INVARIANT DISTRIBUTIONS ON NON-DISTINGUISHED NILPOTENT ORBITS WITH APPLICATION TO THE GELFAND PROPERTY OF $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$

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Abstract. We study invariant distributions on the tangent space to a symmetric space. We prove that an invariant distribution with the property that both its support and the support of its Fourier transform are contained in the set of non-distinguished nilpotent orbits, must vanish. We deduce, using recent developments in the theory of invariant distributions on symmetric spaces that the symmetric pair $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$ is a Gelfand pair. More precisely, we show that for any irreducible smooth admissible Fréchet representation (π, E) of $GL_{2n}(\mathbb{R})$ the ring of continuous functionals $Hom_{Sp_{2n}(\mathbb{R})}(E, \mathbb{C})$ is at most one dimensional. Such a result was previously proven for p-adic fields in [HR] and for $\mathbb{C}$ in [Say1].

Contents

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1. Introduction

Let (V,ω) be a symplectic vector space over $\mathbb{R}$. Consider the standard imbedding $Sp(V) \subset GL(V)$ and the natural action of $Sp(V) \times Sp(V)$ on $GL(V)$. In this paper we prove the following theorem:

Theorem A. *Any $Sp(V) \times Sp(V)$ - invariant distribution on $GL(V)$ is invariant with respect to transposition.*

It has the following corollary in representation theory:

Theorem B. *Let (V,ω) be a symplectic vector space and let E be an irreducible admissible smooth Fréchet representation of $GL(V)$. Then*

$$dim Hom_{Sp(V)}(E,\mathbb{C}) \leq 1$$

In the language of [AGS], Theorem B means that the pair $(GL(V), Sp(V))$ is a Gelfand pair, more precisely satisfies GP1. In particular, Theorem B implies that the spectral decomposition of the unitary representation $L^2(GL(V)/Sp(V))$ is multiplicity free (see e.g. [Lip]).

Theorem B is deduced from Theorem A using the Gelfand-Kazhdan method (adapted to the archimedean case in [AGS]).

The analogue of Theorem A and Theorem B for non-archimedean fields were proven in [HR] using the method of Gelfand and Kazhdan. A simple argument over finite fields is explained in [GG] and using this a simpler proof of the non-archimedean case was written in [OS3]. Recently, one of us, using the ideas of [AG2] extended the result to the case $F = \mathbb{C}$ (see [Say1]).

Our proof of Theorem A is based on the methods of [AG2]. In that work the notion of regular symmetric pair was introduced and shown to be a useful tool in the verification of the Gelfand property . Thus, the main result of the present work is the *regularity* of the symmetric pair $(GL(V), Sp(V))$. In previous works the proof of regularity of symmetric pairs was based either on some simple considerations or on a criterion that requires negativity of certain eigenvalues (this was implicit in [JR], [RR] and was explicated in [AG2], [AG3], [AG4], [Say1]).

The pair $(GL(V), Sp(V))$ does not satisfy the above mentioned criterion and requires new techniques.

1.1. Main ingredients of the proof.

To show regularity we study distributions on the space $\mathfrak{g}^\sigma = \{X \in gl_{2n} : JX = XJ\}$ where $J = \begin{pmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{pmatrix}$. More precisely, we are interested in those distributions that are invariant with respect to the conjugation action of Sp_{2n} and supported on the nilpotent cone. To classify the nilpotent orbits of the action we use the method of [GG] to identify these orbits with nilpotent orbits of the adjoint action of GL_n on its Lie algebra. This allows us to show that there exists a unique *distinguished* nilpotent orbit $\mathcal{O}$ and that this orbit is open in the nilpotent cone. Next, we use the theory of D-modules, as in [AG5], to prove that there are no distributions supported on non-distinguished orbits whose Fourier transform is also supported on non-distinguished orbits (see Theorem 3.0.11).

1.2. **Related works.**
The problem of identifying symmetric pairs that are Gelfand pairs was studied by various authors. In the case of symmetric spaces of rank one this problem was studied extensively in [RR], [vD], [BvD] both in the archimedean and non-archimedean case. Recently, cases of symmetric spaces of high rank were studied in [AGS], [AG2], [AG3], [AG4], [Say2]. However, as hinted above, all these works could treat a restricted class of symmetric pairs, first introduced in [Sek] that are now commonly called *nice* symmetric pairs.

