VANISHING OF CERTAIN EQUIVARIANT DISTRIBUTIONS ON SPHERICAL SPACES

AVRAHAM AIZENBUD AND DMITRY GOUREVITCH

Abstract. We prove vanishing of $\mathfrak{z}$-eigen distributions on a spherical variety of a split real reductive group which are supported away from the open Borel orbits and equivariant with respect to a non-degenerate character of the unipotent radical of the Borel subgroup.

This is a generalization of a result by Shalika, that concerned the group case. Shalika's result was crucial in the proof of his multiplicity one theorem. We view our result as a step in the study of multiplicities of quasi-regular representations on spherical varieties.

As an application we prove non-vanishing of Bessel-like functions.

1. Introduction

1.1. **Main results.** In this paper we prove the following generalization of Shalika's result [Sha74, §2].

Theorem A. *Let a split real reductive group G act transitively on a spherical space X. Let U be the unipotent radical of a Borel subgroup B of G and let ψ be its non-degenerate character. Let Z be the complement to the union of open B-orbits in X. Let $\mathfrak{z}$ be the center of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G.*

Then there are no non-zero $\mathfrak{z}$-eigen (U,ψ)-equivariant distributions supported on Z.

This result in the group case ([Sha74, §2]) was crucial in the proof of Shalika's multiplicity one theorem.

Our proof begins by applying the technique used by Shalika. However, this technique was not enough for this generality and we had to complement it by using integrability of the singular support, as in [AG09].

Theorem A provides a new tool for the study of the multiplicities of the irreducible quotients of the quasi-regular representation of G on Schwartz functions on X, see §§1.3 below for more details.

1.2. **Non-vanishing of spherical Bessel functions.** Another application of Theorem A is to the study of spherical Bessel distributions and functions.

Definition 1.2.1. *Let G be a split real reductive group, and $H \subset G$ be a spherical subgroup. Let (π, V) be a (smooth) irreducible admissible representation of G. Let ϕ be an H-invariant continuous functional on V and v be a (U,ψ)-equivariant continuous functional on the contragredient representation $\tilde{V}$. Define the* spherical Bessel distribution *by*

$$\xi_{v,\phi}(f) := \langle v, \pi^*(f)\phi\rangle.$$

Define the spherical Bessel function *to be the restriction $j_{v,\phi} := \xi_{v,\phi}|_{X-Z}$.*

It is well-known that $j_{v,\phi}$ is a smooth function.

Theorem A easily implies the following corollary.

Date: July 19, 2013.

2010 *Mathematics Subject Classification.* 20G05, 22E45, 46F99.

Key words and phrases. Vanishing of distributions, spherical spaces, multiplicity, Shalika, Bessel function.

Corollary B. *Suppose that* $v, \phi \neq 0$. *Then* $j_{v,\phi} \neq 0$.

The group case of the non-archimedean counterpart of this corollary was proven in [LM, Appendix B].

1.3. **Relation with multiplicities in regular representations of symmetric spaces.** Let (G, H) be a symmetric pair of real reductive groups. Suppose that G is quasi-split and let $B \subset G$ be a Borel subgroup. Let k be the number of open B-orbits on G/H.

Theorem A can be used in order to study the following conjecture.

Conjecture C. *Let* (π, V) *be a (smooth) irreducible admissible representation of* G. *Then the dimension of the space* $(V^*)^H$ *of* H*-invariant continuous functionals on* V *is at most* k. *In particular, any complex reductive symmetric pair is a Gelfand pair.*

We suggest to divide this conjecture into two cases

- π is non-degenerate, i.e. π has a non-zero continuous (U, ψ)-equivariant functional for some non-degenerate character π of U
- π is degenerate.

In the first case, the last conjecture follows from the following one

Conjecture D. *Let* U *be the unipotent radical of* B *and let* ψ *be its non-degenerate character. Let* $\mathfrak{z}$ *be the center of the universal enveloping algebra of the Lie algebra* $\mathfrak{g}$ *of* G. *Let* λ *be a character of* $\mathfrak{z}$.

Then the dimension of the space of $(\mathfrak{z}, \lambda)$*-eigen* (U, ψ)*-equivariant distributions on* G/H *does not exceed* k.

We believe that Theorem A can be useful in approaching this conjecture, since it allows to reduce the study of distributions to the union of open B-orbits.

