Appendix D. Distinguished representations in the Archimedean case, by Avraham Aizenbud and Erez Lapid

In this appendix we consider representations of $G = \mathrm{GL}(n, \mathbb{C})$ and a unitary group $G^x = U(p, q) \subset G$ defined with respect to a Hermitian form x with signature (p, q). Recall that we denote complex conjugation by τ, the diagonal torus of G by M_0 and the upper-triangular Borel subgroup by P_0. For a character χ of M_0 we denote by $I(\chi)$ the representation induced from the character χ on P_0.

Let W_2 be the set of involutions in W. Any $w \in W_2$ can be written as a product of g_w disjoint transpositions where the number of fixed points of w is $f_w = n - 2g_w$. Set $\mathfrak{m}(w) = \binom{f_w}{q-g_w} = \binom{f_w}{p-g_w}$ $(= 0$ if $g_w > \mathfrak{w}(x) = \min(p, q))$.

In this appendix we will prove the following result.

Theorem D.1. *Suppose that π is the Langlands quotient of $I(\chi)$ where $\chi = (\chi_1, ..., \chi_n)$ is a character of M_0 such that $|\chi(t)| = |t_1|^{\lambda_1} \cdots |t_n|^{\lambda_n}$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$\dim \mathrm{Hom}_{G^x}(\pi, \mathbb{C}) \leq \dim \mathrm{Hom}_{G^x}(I(\chi), \mathbb{C}) \leq \sum_{w \in W_2 : w\chi = \chi^\tau} \mathfrak{m}(w). \tag{1}$$

In particular, if π is G^x-distinguished then there exists $w \in W_2$ with $g_w \leq \mathfrak{w}(x)$ such that $w\chi = \chi^\tau$. Hence π is τ-invariant and $\mathfrak{w}(\pi) \leq \mathfrak{w}(x)$.

For $w \in W_2$ set $I_w = \{(i, j) : i > j, w(i) < w(j)\}$ and define for any $\kappa : I_w \to \mathbb{Z}_{\geq 0}$ a character of M_0 by

$$\alpha_\kappa(\mathrm{diag}(t_1, ..., t_n)) = \prod_{(i,j) \in I_w} \left((t_i/t_j)^{\kappa(i,j)} \right).$$

Let

$$S_w(\chi) = \{\kappa : I_w \to \mathbb{Z}_{\geq 0} \mid \chi^\tau w(\chi)^{-1} = \alpha_\kappa^\tau w(\alpha_\kappa)^{-1}\}.$$

Note that if χ satisfies the assumption of Theorem D.1 then

$$S_w(\chi) = \begin{cases} \{\kappa \equiv 0\} & \text{if } w\chi = \chi^\tau, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, Theorem D.1 would follow from the following Proposition which will be proved at the end of the appendix.

Proposition D.2. *Let χ be a character of M_0. Then*

$$\dim \mathrm{Hom}_{G^x}(I(\chi), \mathbb{C}) \leq \sum_{w \in W_2} \mathfrak{m}(w) |S_w(\chi)|.$$

We will prove the Proposition by representing the G^x-invariant linear forms on $I(\chi)$ as equivariant distributions on the Schwartz space of G/G^x and using the analysis of equivariant distributions developed in [AG1].

Henceforth, we will use the following notational conventions. For now, G is an arbitrary group.

- For any G-set X and a point $x \in X$ we denote by $G(x)$ the G-orbit of x and by G^x the stabilizer of x.
- For any representation of G on a vector space V we denote by V^G the subspace of G-invariant vectors in V. For a character χ of G we denote by $V^{G,\chi}$ the subspace of (G, χ)-equivariant vectors in V.
- Given manifolds $L \subset M$ we denote by $N_L^M := (T_M\big|_L)/T_L$ the normal bundle to L in M and by $CN_L^M := (N_L^M)^*$ the conormal bundle. For any point $y \in L$ we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space.
- The symmetric algebra of a vector space V will be denoted by $\mathrm{Sym}(V) = \oplus_{k \geq 0} \mathrm{Sym}^k(V)$.

We will use the theory of Schwartz functions and distributions on Nash manifolds as developed in [AG1] generalizing the usual notions for $\mathbb{R}^n$.[1]

[1] In the present context we will only apply it to smooth real algebraic manifolds.

We denote the Fréchet space of Schwartz functions on a Nash manifold X by $\mathcal{S}(X)$ and the dual space of Schwartz distributions by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$. For a closed subset Z of a smooth manifold X we set $\mathcal{S}^*_X(Z) := \{\xi \in \mathcal{S}^*(X) : \mathrm{Supp}(\xi) \subset Z\}$. More generally, for a locally closed subset $Y \subset X$ we set $\mathcal{S}^*_X(Y) := \mathcal{S}^*_{X \setminus (\overline{Y} \setminus Y)}(Y)$.

