WF-holonomicity of constructible distributions on non-Archimedean local fields

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Distributions

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- Algebraic operations: $+, \cdot, \boxtimes$

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- ☹ No action of differential operators on distributions.

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☹ WF-holonomicity is not stable under Fourier transform.

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It is easy to see that WL is 1-1.

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Main ingredients of the proof.

- *Regularization:* a constructible distribution can be extended from an open set.
- *Resolution of singularities in the constructible (in fact, definable) setting.*
- *Key-Lemma:* a smooth constructible function on an open set can be extended to an holonomic constructible distribution.

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- By the induction assumption $p_*(\mu) - \eta$ is WF-holonomic.

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- Now we prove the Key lemma for the complement of the origin.

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Let f be a constructible function on an open (definable) set $U \subset V$. Then f can be extended to a constructible WF-holonomic distribution on V.

Idea of the Proof.

- WLOG we can assume that the function f has the form: $\psi(p_1)|p_2|val(p_3)$
- Using resolution of singularities we may assume that U is the complement of the coordinate hyper planes and $p_i = u_i m_i$, where u_i are units and m_i are monomials.
- ☹ While u_2 and u_3 can be ignored, u_1 cannot.
- Instead we can swallow it in m_1.
- Now we prove the Key lemma for the complement of the origin.
- We are using an inductive assumption both about the Key lemma and the main theorem.

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- Instead we can swallow it in m_1.
- Now we prove the Key lemma for the complement of the origin.
- We are using an inductive assumption both about the Key lemma and the main theorem.
- Adding 1 point does not affect WF-holonomicity.