1.4. **Acknowledgements.** We thank Erez Lapid for pointing out to us the application to non-vanishing of spherical Bessel functions. D.G. was partially supported by ISF grant 756/12 and a Minerva foundation grant. A.A. was partially supported by NSF grant DMS-1100943;

2. Preliminaries

2.1. **Conventions.**

- By an algebraic manifold we mean a smooth real algebraic variety.
- We will use capital Latin letters to denote Lie groups and the corresponding Gothic letters to denote their Lie algebras.
- Let a Lie group G act on a smooth manifold M. For a vector $v \in \mathfrak{g}$ and a point $x \in M$ we will denote by $vx \in T_xM$ the image of v under the differential of the action map $g \mapsto gx$. Similarly, we will use the notation $\mathfrak{h}x$, for any subspace $\mathfrak{h} \subset \mathfrak{g}$.
- We denote by G_x the stabilizer of x and by $\mathfrak{g}_x$ its Lie algebra.

2.2. **Tangential and transversal differential operators.** In this subsection we shortly review the method of [Sha74, §2]. For a more detailed description see [JSZ11, §§2.1].

Definition 2.2.1. *Let* M *be a smooth manifold and* N *be a smooth submanifold.*

- *A vector field* v *on* M *is called* tangential *to* N *if for every point* $p \in N$, $v_p \in T_pN$ *and* transversal *to* N *if for every point* $p \in N$, $v_p \notin T_pN$.
- *A differential operator* D *is called* tangential *to* N *if every point* $p \in N$ *has an open neighborhood* $U_x \subset N$ *such that* $D|_{U_x}$ *is a finite sum of differential operators of the form* $\phi V_1 \cdot ... \cdot V_r$ *where* ϕ *is a smooth function on* U_x, $r \geq 0$, *and* V_i *are vector fields on* U_x *tangential to* $U_x \cap N$.

Lemma 2.2.2 (cf. the proof of [Sha74, Proposition 2.10])**.** *Let M be a smooth manifold and N be a smooth submanifold. Let D be a differential operator on M tangential to N and V be a vector field on M transversal to N. Let ξ be a distribution on M supported in N such that $D\xi = V\xi$. Then $\xi = 0$.*

2.3. **Singular support.** Let M be an algebraic manifold and ξ be a distribution on M. The *singular support* of ξ is defined to be the singular support of the D-module generated by ξ and denoted $SS(\xi) \subset T^*M$.

We will shortly review the properties of the singular support that are most important for this paper. For more details we refer the reader to [AG09, §§2.3].

Notation 2.3.1. *For a point $x \in M$*

- *we denote by $SS_x(\xi)$ the fiber of x under the natural projection $SS(\xi) \to M$,*
- *for a submanifold $N \subset M$ we denote by $CN_N^M \subset T^*M$ the conormal bundle to N in M, and by $CN_{N,x}^M$ the conormal space at x to N in M.*

Theorem 2.3.2 (Integrability theorem, cf. [Gab81, GQS71, KKS73, Mal78])**.** *Let ξ be a distribution on an algebraic manifold M. Then $SS(\xi)$ is coisotropic in T^*M.*

This theorem implies the following corollary (see [Aiz, §3] for more details).

Corollary 2.3.3. *Let M be an algebraic manifold and $N \subset M$ be a closed algebraic submanifold. Let ξ be a distribution on M supported in N. Suppose that for any $x \in N$, we have $CN_{N,x}^M \not\subseteq SS_x(\xi)$. Then $\xi = 0$.*

3. Proof of the main result

3.1. **Sketch of the proof.** We decompose X into B-orbits. Each B-orbit $\mathcal{O}$ we decompose $\mathcal{O} = \mathcal{O}_s \cup \mathcal{O}_c$ in a certain way. We prove the required vanishing orbit by orbit, using Shalika's method (see §§2.2) for $\mathcal{O}_s$ and singular support analyses (see §§2.3) for $\mathcal{O}_c$.

3.2. Notation and lemmas.

Notation 3.2.1.

- *Fix a torus $T \subset B$ and let $\mathfrak{t} \subset \mathfrak{b}$ denote the corresponding Lie algebras. Let Φ denote the root system, Φ^+ denote the set of positive roots and $\Delta \subset \Phi^+$ denote the set of simple roots. For $\alpha \in \Phi$ let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root space corresponding to α.*
- *Let $C \in \mathfrak{z}$ denote the Casimir element.*
- *We choose $E_\alpha \in \mathfrak{g}_\alpha$, for any $\alpha \in \Phi$ such that $C = \sum_{\alpha\in\Phi^+} E_{-\alpha}E_\alpha + D$, where D is in the universal enveloping algebra of the Cartan subalgebra $\mathfrak{t}$.*
- *Let $\mathcal{O} \subset X$ be a B-orbit. Define*

$$\mathcal{O}_c := \left\{ x \in \mathcal{O} \mid \sum_{\alpha\in\Delta} d\psi(E_\alpha)E_{-\alpha}x \in T_x\mathcal{O} \right\}$$

and $\mathcal{O}_s := \mathcal{O} \setminus \mathcal{O}_c$.

We will need the following lemmas, that will be proved in subsequent subsections.