The pair $(GL(V), Sp(V))$ is *not* a nice symmetric pair and additional methods are needed to study invariant distributions on the corresponding symmetric space. For that, we use the theory of D-modules as in [AG5] and analysis of the nilpotent cone of the pair in question, in order to prove the Gelfand property.

In the non-archimedean case, the pair (GL_{2n}, Sp_{2n}) is a part of a list (GL_{2n}, H_k, ψ_k), $k = 0, 1, ..., n$, of twisted Gelfand pairs that provide a model in the sense of [BGG] to the unitary representations of GL_{2n}. Namely, every irreducible unitarizable representation of GL_{2n} appears exactly once in $\bigoplus_{k=0}^{n} Ind_{H_k}^{GL_{2n}}(\psi_k)$ (see [OS1],[OS2],[OS3]). Considering the strategy taken in those works, a major first step in transferring these results to the archimedean case is taken in the present paper.

1.3. **Structure of the paper.**
In section 2 we give some preliminaries on distributions, symmetric pairs and Gelfand pairs. We introduce the notion of regular symmetric pairs and show that Theorem 7.4.5 of [AG2] and the results of [Say1] allow us to reduce the Gelfand property of the pair in question to proving that the pair is regular. In section 3 we prove the main technical result on distributions, Theorem 3.0.11. It states that under certain conditions there are no distributions supported on non-distinguished nilpotent orbits. The proof is based on the theory of D-modules. In section 4 we use Theorem 3.0.11 to prove that the pair $(GL(V), Sp(V))$ is regular.

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2. Preliminaries

2.1. **Notations on invariant distributions.**

2.1.1. *Schwartz distributions on Nash manifolds.*
We will use the theory of Schwartz functions and distributions as developed in [AG1]. This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic.*

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

Notation 2.1.1. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.*

Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the space of Schwartz distributions on X.

For any Nash vector bundle E over X we denote by $\mathcal{S}(X,E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X,E)$ its dual space.

Notation 2.1.2. *Let X be a smooth manifold and let $Z \subset X$ be a closed subset. We denote $\mathcal{S}^*_X(Z) := \{\xi \in \mathcal{S}^*(X) | \mathrm{Supp}(\xi) \subset Z\}$.*

For a locally closed subset $Y \subset X$ we denote $\mathcal{S}^*_X(Y) := \mathcal{S}^*_{X \setminus (\overline{Y} \setminus Y)}(Y)$. In the same way, for any bundle E on X we define $\mathcal{S}^*_X(Y,E)$.

Remark 2.1.3. *Schwartz distributions have the following two advantages over general distributions:*
(i) For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence

$$0 \to \mathcal{S}^*_X(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

(ii) Fourier transform defines an isomorphism $\mathcal{F} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.

2.1.2. *Basic tools.*
We present here some basic tools on equivariant distributions that we will use in this paper.

Proposition 2.1.4. *Let a Nash group G act on a Nash manifold X. Let $Z \subset X$ be a closed subset.*

Let $Z = \bigcup_{i=0}^{l} Z_i$ be a Nash G-invariant stratification of Z. Let χ be a character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i, Sym^k(CN^X_{Z_i}))^{G,\chi} = 0$. Then $\mathcal{S}^*_X(Z)^{G,\chi} = 0$.

This proposition immediately follows from Corollary 7.2.6 in [AGS].

Theorem 2.1.5 (Frobenius reciprocity). *Let a Nash group G act transitively on a Nash manifold Z. Let $\varphi : X \to Z$ be a G-equivariant Nash map. Let $z \in Z$. Let G_z be its stabilizer. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z, \chi \cdot \Delta_G|_{G_z} \cdot \Delta_{G_z}^{-1}}$ where Δ denotes the modular character.*

For proof see [AG2], Theorem 2.3.8.

2.1.3. *Fourier transform.*
From now till the end of the paper we fix an additive character κ of $\mathbb{R}$ given by $\kappa(x) := e^{2\pi i x}$.

