

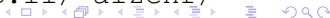
Applications of the Bernstein-Kashiwara Theorem

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Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

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A D -module (or a distribution) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

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Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let \mathcal{E} be an algebraic G -equivariant bundle on X and χ be a character of \mathfrak{g} . Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g}, \chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor \mathcal{E} with a representation of \mathfrak{g} of a fixed dimension.

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- 2 If H is a real spherical subgroup (i.e. HP is open for some minimal parabolic subgroup P) then, for every irreducible admissible representation $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n -dimensional representation τ of \mathfrak{h} we have*

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

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- This implies that gM is smooth.

Relation with multiplicity

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- ξ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G .

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Corollary (Aizenbud, Gourevitch, Minchenko, Sayag)

For any local field F , any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

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