Lemma 3.2.2. *Let $x \in Z$. Then there exists a simple root $\alpha \in \Delta$ such that $\mathfrak{g}_{-\alpha}x \not\subseteq \mathfrak{b}x$.*

Lemma 3.2.3. *Let $x \in X$. Let ξ be a $\mathfrak{z}$-eigen (U, ψ) equivariant distribution on X. Then $SS_x(\xi) \subset CN_{Bx,x}$.*

Lemma 3.2.4. *Let $\mathcal{O} \subset X$ be a B-orbit. Then $\mathcal{O}_s \neq \emptyset$.*

3.3. **Proof of Theorem A.** Suppose that there exists a non-zero $\mathfrak{z}$-eigen (U, ψ)- equivariant distribution ξ supported on Z.

For any B-orbit $\mathcal{O} \subset X$, stratify $\mathcal{O}_c$ to a union of smooth locally closed varieties $\mathcal{O}_c^i$.

The collection $\{\mathcal{O}_c^i \,|\, \mathcal{O} \text{ is a } B\text{-orbit}\} \cup \{\mathcal{O}_s \,|\, \mathcal{O} \text{ is a } B\text{-orbit}\}$ is a stratification of X. Reorder this collection to a sequence $\{S_i\}_{i=1}^N$ of smooth locally closed subvarieties of X s.t. $U_k := \bigcup_{i=1}^k S_i$ is open in X for any $1 \leq k \leq N$.

Let k be the maximal integer such that $\xi|_{U_{k-1}} = 0$. Let $\eta := \xi|_{U_k}$. We will now show that $\eta = 0$, which leads to a contradiction.

Case 1. $S_k = \mathcal{O}_s$ for some orbit $\mathcal{O}$.

Recall that we have the following decomposition of the Casimir element

$$C = \sum_{\alpha \in \Phi^+} E_{-\alpha} E_\alpha + D$$

Since η is $\mathfrak{z}$-eigen and (U, ψ)-equivariant, we have, for some scalar λ,

$$\lambda\eta = C\eta = \sum_{\alpha \in \Phi^+} E_{-\alpha} E_\alpha \eta + D\eta = \sum_{\alpha \in \Phi^+} E_{-\alpha} d\psi(E_\alpha)\eta + D\eta = \sum_{\alpha \in \Delta} E_{-\alpha} d\psi(E_\alpha)\eta + D\eta$$

Let $V := \sum_{\alpha \in \Delta} d\psi(E_\alpha) E_{-\alpha}$ and $D' := \lambda Id - D$. We have $V\eta = D'\eta$, and it is easy to see that D' is tangential to $\mathcal{O}_s$, and V is transversal to $\mathcal{O}_s$. Now, Lemma 2.2.2 implies $\eta = 0$ which is a contradiction.

Case 2. $S_k \subset \mathcal{O}_c$ for some orbit $\mathcal{O}$.

By Corollary 2.3.3 it is enough to show that for any $x \in S_k$ we have

$$CN_{S_k,x}^X \not\subseteq SS_x(\eta). \tag{1}$$

By Lemma 3.2.3, $SS_x(\eta) \subset CN_{\mathcal{O},x}^X$. By Lemma 3.2.4, $S_k \subsetneq \mathcal{O}$, thus $CN_{S_k,x}^X \supsetneq CN_{\mathcal{O},x}^X$ which implies (1).

□

3.4. **Proof of Lemma 3.2.2.** First we need the following lemma and notation

Lemma 3.4.1. *Let $K \subset K_i \subset G$ for $i = 1, \ldots, n$ be algebraic subgroups generating G. Let Y be a transitive G space. Let $y \in Y$. Assume that Ky is Zariski dense in $K_i y$. Then Ky is Zariski dense in Y.*

Proof. By induction we may assume that $n = 2$. Let

$$O_l := \underbrace{K_1 K_2 \cdots K_1 K_2}_{l \text{ times}} y.$$

It is enough to prove that for any l the orbit Ky is dense in O_l. Let us prove it by induction on l. Suppose that we have already proven that Ky is dense in O_{l-1}. Then

$$\overline{Ky} = \overline{K_2 y} = \overline{K_2 K y} = \overline{K_2 O_{l-1} y}.$$

Thus Ky is dense in $K_2 O_{l-1}$. Similarly, $K_2 O_{l-1}$ is dense in $K_1 K_2 O_{l-1} = O_l$. □

Notation 3.4.2. *For a simple root $\alpha \in \Delta$, denote by $P_\alpha \subset G$ the parabolic subgroup whose Lie algebra is $\mathfrak{g}_{-\alpha} \oplus \mathfrak{b}$.*