If U is an open Nash submanifold of X then we have the following exact sequence

$$0 \to \mathcal{S}^*_X(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

For any Nash vector bundle E over X we denote by $\mathcal{S}(X, E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X, E)$ its dual space.

We denote by D_X the bundle of densities over X ([AG1, A.1.1]) and by $\mathcal{G}(X) := \mathcal{S}^*(X, D_X)$ the space of generalized functions on X. More generally we set $\mathcal{G}(X, E) := \mathcal{S}^*(X, E^* \otimes D_X)$ for any Nash vector bundle E over X. Note that $\mathcal{S}(X, E)$ is naturally imbedded into $\mathcal{G}(X, E)$ but not into $\mathcal{S}^*(X, E)$. For any locally closed subset Y of X the spaces $\mathcal{S}^*_X(Y, E), \mathcal{G}_X(Y, E)$ and $\mathcal{G}_X(Y)$ are similarly defined.

Suppose that a group G acts on a Nash manifold X. Then G naturally acts on $\mathcal{S}(X)$ and $\mathcal{S}^*(X)$ and T_X has a natural G-equivariant structure. Therefore all the standard bundles constructed from T_X, such as D_X, also have G-equivariant structure. This gives rise to an action of G on $\mathcal{S}(X, D_X)$ and the dual action on $\mathcal{G}(X)$. Note that the G-action on $\mathcal{G}(X)$ extends the action on $\mathcal{S}(X)$ and similarly the action on $\mathcal{S}^*(X)$ extends the action on $\mathcal{S}(X, D_X)$.

We will use some standard facts about equivariant distributions.

Proposition D.3. *Let a Nash group G act on a Nash manifold X. Let $Z \subset X$ be a closed G-invariant subset with a G-invariant stratification $Z = \bigcup_{i=0}^{l} Z_i$. Let χ be a character of G. Then*

$$\dim(\mathcal{S}^*_X(Z)^{G,\chi}) \leq \sum_{i=0}^{l} \sum_{k=0}^{\infty} \dim(\mathcal{S}^*(Z_i, \mathrm{Sym}^k(CN^X_{Z_i}))^{G,\chi}).$$

The proof is the same as in [AGS, corollary B.2.4].

Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a bundle on N. We denote by $\phi^* : \mathcal{G}(N, E) \to \mathcal{G}(M, \phi^*(E))$ the pull back of generalized functions ([AG3, Notation B.2.5]).

Proposition D.4. *Let M be a Nash manifold. Let K be a Nash group. Let $E \to M$ be a Nash bundle. Consider the standard projection $p : K \times M \to M$. Then the map $p^* : \mathcal{G}(M, E) \to \mathcal{G}(M \times K, p^*E)^K$ is an isomorphism.*

For a proof see [AG3, Proposition B.3.1].

Corollary D.5. *Let G be real algebraic group and $H \subset G$ be its closed subgroup. Then $\mathcal{G}(G)^H \cong \mathcal{G}(G/H)$.*

Proof. By [AG2, Proposition 4.0.6] the map $G \to G/H$ is a Nash locally trivial fibration ([AG2, Definition 2.4.1]). The assertion follows from Proposition D.4 by a partition of unity argument (cf. [AG1, Theorem 5.2.1]). □

The following version of Frobenius reciprocity is a slight generalization of [AG3, Theorem 2.5.7]. For the convenience of the reader we sketch a proof.

Theorem D.6 (Frobenius reciprocity)**.** *Let a Nash group G act transitively on a Nash manifold Z. Let $\varphi : X \to Z$ be a G-equivariant Nash map. Let $z \in Z$ and let X_z be the fiber of z. Let χ be a tempered character of G (*[AG1, Definition 5.1.1]*). Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z, \chi \delta_H^{-1} \delta_G}$.*

Moreover, for any G-equivariant bundle E on X, the space $\mathcal{S}^*(X, E)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z, E|_{X_z})^{G_z, \chi \delta_H^{-1} \delta_G}$. Here δ_G and δ_H are the modulus characters of the groups G and H.