Notation 2.1.6. *Let V be a vector space over $\mathbb{R}$. Let B be a non-degenerate bilinear form on V. Then B defines Fourier transform with respect to the self-dual Haar measure on V. We denote it by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$.*

For any Nash manifold M we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the partial Fourier transform.

If there is no ambiguity, we will write $\mathcal{F}_V$, and sometimes just $\mathcal{F}$, instead of $\mathcal{F}_B$.

We will use the following trivial observation.

Lemma 2.1.7. *Let V be a finite dimensional vector space over $\mathbb{R}$. Let a Nash group G act linearly on V. Let B be a G-invariant non-degenerate symmetric bilinear form on V. Let M be a Nash manifold with an action of G. Let $\xi \in \mathcal{S}^*(V \times M)$ be a G-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also G-invariant.*

2.2. **Gelfand pairs and invariant distributions.**
In this section we recall a technique due to Gelfand and Kazhdan (see [GK]) which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS], section 2.

Definition 2.2.1. *Let G be a reductive group. By an* ***admissible representation of*** *G we mean an admissible smooth Fréchet representation of $G(\mathbb{R})$.*

We now introduce three notions of Gelfand pair.

Definition 2.2.2. *Let $H \subset G$ be a pair of reductive groups.*

- *We say that (G, H) satisfy* **GP1** *if for any irreducible admissible smooth Fréchet representation (π, E) of G we have*
$$\dim Hom_{H(\mathbb{R})}(E, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy* **GP2** *if for any irreducible admissible smooth Fréchet representation (π, E) of G we have*
$$\dim Hom_{H(\mathbb{R})}(E, \mathbb{C}) \cdot \dim Hom_{H(\mathbb{R})}(\widetilde{E}, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy* **GP3** *if for any irreducible* **unitary** *representation $(\pi, \mathcal{H})$ of $G(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ we have*
$$\dim Hom_{H(\mathbb{R})}(\mathcal{H}^\infty, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g.[vD, BvD]).

We have the following straightforward proposition.

Proposition 2.2.3. *$GP1 \Rightarrow GP2 \Rightarrow GP3$.*

We will use the following theorem from [AGS] which is a version of a classical theorem of Gelfand and Kazhdan.

Theorem 2.2.4. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(\mathbb{R})$-invariant distributions ξ on $G(\mathbb{R})$. Then (G, H) satisfies GP2.*

In our case GP2 is equivalent to GP1 by the following proposition.

Proposition 2.2.5. *Suppose $H \subset \mathrm{GL}_n$ is transpose invariant subgroup. Then $GP1$ is equivalent to $GP2$ for the pair (GL_n, H).*

For proof see [AGS], proposition 2.4.1.

Corollary 2.2.6. *Theorem A implies Theorem B.*

2.3. **Symmetric pairs.**
In this subsection we review some tools developed in [AG2] that enable to prove that, granting certain hypothesis, a symmetric pair is a Gelfand pair.

Definition 2.3.1. *A* ***symmetric pair*** *is a triple* (G, H, θ) *where* $H \subset G$ *are reductive groups, and* θ *is an involution of* G *such that* $H = G^{\theta}$*. In cases when there is no ambiguity we will omit* θ

For a symmetric pair (G, H, θ) *we define an anti-involution* $\sigma : G \to G$ *by* $\sigma(g) := \theta(g^{-1})$*, denote* $\mathfrak{g} := LieG$*,* $\mathfrak{h} := LieH$*,* $\mathfrak{g}^{\sigma} := \{a \in \mathfrak{g} | \theta(a) = -a\}$*. Note that* H *acts on* $\mathfrak{g}^{\sigma}$ *by the adjoint action. Denote also* $G^{\sigma} := \{g \in G | \sigma(g) = g\}$ *and define a* ***symmetrization map*** $s : G(\mathbb{R}) \to G^{\sigma}(\mathbb{R})$ *by* $s(g) := g\sigma(g)$*.*

The following lemma is standard:

Lemma 2.3.2. *The symmetrization map* $s : G \to G^{\sigma}$ *is submersive and hence open.*

