Multiplicities in relative representation theory

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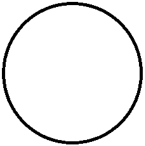
Fourier Siries



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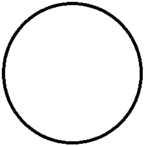

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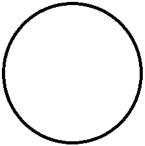

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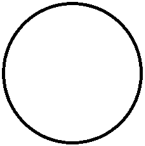

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- Any function can be decomposed into a combination of trigonometric functions.

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Goal

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- *Describe the assignment* $\rho \mapsto \dim Hom(\rho, Func(X))$

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Conclusion

We can interpret harmonic analysis as a generalization of representation theory.

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- *Deduce the result.*

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Definition

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Example

$H \times H/\Delta H = H$ as a $H \times H$-space.

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Let $n > 0$ be an integer. Then

$$\sup_{\substack{p > 2 - \text{ prime} \\ \Gamma - \text{ group} \\ \phi : \Gamma \to GL_n(\mathbb{F}_p); |Ker(\phi)| = p^k \\ \theta : \Gamma \to \Gamma; \theta \circ \theta = id \\ \rho \in \mathrm{irr}(\Gamma)}} \dim \left(\rho^{\Gamma^\theta} \right) < \infty.$$

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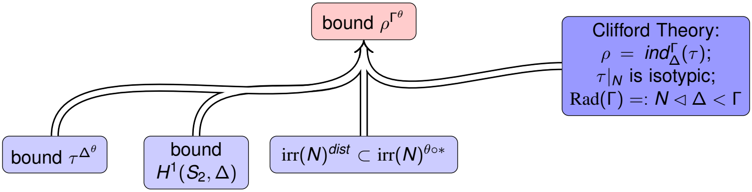
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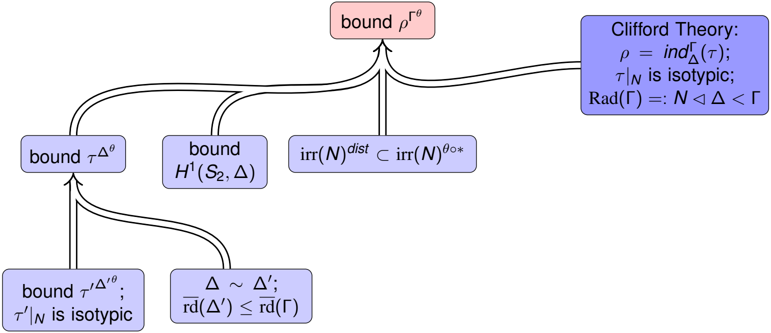

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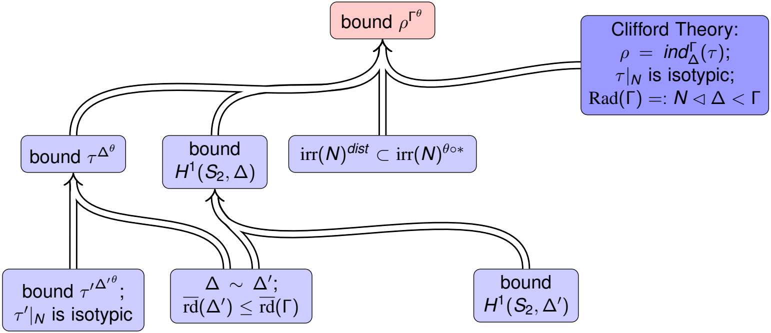

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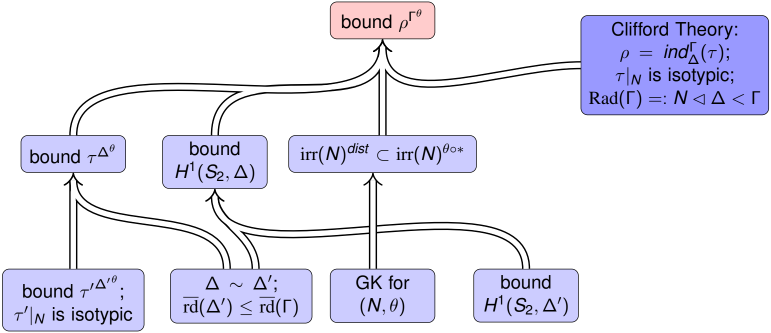

