SCHWARTZ FUNCTIONS ON NASH MANIFOLDS AND APPLICATIONS TO REPRESENTATION THEORY

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Let us start with the following motivating example. Consider the circle S^1, let $N \subset S^1$ be the north pole and denote $U := S^1 \setminus N$. Note that U is diffeomorphic to $\mathbb{R}$ via the stereographic projection. Consider the space $\mathcal{D}(S^1)$ of distributions on S^1, that is the space of continuous linear functionals on the Fréchet space $C^\infty(S^1)$. Consider the subspace $\mathcal{D}_{S^1}(N) \subset \mathcal{D}(S^1)$ consisting of all distributions supported at N. Then the quotient $\mathcal{D}(S^1)/\mathcal{D}_{S^1}(N)$ will not be the space of distributions on U. However, it will be the space $\mathcal{S}^*(U)$ of Schwartz distributions on U, that is continuous functionals on the Fréchet space $\mathcal{S}(U)$ of Schwartz functions on U. In this case, $\mathcal{S}(U)$ can be identified with $\mathcal{S}(\mathbb{R})$ via the stereographic projection.

The space of Schwartz functions on $\mathbb{R}$ is defined to be the space of all infinitely differentiable functions that rapidly decay at infinity together with all their derivatives, i.e. $x^n f^{(k)}$ is bounded for any n, k.

In this talk we extend the notions of Schwartz functions and Schwartz distributions to a larger geometric realm.

As we can see, the definition is of algebraic nature. Hence it would not be reasonable to try to extend it to arbitrary smooth manifolds. However, it is reasonable to extend this notion to smooth algebraic varieties. Unfortunately, sometimes this is not enough. For example, a connected component of real algebraic variety is not always an algebraic variety. By this reason we extend this notion to smooth semi-algebraic manifolds. They are called **Nash manifolds**.

For any Nash manifold M, we will define the spaces $\mathcal{G}(M)$, $\mathcal{T}(M)$ and $\mathcal{S}(M)$ of **generalized Schwartz functions**[1], **tempered functions** and **Schwartz functions** on M. Informally, $\mathcal{T}(M)$ is the ring of functions that have no more than polynomial growth together with all their derivatives, $\mathcal{G}(M)$ is the space of generalized functions with no more than polynomial growth and $\mathcal{S}(M)$ is the space of functions that decay together with all their derivatives faster than any inverse power of a polynomial.

As in the classical case, in order to define generalized Schwartz functions, we have to define Schwartz functions first. Both $\mathcal{G}(M)$ and $\mathcal{S}(M)$ are modules over $\mathcal{T}(M)$.

The triple $\mathcal{S}(M)$, $\mathcal{T}(M)$, $\mathcal{G}(M)$ is analogous to $C_c^\infty(M)$, $C^\infty(M)$ and $C^{-\infty}(M)$ but it has additional nice properties as we will see later.

We will show that for $M = \mathbb{R}^n$, $\mathcal{S}(M)$ is the space of classical Schwartz functions and $\mathcal{G}(M)$ is the space of classical generalized Schwartz functions. For compact Nash manifold M, $\mathcal{S}(M) = \mathcal{T}(M) = C^\infty(M)$.

Main results.

Result 1. *Let M be a Nash manifold and $Z \subset M$ be a closed Nash submanifold. Then the restriction maps $\mathcal{T}(M) \to \mathcal{T}(Z)$ and $\mathcal{S}(M) \to \mathcal{S}(Z)$ are onto.*

Result 2. *Let M be a Nash manifold and $U \subset M$ be a semi-algebraic open subset. Then a Schwartz function on U is the same as a Schwartz function on M which vanishes with all its derivatives on $M \setminus U$.*

This theorem tells us that extension by zero $\mathcal{S}(U) \to \mathcal{S}(M)$ is a closed imbedding, and hence restriction morphism $\mathcal{G}(M) \to \mathcal{G}(U)$ is onto.

[1]Here we distinguish between the (similar) notions of a generalized function and a distribution. They can be identified by choosing a measure. Without fixing a measure, a smooth function defines a generalized function but not a distribution.

Classical generalized functions do not have this property. This was our main motivation for extending the definition of Schwartz functions.

Schwartz sections of Nash bundles. Similar notions will be defined for Nash bundles, i.e. smooth semi-algebraic bundles.