Proof. As in [AG3, Theorem 2.5.7] we will prove an equivalent statement for generalized functions. Namely we will construct canonical isomorphisms $HC : \mathcal{G}(X, E)^{G,\chi} \to \mathcal{G}(X_z, E|_{X_z})^{G_z,\chi}$ and $Fr : \mathcal{G}(X_z, E|_{X_z})^{G_z,\chi} \to \mathcal{G}(X, E)^{G,\chi}$. Consider the natural submersion $a : G \times X_z \to X$ and the projection $p : G \times X_z \to X_z$. Note that the equivariant structure of E gives us an identification $\phi : a^*(E) \to p^*(E|_{X_z})$. consider the tempered function f on $G \times X_z$ given by $f(g, x) = \chi^{-1}(g)$. Consider the map $a^{*,\chi}: HC: \mathcal{G}(X,E)^{G,\chi} \to \mathcal{G}(G\times X_z, p^*(E\big|_{X_z}))^G$ given by $a^{*,\chi}(\xi) = f\phi(a^*(\xi))$. Here the action of G on $G\times X_z$ is on the first coordinate. By Proposition D.4 $\mathcal{G}(G\times X_z, p^*(E\big|_{X_z}))^G \cong \mathcal{G}(X_z, E\big|_{X_z})$. This gives us the map HC. A similar modification to the construction of Fr in [AG3, Theorem 2.5.7] gives rise to Fr in our context. $\square$

Proof of proposition D.2. Let $G = \mathrm{GL}_n(\mathbb{C})$ and $H = U(p,q)$. Note that after identifying D_G and $D_{G/H}$ with the trivial bundle (in a G-equivariant way) we have

$$I(\chi)^* = \mathcal{G}(G)^{P_0,\chi\delta_0^{-1/2}} = \mathcal{S}^*(G)^{P_0,\chi\delta_0^{-\frac{1}{2}}}$$

where P_0 acts on generalized functions on the left. Therefore

$$\mathrm{Hom}_H(I(\chi),\mathbb{C}) = \mathcal{G}(G/H)^{P_0,\chi\delta_0^{-1/2}} = \mathcal{S}^*(G/H)^{P_0,\chi\delta_0^{-\frac{1}{2}}}.$$

We can stratify G/H by P_0-orbits. By [FLO, Remark 2] any such orbit contains a unique element x of the form $x = wa$ where $w \in W_2$ and $a \in M_0$ is such that $a_i = 1$ if $w(i) \neq i$ and $a_i = \pm 1$ otherwise. The number of P_0-orbits on G/H above a given $w \in W_2$ is precisely $\mathfrak{m}(w)$ and moreover,

$$M_0^x = M_0^w = \{t \in M_0 : twt^\tau w = 1\} = \{tw(t^{-1})^\tau w : t \in M_0\}. \tag{2}$$

Using Proposition D.3 it suffices to show that for any w and a as above we have

$$\sum_{k=0}^{\infty} \dim(\mathcal{S}^*(P_0(x), \mathrm{Sym}^k(CN^X_{P_0(x)}))^{P_0,\chi\delta_0^{-1/2}}) \leq |S_w(\chi)|.$$

By Theorem D.6 and the relation $\delta_0^{1/2}\big|_{P_0^x} = \delta_{P_0^x}$ ([LR03, Proposition 4.3.2]) we get

$$\begin{aligned}\mathcal{S}^*(P_0(x), \mathrm{Sym}^k(CN^X_{P_0(x)}))^{P_0,\chi\delta_0^{-1/2}} &= \mathcal{S}^*(\{x\}, \mathrm{Sym}^k(CN^X_{P_0(x),x}))^{P_0,\chi\delta_0^{-1/2}\delta_{P_0^x}^{-1}\delta_0} \\ &= \mathcal{S}^*(\{x\}, \mathrm{Sym}^k(CN^X_{P_0(x),x}))^{P_0,\chi} = (\mathrm{Sym}^k(N^X_{P_0(x),x}) \otimes_{\mathbb{R}} \mathbb{C})^{P_0,\chi}.\end{aligned}$$

We reduce to showing that

$$\dim(\mathrm{Sym}(N^{G/H}_{P_0(x),x}) \otimes_{\mathbb{R}} \mathbb{C})^{P_0^x,\chi} \leq |S_w(\chi)|$$

To that end it suffices to show that

$$\mathrm{Sym}(N^{G/H}_{P_0(x),x}) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\kappa: I_w \to \mathbb{Z}_{\geq 0}} \alpha_\kappa \tag{3}$$

as a representation of M_0^x. Indeed, by (2) we have

$$\alpha_\kappa\big|_{M_0^x} = \chi\big|_{M_0^x} \iff \kappa \in S_w(\chi)$$

and hence it would follow that

$$\dim(\mathrm{Sym}(N^{G/H}_{P_0(x),x}) \otimes_{\mathbb{R}} \mathbb{C})^{P_0^x,\chi} \leq \dim(\mathrm{Sym}(N^{G/H}_{P_0(x),x}) \otimes_{\mathbb{R}} \mathbb{C})^{M_0^x,\chi} \leq |S_w(\chi)|$$

as required.