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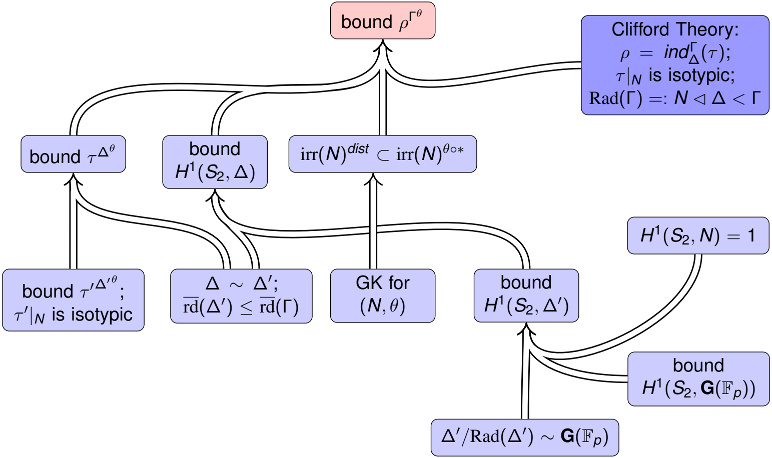

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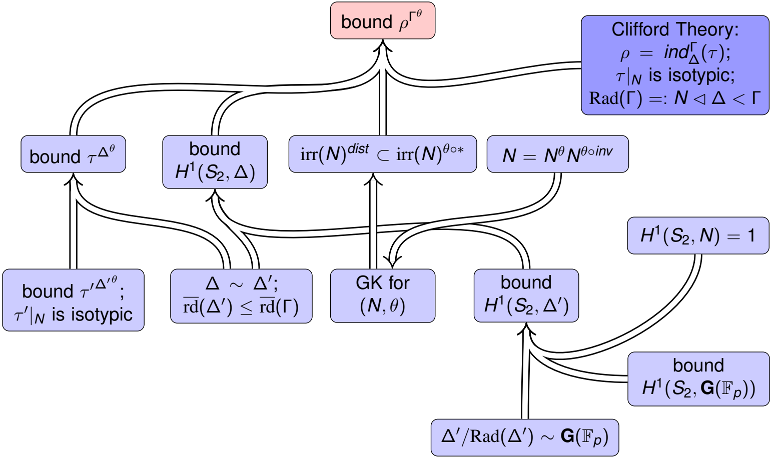

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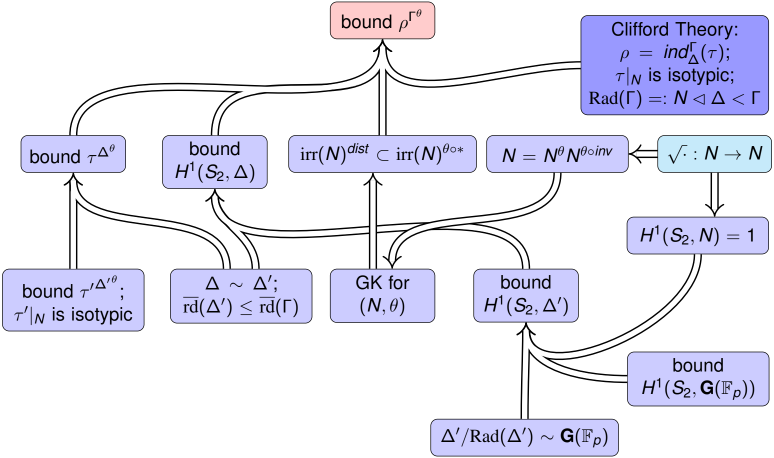
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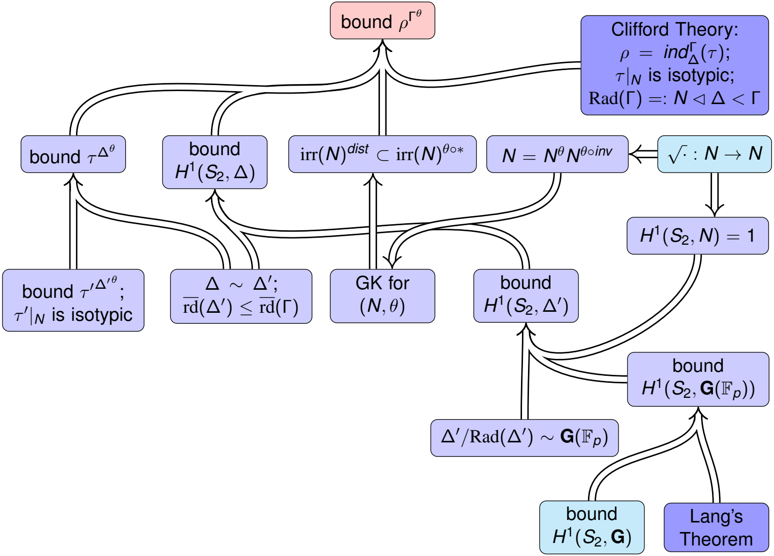
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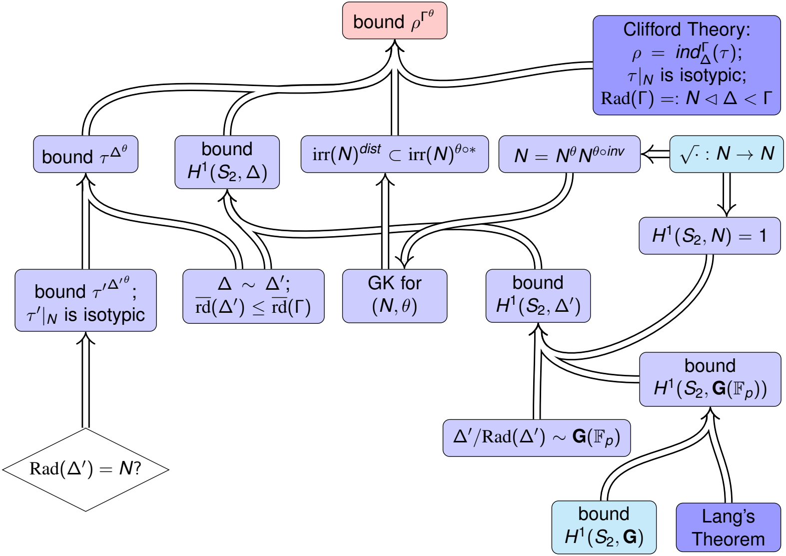

