Relative representation theory over close local fields

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We will discuss two types of local fields:

- zero characteristic – finite extensions of $\mathbb{Q}_p$.
- positive characteristic – finite extensions of $\mathbb{F}_p((t))$.

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Example

- $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p$.
- $\mathcal{P}_{\mathbb{Q}_p} = p\mathbb{Z}_p$.
- $\mathcal{O}_{\mathbb{F}_p((t))} = \mathbb{F}_p[[t]]$.
- $\mathcal{P}_{\mathbb{F}_p((t))} = t\mathbb{F}_p[[t]]$.

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Example

$\mathbb{Q}_p(\sqrt[n]{p})$ is n-close to $\mathbb{F}_p((t))$.

Congruence subgroups

Let G be a reductive group defined over $\mathbb{Z}$ and let F be a local field.

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$K_0 := K_0(F) := K_0(G, F) := G(\mathcal{O}_F) \subset G(F)$

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Example

$F = \mathbb{Q}_p$, $G = GL_n$, $K_0 = GL_n(\mathbb{Z}_p)$, $K_n = Id + p^n Mat_n(\mathbb{Z}_p)$, $K_0/K_n = GL_n(\mathbb{Z}/p^n\mathbb{Z})$.

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Remark

If F and F' are n-close then $K_0(F)/K_n(F) \simeq K_0(F')/K_n(F')$.

Smooth representations

Definition

- We denote the category of smooth representation of $G(F)$, (i.e. complex linear representations V such that any $v \in V$ has an open stabilizer) by $\mathcal{M}(G(F))$.

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Notation

the Hecke algebra, denoted by $\mathcal{H}_K(G(F))$, is the algebra of double K-invariant measures on $G(F)$. The algebra structure on $\mathcal{H}_K(G(F))$ is defined by convolution.

Theorem (Bernstein 1980)

- $\mathcal{M}_{K_n}(G(F))$ *is a direct summand of* $\mathcal{M}(G(F))$.
- $\mathcal{M}_{K_n}(G(F))$ *is equivalent to the category of* $\mathcal{H}_{K_n}(G(F))-$*modules*
- *The algebra* $\mathcal{H}_{K_n}(G(F))$ *is finite over its center which is finitely generated.*

Kazhdan's Theorem

Theorem (Kazhdan - 1984)

For any natural n there exist N s.t. if F and F′ are N−close then

$$\mathcal{H}_{K_n(F)}(G(F)) \simeq \mathcal{H}_{K_n(F')}(G(F')).$$

Ingredients of the proof

- A construction of a linear isomorphism

$$\phi : \mathcal{H}_{K_n(F)}(G(F)) \to \mathcal{H}_{K_n(F')}(G(F'))$$

whenever F and F' are $n-$close.

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- The algebra $\mathcal{H}_{K_n}(G(F))$ is finitely presented.

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Let $H \subset G$ be a spherical pair (i.e. the Borel subgroup of G acts on G/H with finite number of orbits). One can consider harmonic analysis over G/H as a generalization of representation theory.

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Example

Schur's lemma is analogous to the Gelfand property of (G, H):

$$\forall \pi \in irr(G) : \dim \mathrm{Hom}(\mathcal{S}(G/H), \pi) \leq 1$$

or equivalently

$$\forall \pi \in irr(G) : \dim \mathrm{Hom}_H(\pi|_H, \mathbb{C}) \leq 1$$

Relative version of Kazhdan's theorem

Theorem (Aizenbud, Avni, Gourevitch - 2009)

Under certain conditions on the pair (G,H), for any natural n the exist N s.t. if F and F' are $N-$close then

$$\mathcal{S}(G(F)/H(F))^{K_n(F)} \simeq \mathcal{S}(G(F')/H(F'))^{K_n(F')}$$

as a

$$\mathcal{H}_{K_n(F)}(G(F)) \simeq \mathcal{H}_{K_n(F')}(G(F')) - module$$

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$$\psi(xy) = \phi(x)\psi(y).$$
- The module $\mathcal{S}(G(F)/H(F))^{K_n(F)}$ is finitely generated over the algebra $\mathcal{H}_{K_n}(G(F))$.

Proof of the 3-rd ingredient

Observation

If the module $\mathcal{S}(G(F)/H(F))^{K_n(F)}$ is finitely generated over the algebra $\mathcal{H}_{K_n}(G(F))$ for all n then

$$\forall \pi \in irr(G) : \dim Hom_H(\pi|_H, \mathbb{C}) < \infty.$$

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Theorem

The following are equivalent

- $\forall \pi \in irr(G) : \dim Hom_H(\pi|_H, \mathbb{C}) < \infty$
- *The module $\mathcal{S}(G(F)/H(F))^{K_n(F)}$ is finitely generated over the algebra $\mathcal{H}_{K_n}(G(F))$ for all n.*

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1. $\forall \pi \in irr(G) : \dim Hom_H(\pi|_H, \mathbb{C}) < \infty$
2. $\forall M$ *and a cuspidal* $\rho \in irr(M) : \dim Hom_{H_M}(\rho|_{H_M}, \mathbb{C}) < \infty$
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(2) $\Rightarrow$ (3) - follows from the theory of Bernstein center.
The hard part, (1) $\Rightarrow$ (2), follows from estimation of co-homologies. □

Applications

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Set $G := GL_{m+k}$ and $H = GL_m \times GL_k$

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Theorem (Jacquet, Rallis - 1996)

Let F be a local field of characteristic 0. Then the pair $(G(F), H(F))$ is a Gelfand pair i.e. for any irreducible representations π of $G(F)$ we have

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$\mathrm{Hom}_{H(F)}(\pi|_{H(F)}, \mathbb{C}) \cong \mathrm{Hom}_{G(F)}(\pi, C^\infty(G(F)/H(F))) \cong \mathrm{Hom}_{G(F)}(\mathcal{S}(G(F)/H(F)), \tilde{\pi})$

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$\square$

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Theorem (Aizenbud, Gourevitch, Rallis, Schiffmann - 2007)
Let F be a local field of characteristic 0. Then the pair $(GL_{n+1}(F), GL_n(F))$ is a strong Gelfand pair i.e. for any irreducible representations π of $GL_{n+1}(F)$ and τ of $GL_n(F)$ we have

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$\mathrm{Hom}_H(\pi|_H, \tau) \cong \mathrm{Hom}_H(\pi|_H, \tilde{\tilde{\tau}}) \cong \mathrm{Hom}_H(\pi|_H, \tilde{\tau}^*) \cong \mathrm{Hom}_{\Delta H}(\pi \otimes \tilde{\tau}, \mathbb{C})$ □