Multiplicity one theorem for GL(n) and other Gelfand pairs

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- the algebra of bi-H-invariant functions on G, $C(H \backslash G/H)$, is commutative w.r.t. convolution.

Philosophical motivation

Observation

Representation theory of G

$$\Updownarrow$$

Harmonic analysis on G *w.r.t. the two sided action of* $G \times G$

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- the algebra of Ad(H)-invariant functions on G, $C(G//H) := C(G/Ad(H))$, is commutative w.r.t. convolution.

Gelfand trick

Proposition (Gelfand)

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2)\sigma(g_1)$ and $\sigma^2 = Id$) and assume $\sigma(H) = H$. Suppose that $\sigma(f) = f$ for all bi H-invariant functions $f \in C(H \backslash G / H)$. Then (G, H) is a Gelfand pair.

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Sum up

Rep. theory: $\forall \rho \dim \rho^H \leq 1$

$\Longleftrightarrow$

Algebra: $C(H \backslash G / H)$ is commutative

$\Longleftarrow$

"Analysis": $\exists$ anti-involution σ s.t. $f = \sigma(f)$ $\forall f \in C(H \backslash G / H)$

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Geometry: $\exists$ anti-involution σ that preserves H double cosets

Non compact setting

Setting

In the non compact case we will consider complex smooth admissible representations of algebraic reductive groups over local fields.

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Definition

A local field is a locally compact non-discrete topological field. There are 2 types of local fields of characteristic zero:

- Archimedean: $\mathbb{R}$ and $\mathbb{C}$
- non-Archimedean: $\mathbb{Q}_p$ and their finite extensions

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Definition

A linear algebraic group is a subgroup of GL_n defined by polynomial equations.

Reductive groups

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GL_n, O_n, U_n, Sp_{2n},..., semisimple groups,

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Fact

Reductive groups are unimodular.

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Over Archimedean F, by smooth representation V we mean a complex Fréchet representation V such that for any $v \in V$ the map $G \to V$ defined by v is smooth.

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Over non-Archimedean F, by smooth representation V we mean a complex linear representation V such that for any $v \in V$ there exists an open compact subgroup $K < G$ such that $Kv = v$.

Distributions

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We will consider the space $(C_c^\infty(M))^*$ of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

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Definition

An ℓ-space is a Hausdorff locally compact totally disconnected topological space. For an ℓ-space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X. We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X.

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$$dimHom_H(\rho, \mathbb{C}) \cdot dimHom_H(\widetilde{\rho}, \mathbb{C}) \leq 1$$

Usually, this implies that

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Gelfand-Kazhdan distributional criterion

Theorem (Gelfand-Kazhdan,...)

Let σ be an involutive anti-automorphism of G and assume $\sigma(H) = H$.
Suppose that $\sigma(\xi) = \xi$ for all bi H-invariant distributions ξ on G.
Then (G, H) is a Gelfand pair.

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Proposition

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Corollary

Let σ be an involutive anti-automorphism of G s.t. $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all distributions ξ on G invariant with respect to conjugation by H. Then (G, H) is a strong Gelfand pair.

Rep. theory:
$\forall \rho \dim \rho^H \leq 1$

$\Longleftrightarrow$

Algebra:
$C(H\backslash G/H)$
is commutative

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"Analysis":
$\exists\ \sigma$ s.t. $f = \sigma(f)$
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~~Algebra~~

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$\forall \xi \in \mathcal{S}^*(G)^{H \times H}$

$\Uparrow$ *p–adic*

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Multiplicity one theorem for GL_n

Let F be a local field of characteristic zero.

Theorem (A.-Gourevitch-Rallis-Schiffmann-Sun-Zhu)

Every $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is transposition invariant.

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Theorem (A.-Gourevitch-Rallis-Schiffmann-Sun-Zhu)

Let π be an irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$ and τ be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Then

$$\dim Hom_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1.$$

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Similar theorems hold for orthogonal and unitary groups.

Reformulation

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- $\widetilde{G} := GL_n(F) \rtimes \{1, \sigma\}$
- Define a character χ of $\widetilde{G}$ by $\chi(GL_n(F)) = \{1\}$, $\chi(\widetilde{G} - GL_n(F)) = \{-1\}$.

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Proposition

Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{S}^*(U)^{G,\chi} = 0$ and $\mathcal{S}^*_X(Z)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.