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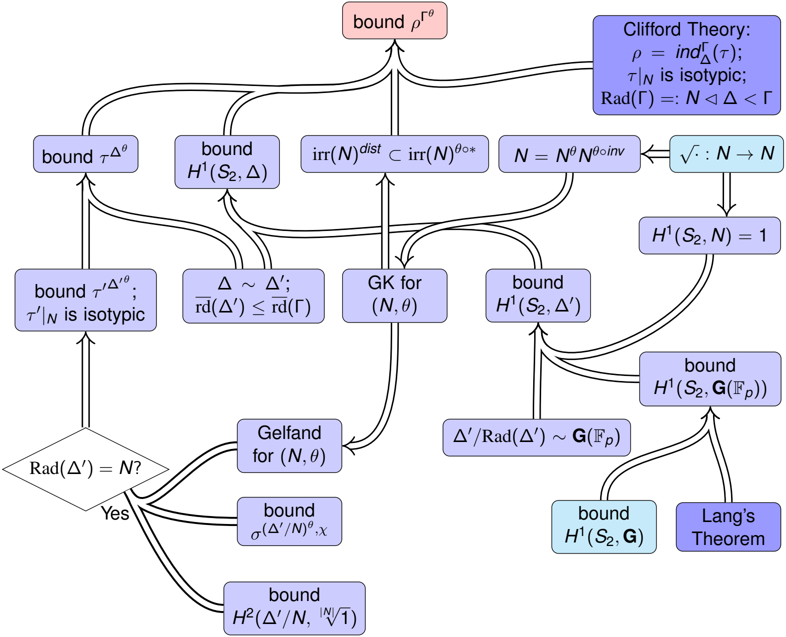