Definition 2.3.3. *Let* (G_1, H_1, θ_1) *and* (G_2, H_2, θ_2) *be symmetric pairs. We define their* ***product*** *to be the symmetric pair* $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$*.*

Definition 2.3.4. *We call a symmetric pair* (G, H, θ) ***good*** *if for any closed* $H(\mathbb{R}) \times H(\mathbb{R})$ *orbit* $O \subset G(\mathbb{R})$*, we have* $\sigma(O) = O$*.*

Definition 2.3.5. *We say that a symmetric pair* (G, H, θ) *is a* ***GK pair*** *if any* $H(\mathbb{R}) \times H(\mathbb{R})$ *- invariant distribution on* $G(\mathbb{R})$ *is* σ *- invariant.*

Definition 2.3.6. *We define an involution* $\theta : GL_{2n} \to GL_{2n}$ *by* $\theta(x) = Jx^tJ^{-1}$ *where* $J = \begin{pmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{pmatrix}$*. Note that* $(GL_{2n}, Sp_{2n}, \theta)$ *is a symmetric pair.*

Theorem A can be rephrased in the following way:

Theorem A'. *The pair* (GL_{2n}, Sp_{2n}) *defined over* $\mathbb{R}$ *is a GK pair.*

2.3.1. *Descendants of symmetric pairs.*

Proposition 2.3.7. *Let* (G, H, θ) *be a symmetric pair. Let* $g \in G(\mathbb{R})$ *such that* HgH *is closed. Let* $x = s(g)$*. Then* x *is semisimple.*

For proof see e.g. [AG2], Proposition 7.2.1.

Definition 2.3.8. *In the notations of the previous proposition we will say that the pair* $(G_x, H_x, \theta|_{G_x})$ *is a descendant of* (G, H, θ)*. Here* G_x *(and similarly for* H*) denotes the stabilizer of* x *in* G*.*

2.3.2. *Regular symmetric pairs.*

Notation 2.3.9. *Let* V *be an algebraic finite dimensional representation over* $\mathbb{R}$ *of a reductive group* G*. Denote* $Q(V) := V/V^G$*. Since* G *is reductive, there is a canonical embedding* $Q(V) \hookrightarrow V$*.*

Notation 2.3.10. *Let* (G, H, θ) *be a symmetric pair. We denote by* $\mathcal{N}_{G,H}$ *the subset of all the nilpotent elements in* $Q(\mathfrak{g}^{\sigma})$*. Denote* $R_{G,H} := Q(\mathfrak{g}^{\sigma}) - \mathcal{N}_{G,H}$*.*

Our notion of $R_{G,H}$ coincides with the notion $R(\mathfrak{g}^{\sigma})$ used in [AG2], Notation 2.1.10. This follows from Lemma 7.1.11 in [AG2].

Definition 2.3.11. *Let π be an action of a real reductive group G on a smooth affine variety X. We say that an algebraic automorphism τ of X is G-**admissible** if*
(i) $\pi(G(\mathbb{R}))$ is of index at most 2 in the group of automorphisms of X generated by $\pi(G(\mathbb{R}))$ and τ.
(ii) For any closed $G(\mathbb{R})$ orbit $O \subset X(\mathbb{R})$, we have $\tau(O) = O$.

Definition 2.3.12. *Let (G, H, θ) be a symmetric pair. We call an element $g \in G(\mathbb{R})$ **admissible** if*
(i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
(ii) $Ad(g)|_{\mathfrak{g}^\sigma}$ is H-admissible.

Definition 2.3.13. *We call a symmetric pair (G, H, θ) **regular** if for any admissible $g \in G(\mathbb{R})$ such that every $H(\mathbb{R})$-invariant distribution on $R_{G,H}$ is also $Ad(g)$-invariant, we have*
(*) every $H(\mathbb{R})$-invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also $Ad(g)$-invariant.