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$0 \to \mathcal{S}^*_X(Z)^{G,\chi} \to \mathcal{S}^*(X)^{G,\chi} \to \mathcal{S}^*(U)^{G,\chi}$. □

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$$gr_k(\mathcal{S}^*_X(Z)) = \mathcal{S}^*(Z, Sym^k(CN_Z^X))$$

Frobenius descent

$$\begin{array}{ccc} X_z & \longrightarrow & X \\ \downarrow & & \downarrow \\ z & \longrightarrow & Z \end{array}$$

Theorem (Bernstein, Baruch, ...)

Let $\psi : X \to Z$ be a map.
Let G act on X and Z such that $\psi(gx) = g\psi(x)$.
Suppose that the action of G on Z is transitive.
Suppose that both G and $Stab_G(z)$ are unimodular. Then

$$\mathcal{S}^*(X)^{G,\chi} \cong \mathcal{S}^*(X_z)^{Stab_G(z),\chi}.$$

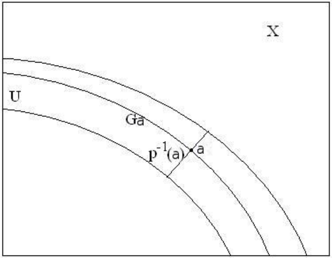
Generalized Harish-Chandra descent

Theorem (A.-Gourevitch)

Let a reductive group G act on a smooth affine algebraic variety X. Let χ be a character of G. Suppose that for any $a \in X$ s.t. the orbit Ga is closed we have

$$\mathcal{S}^*(N_{Ga,a}^X)^{G_a,\chi} = 0.$$

Then $\mathcal{S}^*(X)^{G,\chi} = 0$.



Fourier transform

Let V be a finite dimensional vector space over F and Q be a non-degenerate quadratic form on V. Let $\widehat{\xi}$ denote the Fourier transform of ξ defined using Q.

Proposition

Let G act on V linearly and preserving Q. Let $\xi \in \mathcal{S}^*(V)^{G,\chi}$. Then $\widehat{\xi} \in \mathcal{S}^*(V)^{G,\chi}$.

Fourier transform and homogeneity

- We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree** d if for any $t \in F^\times$,

$$h_t(\xi) = u(t)|t|^d \xi,$$

where h_t denotes the homothety action on distributions and u is some unitary character of $F^\times$.

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Theorem (Homogeneity theorem – Jacquet, Rallis, Schiffmann,...)

*Assume F is **non-Archimedean**. Let $\xi \in \mathcal{S}^*_V(Z(Q))$ be s.t. $\widehat{\xi} \in \mathcal{S}^*_V(Z(Q))$. Then ξ is abs-homogeneous of degree $\frac{1}{2}\dim V$.*

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Theorem (Archimedean homogeneity theorem – A.-Gourevitch)

Let F be any local field. Let $L \subset \mathcal{S}_V^*(Z(Q))$ be a non-zero linear subspace s. t. $\forall \xi \in L$ we have $\widehat{\xi} \in L$ and $Q\xi \in L$.
Then there exists a non-zero distribution $\xi \in L$ which is abs-homogeneous of degree $\frac{1}{2}\dim V$ or of degree $\frac{1}{2}\dim V + 1$.

Singular Support and Wave Front Set

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In the non-Archimedean case we define the singular support to be the Zariski closure of the wave front set.

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- Let V be a linear space. Let $Z \subset V^*$ be a closed subvariety, invariant with respect to homotheties. Let $\xi \in \mathcal{S}^*(V)$. Suppose that $\mathrm{Supp}(\widehat{\xi}) \subset Z$. Then $SS(\xi) \subset V \times Z$.

Properties and the Integrability Theorem

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- Integrability theorem:
  Let $\xi \in \mathcal{S}^*(X)$. Then $SS(\xi)$ is (weakly) coisotropic.

Coisotropic varieties

Definition

Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it ***M*-coisotropic** if the following equivalent conditions hold.

- At every smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^{\perp}$. Here, $(T_z Z)^{\perp}$ denotes the orthogonal complement to $T_z Z$ in $T_z M$ with respect to ω.
- The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.

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If there is no ambiguity, we will call Z a coisotropic variety.