For any Nash bundle E over M we will define the spaces $\mathcal{G}(M,E)$, $\mathcal{T}(M,E)$ and $\mathcal{S}(M,E)$ of generalized Schwartz, tempered and Schwartz sections of E.

As in the classical case, a generalized Schwartz function is not exactly a functional on the space of Schwartz functions, but a functional on Schwartz densities, i.e. Schwartz sections of the bundle of densities.

Therefore, we will define generalized Schwartz sections by $\mathcal{G}(M,E) = (\mathcal{S}(M,\widetilde{E}))^*$, where $\widetilde{E} = E^* \otimes D_M$ and D_M is the bundle of densities on M.

Let $Z \subset M$ be a closed Nash submanifold, and $U = M \setminus Z$. Result 2 tells us that the quotient space of $\mathcal{G}(M)$ by the subspace $\mathcal{G}(M)_Z$ of generalized Schwartz functions supported in Z is $\mathcal{G}(U)$. Hence it is useful to study the space $\mathcal{G}(M)_Z$. As in the classical case, $\mathcal{G}(M)_Z$ has a filtration by the degree of transversal derivatives of delta functions. The quotients of the filtration are generalized Schwartz sections over Z of symmetric powers of normal bundle to Z in M, after a twist.

This result can be extended to generalized Schwartz sections of arbitrary Nash bundles.

Restricted topology and sheaf properties. Similarly to algebraic geometry, the reasonable topology on Nash manifolds to consider is a topology in which open sets are open semi-algebraic sets. Unfortunately, it is not a topology in the usual sense of the word, it is only what is called **restricted topology**. This means that the union of an infinite number of open sets does not have to be open. The only open covers considered in the restricted topology are finite open covers.

The restriction of a generalized Schwartz function (respectively of a tempered function) to an open subset is again a generalized Schwartz (respectively a tempered function). This means that they form pre-sheaves. We will show that they are actually sheaves, which means that for any finite open cover $M = \bigcup_{i=1}^{n} U_i$, a function α on M is tempered if and only if $\alpha|_{U_i}$ is tempered for all i. It is of course not true for infinite covers. We denote the sheaf of generalized Schwartz functions by $\mathcal{G}_M$ and of the sheaf of tempered functions by $\mathcal{T}_M$. By result 2, $\mathcal{G}_M$ is a flabby sheaf.

Similarly, for any Nash bundle E over M we will define the sheaf $\mathcal{T}_M^E$ of tempered sections and the sheaf $\mathcal{G}_M^E$ of generalized Schwartz sections.

As we have mentioned before, Schwartz functions behave similarly to compactly supported smooth functions. In particular, they cannot be restricted to an open subset, but can be extended by zero from an open subset. This means that they do not form a sheaf, but an object dual to a sheaf, a so-called cosheaf. We denote the cosheaf of Schwartz functions by $\mathcal{S}_M$. We will prove that $\mathcal{S}_M$ is actually a cosheaf and not just pre-cosheaf by proving a Schwartz version of the partition of unity theorem. Similarly, for any Nash bundle E over M we will define the cosheaf $\mathcal{S}_M^E$ of Schwartz sections.

Possible applications. Schwartz functions are used in the representation theory of algebraic groups. Our definition coincides with Casselman's definition (cf. [Cas1]) for algebraic groups. Our paper allows to use Schwartz functions in more situations in the representation theory of algebraic groups, since an orbit of an algebraic action is a Nash manifold, but does not have to be an algebraic group or even an algebraic variety.

Generalized Schwartz sections can be used for "devisage". We mean the following. Let $U \subset M$ be an open (semi-algebraic) subset. Instead of dealing with generalized Schwartz sections of a bundle on M, we can deal with generalized Schwartz sections of its restriction to U and generalized Schwartz sections of some other bundles on $M \setminus U$.

For example if we are given an action of an algebraic group G on an algebraic variety M, and a G-equivariant bundle E over M, then devisage to orbits helps us to investigate the space of G-invariant generalized sections of E.

Summary. To sum up, for any Nash manifold M we define a sheaf $\mathcal{T}_M$ of algebras on M (in the restricted topology) consisting of tempered functions, a sheaf $\mathcal{G}_M$ of modules over $\mathcal{T}_M$ consisting of generalized Schwartz functions, and a cosheaf $\mathcal{S}_M$ of modules over $\mathcal{T}_M$ consisting of Schwartz functions.