Clearly, the product of regular pairs is regular (see [AG2], Proposition 7.4.4).
We will deduce Theorem A' (and hence Theorem A) from the following Theorem:

Theorem C. *The pair (GL_{2n}, Sp_{2n}) defined over $\mathbb{R}$ is regular.*

The deduction is based on the following theorem (see [AG2], Theorem 7.4.5.):

Theorem 2.3.14. *Let (G, H, θ) be a good symmetric pair such that all its descendants (including itself) are regular. Then it is a GK pair.*

Corollary 2.3.15. *Theorem C implies Theorem A.*

Proof. The pair (GL_{2n}, Sp_{2n}) is good by Corollary 3.1.3 of [Say1]. In [Say1] it is shown that all the descendance of the pair (GL_{2n}, Sp_{2n}) are products of pairs of the form (GL_{2m}, Sp_{2m}) and $((GL_{2m})_{\mathbb{C}/\mathbb{R}}, (Sp_{2m})_{\mathbb{C}/\mathbb{R}})$, here $G_{\mathbb{C}/\mathbb{R}}$ denotes the restriction of scalars (in particular $G_{\mathbb{C}/\mathbb{R}}(\mathbb{R}) = G(\mathbb{C})$). By Corollary 3.3.1. of [Say1] the pair $((GL_{2m})_{\mathbb{C}/\mathbb{R}}, (Sp_{2m})_{\mathbb{C}/\mathbb{R}})$ is regular. Now clearly Theorem C implies Theorem A' and hence Theorem A. ☐

We will also need the following Proposition, whose proof we include for completeness.

Proposition 2.3.16. *Let $\pi : \mathfrak{g}^\sigma \to Spec(\mathcal{O}(\mathfrak{g}^\sigma))^H$ be the projection, where $\mathcal{O}(\mathfrak{g}^\sigma)$ denote the space of regular functions on the algebraic variety $\mathfrak{g}^\sigma$.*
Let $x \in \mathcal{N}_{G,H}$ be a smooth point. Then π submersive at x.

Proof. Let $\mathcal{J} = \{f \in \mathcal{O}(\mathfrak{g}^\sigma)^H : f(0) = 0\}$. By Theorem 14 of [KR], $\mathcal{J}$ is a radical ideal. Using the Nullstellensatz, this implies that $Ker(d_x\pi) = T_x(\mathcal{N}_{G,H})$. This proves that π is submersive. ☐

2.4. Singular support of distributions.

In this subsection we introduce the notion *Singular Support* of a distribution ξ and list some of its properties. In the literature this notion is sometimes also called *Characteristic Variety*. For more details see [AG5].

Notation 2.4.1. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Let M_ξ be the D_X submodule of $\mathcal{S}^*(X(\mathbb{R}))$ generated by ξ. We denote by $SS(\xi) \subset T^*X$ the singular support of M_ξ (for the definition see* [Bor]*). We will call it the* **singular support** *of ξ.*

Remark 2.4.2.
(i) A similar, but not equivalent notion is sometimes called in the literature a 'wave front of ξ'.
(ii) In some of the literature, singular support of a distribution is a subset of X not to be confused with our $SS(\xi)$ which is a subset of T^*X. We use terminology from the theory of D-modules where the set $SS(\xi)$ is called both the characteristic variety and the singular support of the D-module M_ξ.

Notation 2.4.3. *Let (V, B) be a quadratic space. Let X be a smooth algebraic variety. Consider B as a map $B : V \to V^*$. Identify $T^*(X \times V)$ with $T^*X \times V \times V^*$. We define $F_V : T^*(X \times V) \to T^*(X \times V)$ by $F_V(\alpha, v, \phi) := (\alpha, -B^{-1}\phi, Bv)$.*

Definition 2.4.4. *Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M-**co-isotropic** if one of the following equivalent conditions holds.*

(1) *The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.*
(2) *At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$. Here, $(T_zZ)^\perp$ denotes the orthogonal complement to (T_zZ) in (T_zM) with respect to ω.*
(3) *For a generic smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$.*

If there is no ambiguity, we will call Z a co-isotropic variety.

Note that every non-empty M-co-isotropic variety is of dimension at least $\frac{1}{2}dimM$.

Notation 2.4.5. *For a smooth algebraic variety X we always consider the standard symplectic form on T^*X. Also, we denote by $p_X : T^*X \to X$ the standard projection.*

Let X be a smooth algebraic variety. Below is a list of properties of the Singular support. Proofs can be found in [AG5] section 2.3 and Appendix B.

