $s|_{L \cap U_i} = s_i|_{L \cap U_i}$.*

A.1 Proof of Theorem A.0.2

Case 1. $M \subset N \times [0, 1]$, f is the standard projection.
We fix here a certain well-defined semi-algebraic way to choose a section. One could do it in lots of different ways. For any $y \in N$ define $F_y := p(f^{-1}(y))$ where $p : M \to [0, 1]$ is the standard projection. $F_y \subset [0, 1]$ is a semi-algebraic set, hence a finite union of intervals. Let $\overline{F_y}$ be its closure in the usual topology. Denote $s_1(y) := \min \overline{F_y}$. Note that $s_1(y)$ is an end of some interval in F_y. Denote this interval by I_y. Let $s_2(y)$ be the center of I_y. Now define $s(y) := (y, s_2(y))$. By Seidenberg-Tarski theorem s is semi-algebraic, and it is obviously a section of f.

Case 2. $M \subset N \times \mathbb{R}$, f is the standard projection.
We semi-algebraically embed $\mathbb{R}$ into $[0, 1]$ using the stereographic projection and reduce this case to the previous one.

Case 3. For $M \subset N \times \mathbb{R}^n$, f is the standard projection.
Follows by induction from case 2.

Case 4. General case. Follows from case 3 by considering the graph of f. □

A.2 Proof of Theorem A.0.3

In order to prove this theorem, we will need the following two theorems from [BCR].

Theorem A.2.1 (Sard's theorem) *Let $f : M \to N$ be a Nash map of Nash manifolds. Then the set of its critical values is a semi-algebraic subset in M of codimension 1.*

The proof is written on page 235 of [BCR] (theorem 9.6.2).

Theorem A.2.2 (Nash stratification) *Let $M \subset \mathbb{R}^n$ be a semi-algebraic set. Then it has a finite stratification by Nash manifolds $M = \bigcup N_i$.*

The proof is written on page 212 of [BCR] (theorem 9.1.8).

Proof of Theorem A.0.3. It easily follows by induction from the last two theorems and the following observation. Let $f : M \to N$ be a semi-algebraic map between Nash manifolds. Suppose that the graph Γ_f of f is a Nash manifold. Then the set of irregular points of f is exactly the set of critical values of the standard projection $p : \Gamma_f \to M$. □

A.3 Proof of Proposition A.0.4

Notation A.3.1 *Let $x \in \mathbb{R}^n$, $r \in \mathbb{R}$. We denote by $B(x, r)$ the open ball with center x and radius r.*

Definition A.3.2 *A Nash map $e : M \to N$ is called* ***ètale*** *if for any $x \in M$, $de_x : T_xM \to T_{e(x)}N$ is an isomorphism.*

We will need a lemma from [AG] (Theorem 3.6.2).

Lemma A.3.3 *Let $N \subset \mathbb{R}^n$ be an affine Nash manifold and $L \subset N$ be a Nash submanifold. Then there exists a Nash positive function $f_L^N : L \to \mathbb{R}$ and a Nash embedding $\phi_L^N : U_{f_L^N} \hookrightarrow N$ such that $\phi(x, 0) = x$, where $U_f := \{(x, y) \in N_L^N \| ||y|| < f(x)\}$ and $||y||$ is the norm induced from $\mathbb{R}^n$ to the normal space at x.*

Proof of the proposition.
Warning: proofs for cases 1 and 2 are technical and boring. The reader will suffer less if he will do them himself.

Case 1. The map ν is ètale.
It is enough to prove for affine M and N. Embed $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^l$. Consider the graphs $\Gamma(\nu) \subset M \times N$ and $\Gamma(s) \subset \Gamma(\nu)$. Note that $N_{\Gamma(s)}^{\Gamma(\nu)}$ is naturally embedded to $\mathbb{R}^{2(k+l)}$. From analysis we know that for any $y \in M$ there exists $r \in \mathbb{R}$ such that $\nu|_{B(y,r)\cap M}$ is an embedding. For any $((m, n), v) \in N_{\Gamma(s)}^{\Gamma(\nu)}$ denote $B_{((m,n),v)}(r) := B(((m, n), v), r) \cap N_{\Gamma(s)}^{\Gamma(\nu)}$. Consider the function $g : \Gamma(s) \to \mathbb{R}$ defined by $g(m, n) = \sup\{r \in \mathbb{R} | (pr \circ \phi_{\Gamma(s)}^{\Gamma(\nu)})|_{B_{((m,n),0)}(r)}$ is an embedding $\}/2$,where $pr : \Gamma(\nu) \to N$ is the standard projection. Denote $h = min(\nu_{\Gamma(s)}^{\Gamma(\nu)}, g)$. It is easy to see that $\phi_{\Gamma(s)}^{\Gamma(\nu)}(U_h)$ is the graph of the required section.

Case 2. $N \subset \mathbb{R}^l$ is affine, $M \subset \mathbb{R}^k \times N$ open, and ν is the standard projection.
Consider the function $g : L \to \mathbb{R}$ defined by $g(x) = \sup\{r \in \mathbb{R} | B(s(x), r) \cap N \times \mathbb{R}^k \subset M\}/2$. For any $x \in L$ define $B_x = \nu(B(s(x), g(x)) \cap M)$. For any $(x, v) \in N_L^N$ define $B_{(x,v)}(r) := B((x, v), r) \cap N_L^N$. Define $g_2 : L \to \mathbb{R}$ by $g_2(x) = \sup\{r \in \mathbb{R} | \phi_L^N(B_{(x,0)}(r)) \subset B_x\}/2$. Denote $h = min(\nu_L^N, g_2)$. Now we define $s' : \phi_L^N(U_h) \to M$ by $s'(x) = (p(s(\pi((\phi_L^N)^{-1}(x)))), x)$, where $p : \mathbb{R}^k \times N \to \mathbb{R}^k$ is the standard projection, $\pi : CN_L^N \to L$ is the standard projection.

Case 3. For $N \subset \mathbb{R}^l$ affine, $M \subset \mathbb{R}^k \times N$ any Nash submanifold, and ν is the standard projection.
Denote $m := dim(M)$ and $n := dim(N)$. Let κ be the set of all coordinate subspaces of $\mathbb{R}^k$ of dimension $n - l$. For any $V \in \kappa$ consider the projection $p : M \to N \times V$. Define

$$U_V := \{x \in M | dp_x \text{ is an isomorphism } \}.$$

It is easy to see that $p|_{U_V}$ is ètale and $\{U_V\}_{V\in\kappa}$ gives a finite cover of M. Now this case follows from the previous two ones.

Case 4. General case.
It is enough to prove for affine M and N. Now we can replace M by $\Gamma(\nu)$ and reduce to case 3. □

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