Moreover, for any Nash bundle E over M we define sheaves $\mathcal{T}_M^E$ and $\mathcal{G}_M^E$ of modules over $\mathcal{T}_M$ consisting of tempered and generalized Schwartz sections of E respectively and a cosheaf $\mathcal{S}_M^E$ of modules over $\mathcal{T}_M$ consisting of Schwartz sections of E.

Let us list the main properties of these objects that we will prove in this paper:

1. *Compatibility: For open semi-algebraic subset* $U \subset M$, $\mathcal{S}_M^E|_U = \mathcal{S}_U^{E|_U}$, $\mathcal{T}_M^E|_U = \mathcal{T}_U^{E|_U}$, $\mathcal{G}_M^E|_U = \mathcal{G}_U^{E|_U}$.

2. $\mathcal{S}(\mathbb{R}^n)$ = *Classical Schwartz functions on* $\mathbb{R}^n$.

3. *For compact* M, $\mathcal{S}(M,E) = \mathcal{T}(M,E) = C^\infty(M,E)$.

4. $\mathcal{G}_M^E = (\mathcal{S}_M^{\widetilde{E}})^*$, *where* $\widetilde{E} = E^* \otimes D_M$ *and* D_M *is the bundle of densities on* M.

5. *Let* $Z \subset M$ *be a closed Nash submanifold. Then the restriction maps* $\mathcal{S}(M,E)$ *onto* $\mathcal{S}(Z,E|_Z)$ *and* $\mathcal{T}(M,E)$ *onto* $\mathcal{T}(Z,E|_Z)$.

6. *Let* $U \subset M$ *be a semi-algebraic open subset, then*

$$\mathcal{S}_M^E(U) \cong \{\phi \in \mathcal{S}(M,E)| \quad \phi \textit{ is 0 on } M \setminus U \textit{ with all derivatives}\}.$$

7. *Let* $Z \subset M$ *be a closed Nash submanifold. Consider* $\mathcal{G}(M,E)_Z = \{\xi \in \mathcal{G}(M,E)|\xi$ *is supported in* $Z\}$. *It has a canonical filtration such that its factors are canonically isomorphic to* $\mathcal{G}(Z, E|_Z \otimes S^i(N_Z^M) \otimes D_M^*|_Z \otimes D_Z)$ *where* N_Z^M *is the normal bundle of* Z *in* M *and* S^i *means i-th symmetric power.*

Remarks.

Remark 0.0.1. *Harish-Chandra has defined a Schwartz space for every reductive Lie group. However, Harish-Chandra's Schwartz space does not coincide with the space of Schwartz functions that we define in this paper even for the algebraic group* $\mathbb{R}^\times$.

Remark 0.0.2. *There is a different approach to the concept of Schwartz functions. Namely, if* M *is embedded as an open subset in a compact manifold* K *then one can define the space of Schwartz functions on* M *to be the space of all smooth functions on* K *that vanish outside* M *together with all their derivatives. This approach is implemented in* [CHM], [KS], [Mor] *and* [Pre]. *In general, this definition depends on the embedding into* K. *Our results show that for Nash manifolds* M *and* K *it coincides with our definition and hence does not depend on the embedding.*

Remark 0.0.3. *After the completion of this project we found out that many of the properties of Schwartz functions on affine Nash manifolds have been obtained already in* [dCl].

An application to representation theory. Using the theory of Schwartz functions we showed the following theorem:

Theorem 0.0.4. *Let* F *be either the field of real or complex numbers and consider the standard imbedding* $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. *We consider the two-sided action of* $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ *on* $\mathrm{GL}_{n+1}(F)$ *defined by* $(g_1, g_2)h := g_1 h g_2^{-1}$. *Then any* $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ *invariant distribution on* $\mathrm{GL}_{n+1}(F)$ *is invariant with respect to transposition.*

This theorem has the following corollary in representation theory.

Theorem 0.0.5. . *Let* (π, E) *be an irreducible admissible continuous representation of* $\mathrm{GL}_{n+1}(F)$ *on a Hilbert space* E. *Then*

$$dim Hom_{\mathrm{GL}_n(F)}(E^\infty, \mathbb{C}) \leq 1 \tag{1}$$

Clearly, the last theorem implies in particular that (1) holds for unitary irreducible representations of $\mathrm{GL}_{n+1}(F)$. That is, the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a generalized Gelfand pair in the sense of [vD].

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