Property 2.4.6. *Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))(\mathbb{R})$, where $\overline{\mathrm{Supp}(\xi)}_{Zar}$ denotes the Zariski closure of $\mathrm{Supp}(\xi)$.*

Property 2.4.7.
Let an algebraic group G act on X. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))^{G(\mathbb{R})}$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g}\, \phi(\alpha(x)) = 0\}.$$

Property 2.4.8. *Let (V, B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X\times V}^{-1}(Z))$.*

Finally, the following is a corollary of the integrability theorem ([KKS], [Mal], [Gab]):

Property 2.4.9. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $SS(\xi)$ is co-isotropic.*

3. Invariant distributions supported on non-distinguished nilpotent orbits in symmetric pairs

For this section we fix a symmetric pair (G, H, θ).

Definition 3.0.10. *We say that a nilpotent element $x \in \mathfrak{g}^\sigma$ is distinguished if*

$$\mathfrak{g}_x \cap Q(\mathfrak{g}^\sigma) \subset \mathcal{N}_{G,H}$$

Theorem 3.0.11. *Let $A \subset \mathcal{N}_{G,H}$ be an H invariant closed subset and assume that all elements of A are non-distinguished. Let $W = \mathcal{S}^*_{\mathfrak{g}^\sigma}(A)^H$. Then $W \cap \mathcal{F}(W) = 0$.*

Remark 3.0.12. *We believe that the methods of* [SZ] *allow to show the same result without the assumption of H-invariance.*

The proof is based on the following proposition:

Proposition 3.0.13. *Let $A \subset \mathcal{N}_{G,H}$ be an H invariant closed subset and assume that all elements of A are non-distinguished. Denote by*

$$B = \{(\alpha, \beta) \in A \times A : [\alpha, \beta] = 0\} \subset Q(\mathfrak{g}^\sigma) \times Q(\mathfrak{g}^\sigma).$$

Identify $T^*(Q(\mathfrak{g}^\sigma))$ with $Q(\mathfrak{g}^\sigma) \times Q(\mathfrak{g}^\sigma)$. Then there is no non-empty $T^*(Q(\mathfrak{g}^\sigma))$-co-isotropic subvariety of B.

Proof. Stratify A by its orbits $\mathcal{O}_1, ..., \mathcal{O}_r$. Let $p : A \times A \to A$ be the projection onto the first factor. By inductive argument it is enough to show that, for any orbit $\mathcal{O}$, $p^{-1}(\mathcal{O}) \cap B$ does not include a non empty co-isotropic subvariety. Consider the set

$$C_{\mathcal{O}} = \{(a, b) : a \in \mathcal{O}, b \in Q(\mathfrak{g}^\sigma), [a, b] = 0\}.$$

Then $dim(C_{\mathcal{O}}) = dim(Q(\mathfrak{g}^\sigma))$. Since $\mathcal{O}$ is not distinguished, $p^{-1}(\mathcal{O}) \cap B$ is a closed subvariety of $C_{\mathcal{O}}$ which does not include any of the irreducible components of $C_{\mathcal{O}}$. This finishes the proof. ☐

Proof of Theorem 3.0.11. Let $\xi \in W \cap \mathcal{F}(W)$ and let B be as in proposition 3.0.13. By properties 2.4.6, 2.4.7, 2.4.8 we conclude that $SS(\xi) \subset B$. But by Property 2.4.9 it is co-isotropic and hence by Proposition 3.0.13 it is empty. Thus $\xi = 0$. ☐

4. Regularity

In this section we prove the main result of the paper:

Theorem C. *The pair (GL_{2n}, Sp_{2n}) defined over $\mathbb{R}$ is regular.*

For the rest of this section we let (G, H) to be the symmetric pair $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$.