- Every non-empty coisotropic subvariety of M has dimension at least $\frac{\dim M}{2}$.

Weakly coisotropic varieties

Definition

Let X be a smooth algebraic variety. Let $Z \subset T^*X$ be an algebraic subvariety. We call it T^*X-**weakly coisotropic** if one of the following equivalent conditions holds.

- For a generic smooth point $a \in p_X(Z)$ and for a generic smooth point $y \in p_X^{-1}(a) \cap Z$ we have $CN^X_{p_X(Z),a} \subset T_y(p_X^{-1}(a) \cap Z)$.
- For any smooth point $a \in p_X(Z)$ the fiber $p_X^{-1}(a) \cap Z$ is locally invariant with respect to shifts by $CN^X_{p_X(Z),a}$.

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- Every non-empty weakly coisotropic subvariety of T^*X has dimension at least $\dim X$.

Definition

Let X be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety and $R \subset T^*X$ be any subvariety. We define **the restriction** $R|_Z \subset T^*Z$ **of** R **to** Z by

$$R|_Z := q(p_X^{-1}(Z) \cap R),$$

where $q : p_X^{-1}(Z) \to T^*Z$ is the projection.

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Lemma

Let X be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety. Let $R \subset T^*X$ be a (weakly) coisotropic variety. Then, under some transversality assumption, $R|_Z \subset T^*Z$ is a (weakly) coisotropic variety.

Notation

$$S := \{(A, v, \phi) \in X_n | A^n = 0 \text{ and } \phi(A^i v) = 0 \, \forall 0 \leq i \leq n\}.$$

Harish-Chandra descent and homogeneity

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Reduction to the geometric statement

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$$T' = \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \,|\, \forall i, j \in \{1, 2\}$$
$$(A_i, v_j, \phi_j) \in S' \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

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It is enough to show:

Theorem (The geometric statement)

There are no non-empty $X \times X$-weakly coisotropic subvarieties of T'.

Summary

Flowchart

$$sl(V) \times V \times V^* \xrightarrow[\text{descent}]{\text{H.Ch.}} S \xrightarrow[\text{homogeneity theorem}]{\text{Fourier transform and}} S' \xrightarrow[\text{integrability theorem}]{\text{Fourier transform and}} T'$$

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Let $A \in sl(V)$ be a nilpotent Jordan block. Denote $R_A := (T' - T'')|_{\{A\} \times V \times V^*}$.

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Lemma (Key Lemma)

There are no non-empty $V \times V^* \times V \times V^*$-weakly coisotropic subvarieties of R_A.

Proof of the Key Lemma

Notation

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It is enough to show that $f(R_A \cap L_{ii}) = \{0\}$.

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Summary

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$sl(V) \times V \times V^*$

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Aim

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$$\Updownarrow$$

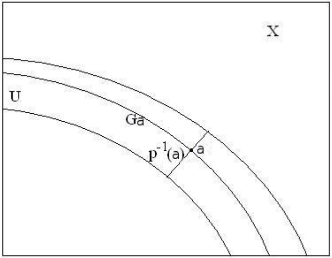
$\chi|_{G_a} \neq 1$ for any semi simple $a \in X$ (i.e. $a \in X$ with closed orbit Ga)

Luna's slice theorem

Luna's slice theorem

Theorem (Luna)

Let a reductive group G act on a smooth affine algebraic variety X. Let $a \in X$ be a semi-simple point. Then there exists an invariant (etale) neighborhood U of Ga with an equivariant projection $p : U \to Ga$ s.t. the fiber $p^{-1}(a)$ is G-isomorphic to an (etale) neighborhood of 0 in the normal space $N^X_{Ga,a}$.



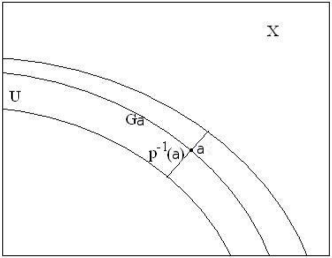
Generalized Harish-Chandra descent

Theorem (A.-Gourevitch)

Let a reductive group G act on a smooth affine algebraic variety X. Let χ be a character of G. Suppose that for any $a \in X$ s.t. the orbit Ga is closed, we have

$$\mathcal{S}^*(N_{Ga,a}^X)^{G_a,\chi} = 0.$$

Then $\mathcal{S}^*(X)^{G,\chi} = 0$.