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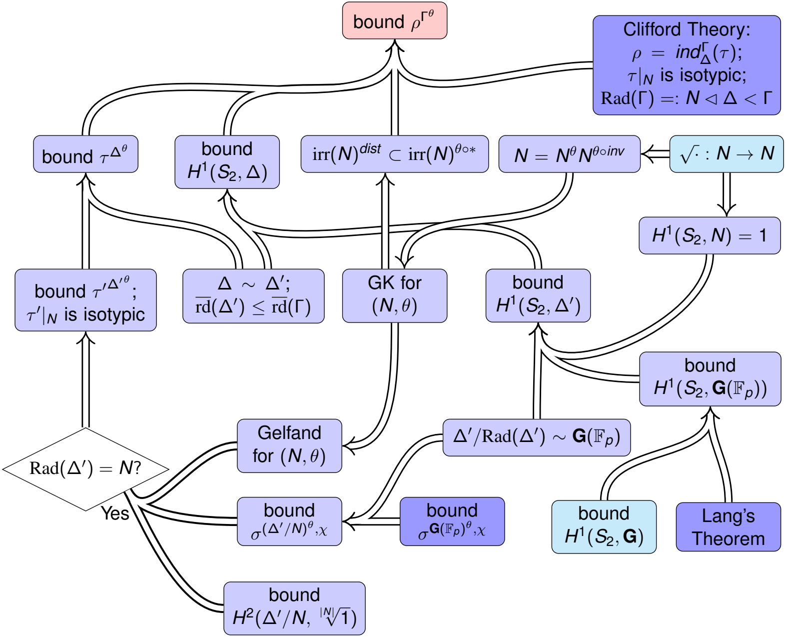
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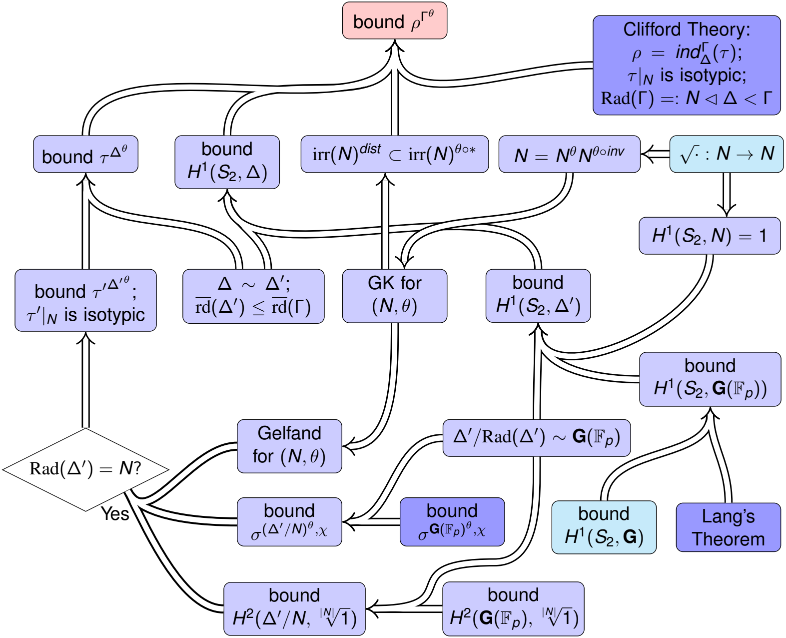
bound $\sigma^{(\Delta'/N)^\theta, \chi}$
bound $\sigma^{\mathbf{G}(\mathbb{F}_p)^\theta, \chi}$
bound $H^1(S_2, \mathbf{G})$
Lang's Theorem
bound $H^2(\Delta'/N, \sqrt[|N|]{1})$



bound ρ^{Γ^θ}
Clifford Theory: $\rho = ind_\Delta^\Gamma(\tau)$; $\tau|_N$ is isotypic; $\mathrm{Rad}(\Gamma) =: N \lhd \Delta < \Gamma$
bound τ^{Δ^θ}
bound $H^1(S_2, \Delta)$
$\mathrm{irr}(N)^{dist} \subset \mathrm{irr}(N)^{\theta\circ *}$
$N = N^\theta N^{\theta\circ inv}$
$\sqrt{\cdot} : N \to N$
$H^1(S_2, N) = 1$
bound $\tau'^{\Delta'^\theta}$; $\tau'|_N$ is isotypic
$\Delta \sim \Delta'$; $\overline{\mathrm{rd}}(\Delta') \leq \overline{\mathrm{rd}}(\Gamma)$
GK for (N, θ)
bound $H^1(S_2, \Delta')$
bound $H^1(S_2, \mathbf{G}(\mathbb{F}_p))$
Gelfand for (N, θ)
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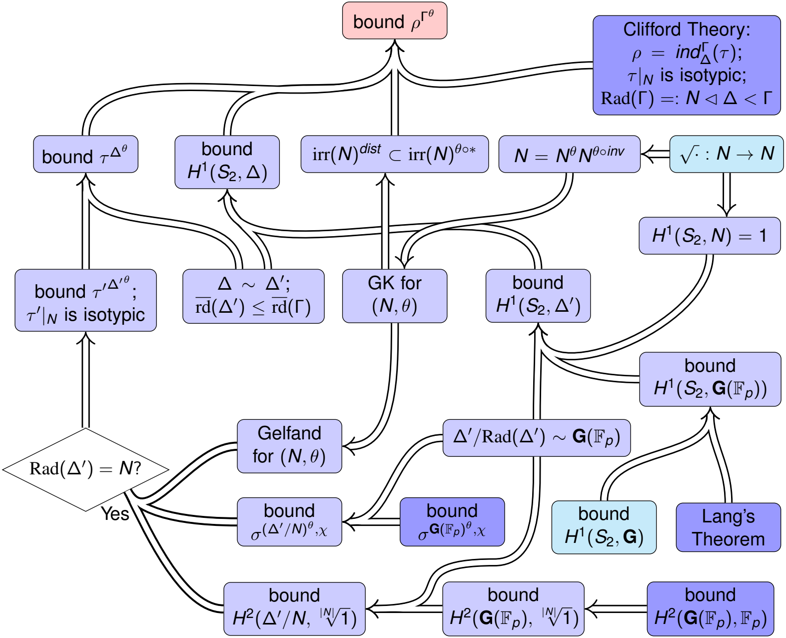