4.1. H orbits on $\mathfrak{g}^\sigma$.

Proposition 4.1.1. *There exists a unique distinguished H-orbit in $\mathcal{N}_{G,H}(\mathbb{R})$. This orbit is open in $\mathcal{N}_{G,H}(\mathbb{R})$ and invariant with respect to any admissible $g \in G$.*

For the proof we will use the following Proposition (this is Proposition 2.1 of [GG]):

Proposition 4.1.2. *Let F be an arbitrary field. For $x \in GL_n(F)$ define*

$$\gamma(x) = \begin{pmatrix} x & 0 \\ 0 & I_n \end{pmatrix}$$

Then γ induces a bijection between the set of conjugacy classes in $GL_n(F)$ and the set of orbits of $Sp_{2n}(F) \times Sp_{2n}(F)$ in $GL_{2n}(F)$.

Corollary 4.1.3. *Let $d : \mathrm{gl}_n \to \mathfrak{g}^\sigma$ be defined by*

$$d(X) = \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix}.$$

Then d induces a bijection between nilpotent conjugacy classes in gl_n and H orbits in $\mathcal{N}_{G,H}$.

Proof. Let $s : GL_{2n} \to GL_{2n}^\sigma$ be given by $s(g) = g\sigma(g)$. Let $W = s(GL_{2n}(\mathbb{R}))$. By Proposition 4.1.2, the map $s \circ \gamma$ induces a bijection between conjugacy classes in $GL_n(\mathbb{R})$ and H orbits on W.

Let $e : \mathcal{N} \to GL_n$ be given by $e(X) = 1 + X$ where $\mathcal{N}$ is the cone of nilpotent elements in gl_n. Let $\ell : W \to \mathfrak{g}^\sigma$ given by $\ell(w) = w - 1$.

Then, it is easy to see that the map $d|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}_{G,H}$ coincides with the composition $\ell \circ s \circ \gamma \circ e$.

To finish the proof of the Proposition it is enough to show that $\ell(W)$ contains all nilpotent elements. Indeed, by lemma 2.3.2 the set $W = s(GL_{2n}(\mathbb{R}))$ is open and thus $\ell(W)$ is open and hence contains all nilpotent elements. $\square$

We are now ready to prove the proposition.

Proof of Proposition 4.1.1. It is easy to see that if X is non regular nilpotent then $d(X)$ is not distinguished. Also, a simple verification shows that if $X = J_n$ is a standard Jordan block then $d(J_n)$ is distinguished. Thus we only need to show that $C = Ad(H)d(J_n)$ is open in $\mathcal{N}_{G,H}$. For this we will show that C is dense in $\mathcal{N}_{G,H}$. Indeed, $\bar{C} \supset d(\overline{Ad(GL_n)J_n}) = d(\mathcal{N})$, where $\mathcal{N}$ is the set of nilpotent elements in gl_n. But C is $Ad(H)$-invariant and this implies that $\bar{C} = \mathcal{N}_{G,H}$ $\square$

4.2. **Proof of Theorem C.**
Theorem C follows from Theorem 3.0.11 and the next Proposition:

Proposition 4.2.1. *Let $g \in G$ be an admissible element. Let A be the union of all non-distinguished elements. Note that A is closed. Let ξ be any H-invariant distribution on $\mathfrak{g}^\sigma$ which is anti-invariant with respect to $Ad(g)$. Then $Supp(\xi) \subset A$.*

Proof. Let $O_0 \subset \mathcal{N}_{G,H}$ be the distinguished orbit. Let $\widetilde{H} = \langle Ad(H), Ad(g)\rangle$ be the group of automorphisms of $\mathfrak{g}^\sigma$ generated by the adjoint action of H and g. Let χ be the character of $\widetilde{H}$ defined by $\chi(\widetilde{H} - H) = -1$. We need to show

$$\mathcal{S}^*_{Q(g^\sigma)}(O_0)^{\widetilde{H},\chi} = 0$$

By Proposition 2.1.4 it is enough to show

$$\mathcal{S}^*(O_0, Sym^k(CN_{O_0}^{Q(\mathfrak{g}^\sigma)}))^{\widetilde{H},\chi} = 0$$

Notice that $\widetilde{H}$ acts trivially on $Spec(O(\mathfrak{g}^\sigma))^H$. Hence, by Proposition 2.3.16 the bundle $N_{O_0}^{Q(\mathfrak{g}^\sigma)}$ is trivial as a $\widetilde{H}$ bundle. This completes the proof. $\square$

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