Applications of the Bernstein-Kashiwara Theorem

Avraham Aizenbud

Weizmann Institute of Science

J. w. Dmitry Gourevitch, Bernhad Krötz, Andrey Minchenko,

Gang Liu, Eitan Sayag

http://www.wisdom.weizmann.ac.il/~aizenr/

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A D-module (or a distribution) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

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Corollary (Aizenbud, Gourevitch, Minchenko)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a character of $\mathfrak{g}$. Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g},\chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.

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2. *If H is a real spherical subgroup (i.e. HP is open for some minimal parabolic subgroup P) then, for every irreducible admissible representation $\pi \in Irr(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n-dimensional representation τ of $\mathfrak{h}$ we have*

$$\dim Hom_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

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- *ξ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G.*

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Corollary (Aizenbud, Gourevitch, Minchenko, Sayag)

For any local field F, any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

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