Proof of Lemma 3.2.2. Assume the contrary. Then for any $\alpha \in \Delta$, $T_x P_\alpha x = T_x B x$. Thus Bx is Zariski dense in $P_\alpha x$. By the lemma this implies that Bx is dense in X, which contradicts the condition $x \in Z$. □

3.5. **Proof of Lemma 3.2.3.**

Proof. Let H be the stabilizer of x and let $\mathfrak{h}$ be its Lie algebra. Identify T_xX with $\mathfrak{g}/\mathfrak{h}$ and T_x^*X with $\mathfrak{h}^\perp \subset \mathfrak{g}^*$. Then $CN_{Bx,x}^X = (\mathfrak{t}+\mathfrak{u}+\mathfrak{h})^\perp$. Since ξ is $\mathfrak{u}$-equivariant, $SS_x(\xi) \subset (\mathfrak{u}+\mathfrak{h})^\perp$. Since ξ is also $\mathfrak{z}$-eigen, $SS_x(\xi) \subset \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone. Now, $(\mathfrak{u}+\mathfrak{h})^\perp \cap \mathcal{N}(\mathfrak{g}^*) = (\mathfrak{t}+\mathfrak{u}+\mathfrak{h})^\perp$.

□

3.6. **Proof of Lemma 3.2.4.**

Let $p : B \to \mathcal{O}$ denote the action map. It is enough to show that $p^{-1}(\mathcal{O}_c) \neq B$.

$$p^{-1}(\mathcal{O}_c) = \{b \in B \mid \sum \psi(E_\alpha)E_{-\alpha} \in \mathfrak{g}_{bx}+\mathfrak{b}\} = \{b \in B \mid \sum \psi(E_\alpha)ad(b^{-1})E_{-\alpha} \in \mathfrak{g}_x+\mathfrak{b}\} =$$
$$\{tu \in B \mid \sum \psi(E_\alpha)ad(t^{-1})E_{-\alpha} \in \mathfrak{g}_x+\mathfrak{b}\} = \{tu \in B \mid \sum \psi(E_\alpha)\alpha(t)E_{-\alpha} \in \mathfrak{g}_x+\mathfrak{b}\}$$

By Lemma 3.2.2 we can choose $\alpha \in \Delta$ such that $\mathfrak{g}_{-\alpha}x \nsubseteq \mathfrak{b}x$. For any $\varepsilon > 0$ there exists $t \in T$ s.t. $\alpha(t) = 1$ and $\forall \beta \neq \alpha \in \Delta$ we have $|\beta(t)| < \varepsilon$. It is easy to see that for ε small enough, $t \notin p^{-1}(\mathcal{O}_c)$.

References

[Aiz] *A partial analog of integrability theorem for distributions on p-adic spaces and applications,* Israel Journal of Mathematics (2013), **193**, Issue 1, pp 233-262. See also arXiv:0811.2768[math.RT].

[AG09] A. Aizenbud, D. Gourevitch, *Multiplicity one theorem for* $(GL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$, Selecta Mathematica, 15/2, pp. 271-294 (2009). See also arXiv:0808.2729 [math.RT].

[Gab81] O.Gabber, *The integrability of the characteristic variety.* Amer. J. Math. 103/3 (1981).

[GQS71] V. Guillemin, D. Quillen, S. Sternberg, *The integrability of characteristics.* Comm. Pure Appl. Math. 23/1 (1970).

[JSZ11] D. Jiang, B. Sun, and C.-B. Zhu *Uniqueness of Ginzburg-Rallis models: the Archimedean case,* Transactions of the AMS, **363**, n. 5, (2011), 2763-2802.

[KKS73] M. Kashiwara, T. Kawai, and M. Sato, *Hyperfunctions and pseudo-differential equations* (Katata, 1971), pp. 265–529, Lecture Notes in Math., **287**, Springer, Berlin, 1973;

[LM] E.Lapid, Z. Mao *On Whittaker - Fourier coefficients of automorphic forms on* $\widetilde{Sp_n}$, preprint available at http://www.math.huji.ac.il/~erezla/papers/master010413.pdf.

[Mal78] B. Malgrange *L'involutivite des caracteristiques des systemes differentiels et microdifferentiels* Séminaire Bourbaki 30è Année (1977/78), Exp. No. 522, Lecture Notes in Math., **710**, Springer, Berlin, 1979;

[Sha74] J.A. Shalika, Multiplicity one theorem for GLn, Ann. Math. **100** (1974), 171-193.

Avraham Aizenbud, Faculty of Mathematics and Computer Science, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel
E-mail address: aizenr@gmail.com
URL: http://math.mit.edu/~aizenr/

Dmitry Gourevitch, Faculty of Mathematics and Computer Science, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel
E-mail address: dimagur@weizmann.ac.il
URL: http://www.wisdom.weizmann.ac.il/~dimagur