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by induction we may assume:

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Using Fourier transform, we reduce the problem to showing

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Using homogeneity theorem, we reduce the problem to showing

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We call the property (2) regularity. We conjecture that all symmetric pairs are regular. This will imply that any good symmetric pair is a Gelfand pair.

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Regular symmetric pairs

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$(G \times G, \Delta G)$	A.-Gourevitch	A.-Gourevitch
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$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	
$(O_{n+k}, O_n \times O_k)$	A.-Gourevitch	
(GL_n, O_n)		
(GL_{2n}, Sp_{2n})	Heumos - Rallis	A.-Sayag
$(sp_{2m}, sl_m \oplus \mathfrak{g}_a)$	A.	Sayag (based on work of Sekiguchi)
(e_6, sp_8)		
$(e_6, sl_6 \oplus sl_2)$		
(e_7, sl_8)		
(e_8, so_{16})		
$(f_4, sp_6 \oplus sl_2)$		
$(g_2, sl_2 \oplus sl_2)$		

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  The same for the pair $(GL(n,\mathbb{C}), U(n))$.

More modern applications

Automorphic forms

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- Automorphic multiplicity one $\Leftrightarrow$ $(GL_n(\mathbb{A}), GL_n(\mathbb{Q}))$ is a Gelfand pair $\Leftarrow$ (GL_n, U_n, ψ) is a Gelfand pair.
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  (G, H) is a Gelfand pair $\Rightarrow$ H-period splits into local factors.

Other applications of invariant distributions

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- Study of the principal series

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  any $Ad(GL_{n-1})$ invariant distribution on GL_n is transposition invariant
- Trace formula and relative trace formula:
  Smooth matching

Classical examples

Pair	Anti-involution
$(G \times G, \Delta G)$	$(g, h) \mapsto (h^{-1}, g^{-1})$
$(O(n+k), O(n) \times O(k))$	$g \mapsto g^{-1}$
$(U(n+k), U(n) \times U(k))$	$g \mapsto g^{-1}$
$(GL(n, \mathbb{R}), O(n))$	$g \mapsto g^t$
(G, G^θ), where G - Lie group, θ- involution, G^θ is compact	$g \mapsto \theta(g^{-1})$
(G, K), where G - is a reductive group, K - maximal compact subgroup	Cartan anti-involution

Results on Gelfand pairs

Pair	p-adic case	real case
$(G,(N,\psi))$	Gelfand-Kazhdan	Shalika, Kostant
$(GL_n(E),GL_n(F))$	Flicker	A.-Gourevitch
$(GL_{n+k},GL_n\times GL_k)$	Jacquet-Rallis	
$(O_{n+k},O_n\times O_k)$ over $\mathbb{C}$	————	
(GL_n,O_n) over $\mathbb{C}$		
(GL_{2n},Sp_{2n})	Heumos-Rallis	A.-Sayag
$(GL_{2n},(\begin{pmatrix} g & u \\ 0 & g \end{pmatrix},\psi))$	Jacquet-Rallis	A.-Gourevitch -Jacquet
$(GL_n,(\begin{pmatrix} SP & u \\ 0 & N \end{pmatrix},\psi))$	Offen-Sayag	A.-Offen-Sayag

- real: $\mathbb{R}$ and $\mathbb{C}$
- p-adic: $\mathbb{Q}_p$ and its finite extensions.

Results on Gelfand pairs

<table>
<tr><th>Pair</th><th>p-adic</th><th>$\mathrm{char} F > 0$</th><th>real</th></tr>
<tr><td>$(G, (N, \psi))$</td><td>Gelfand-Kazhdan</td><td>Gelfand-Kazhdan</td><td>Shalika, Kostant</td></tr>
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- $\mathrm{char} F > 0$: $\mathbb{F}_q((t))$

Results on strong Gelfand pairs

<table>
<tr><th>Pair</th><th>p-adic</th><th>$\mathrm{char} F > 0$</th><th>real</th></tr>
<tr><td>(GL_{n+1}, GL_n)</td><td rowspan="3">A.-Gourevitch-Rallis-Schiffmann</td><td>A.-Avni-Gourevitch, Henniart</td><td>A.-Gourevitch, Sun-Zhu</td></tr>
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