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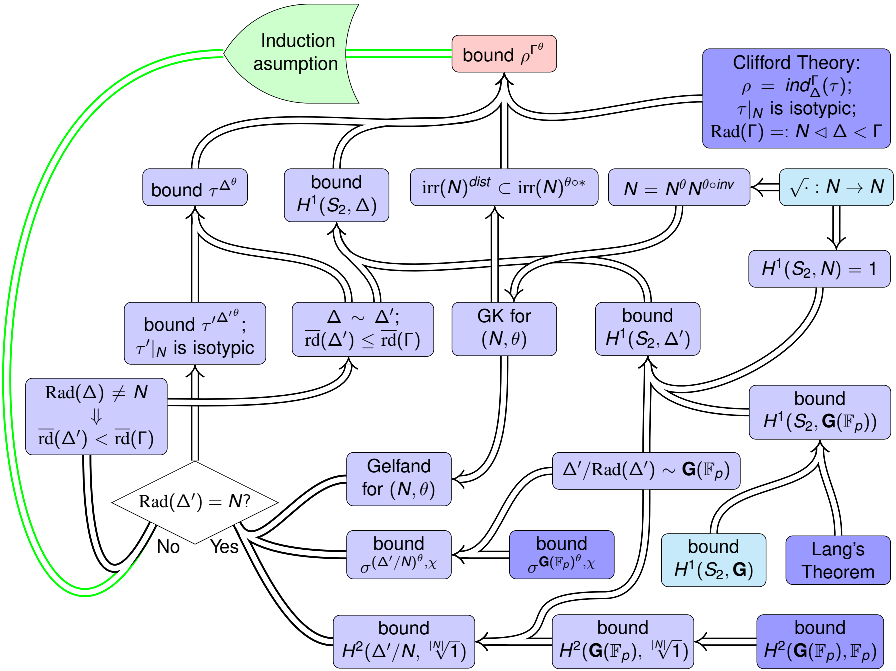

Induction asumption
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GK for (N, θ)
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$\mathrm{Rad}(\Delta) \neq N$ $\Downarrow$ $\overline{\mathrm{rd}}(\Delta') < \overline{\mathrm{rd}}(\Gamma)$
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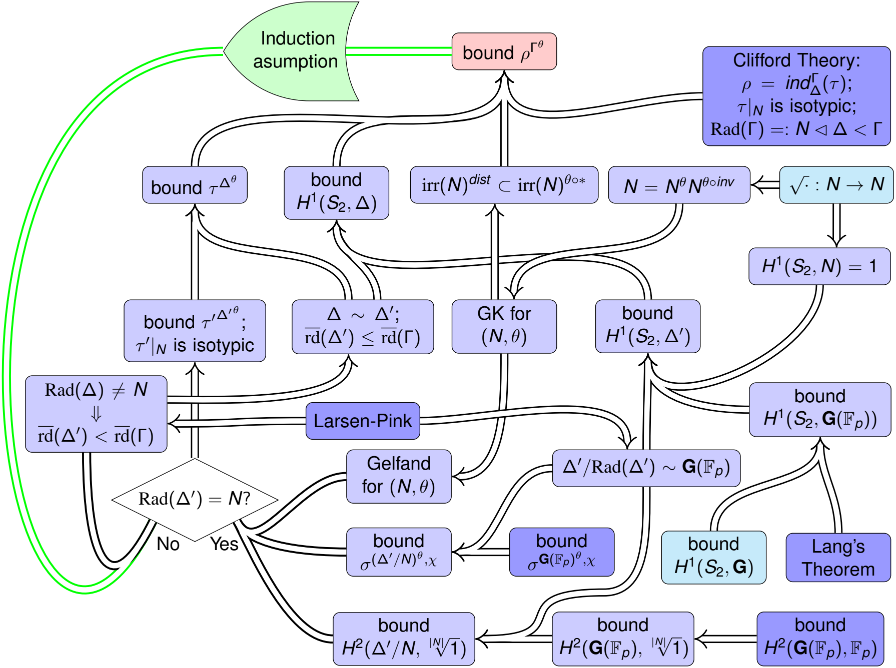
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