It remains to show (3). We will deduce it by showing that

$$N^{G/H}_{P_0(x),x} \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{\imath \in I_w} \alpha_{\delta_\imath}$$

as a representation of M_0^x where $\delta_\imath$ is defined by $\delta_\imath(\jmath) = \delta_{\imath,\jmath}$.

We have

$$N_{P_0(x),x} = \mathrm{Herm}\,/\,\mathrm{Im}(\phi)$$

where Herm is the space of $n\times n$ hermitian matrices and $\phi : \mathrm{Lie}(P_0) \to \mathrm{Herm}$ is defined by $\phi(b) = bwa + wa^t b^\tau$.

It is easy to see that

$$\begin{aligned}\mathrm{Im}(\phi) = \mathrm{Span}_{\mathbb{C}}(\{e_{i,w(j)}, e_{w(j),i} : j \geq i\}) \cap \mathrm{Herm} &= \mathrm{Span}_{\mathbb{C}}(\{e_{i,j}, e_{j,i} : w(j) \geq i\}) \cap \mathrm{Herm} \\ &= \mathrm{Span}_{\mathbb{C}}(\{e_{i,j} : w(j) \geq i \text{ or } w(i) \geq j\}) \cap \mathrm{Herm},\end{aligned}$$

where $e_{i,j}$ is the standard basis for $n\times n$ matrices. Therefore

$$N_{P_0(x),x} \cong \mathrm{Span}_{\mathbb{C}}(\{e_{i,j} : i > w(j), j > w(i)\}) \cap \mathrm{Herm} =$$
$$= \mathrm{Span}_{\mathbb{C}}(\{e_{i,w(j)} : i > j, w(j) > w(i)\}) \cap \mathrm{Herm} = \mathrm{Span}_{\mathbb{C}}(\{e_{i,w(j)} : (i,j) \in I_w\}) \cap \mathrm{Herm}$$
$$\cong \bigoplus_{\{(i,j)\in I_w : i=w(j)\}} \mathrm{Span}_{\mathbb{R}}(e_{i,w(j)}) \oplus \bigoplus_{\{(i,j)\in I_w : i<w(j)\}} \mathrm{Span}_{\mathbb{R}}(e_{i,w(j)} + e_{w(j),i}, \sqrt{-1}(e_{i,w(j)} - e_{w(j),i})).$$

By (2) the action of M_0^x on $e_{i,w(j)}$ is given by $\alpha_{\delta_{(i,j)}} = t_i/t_j$. Thus as a representation of M_0^x we have

$$N_{P_0(x),x} \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{\{(i,j)\in I_w, i=w(j)\}} \alpha_{\delta_{(i,j)}} \oplus \bigoplus_{\{(i,j)\in I_w, i<w(j)\}} (\alpha_{\delta_{(i,j)}} \oplus \alpha^{\tau}_{\delta_{(i,j)}})$$
$$= \bigoplus_{\{(i,j)\in I_w, i=w(j)\}} \alpha_{\delta_{(i,j)}} \oplus \bigoplus_{\{(i,j)\in I_w, i<w(j)\}} (\alpha_{\delta_{(i,j)}} \oplus \alpha_{\delta_{(w(j),w(i))}})$$
$$= \bigoplus_{\{(i,j)\in I_w, i=w(j)\}} \alpha_{\delta_{(i,j)}} \oplus \bigoplus_{\{(i,j)\in I_w, i<w(j)\}} \alpha_{\delta_{(i,j)}} \oplus \bigoplus_{\{(i,j)\in I_w, i>w(j)\}} \alpha_{\delta_{(i,j)}} = \bigoplus_{\imath\in I_w} \alpha_{\delta_\imath}$$

as required. ☐

Theorem D.7. *For any $\lambda \in \mathfrak{a}^*_{M_0,\mathbb{C}}$ with $\Re\lambda_1 > \cdots > \Re\lambda_n$ the map $\alpha \mapsto J(\alpha,\lambda)$ defines an isomorphism $\mathcal{E}_{M_0}(X_{M_0}, 1^*_{M_0}) \to \mathcal{E}_G(X, I(1_{M_0},\lambda)^*)$.*

Proof. We showed that only open orbits contribute. Then we continue as in the proof of [FLO, Lemma 11.3] ☐

References

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