Category Theory Spring 2015 Exercise 2

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Note on submission: Questions marked with **[S]** are important and for submission and grading. Questions marked with * are somewhat more difficult, usually requiring things that did not appear in the course - these are good exercises but not required for grading. Of course you may submit the other questions as well.

Categories:

- **Set** - category of sets
- **Grp** - category of groups
- **AbGrp** - category of abelian groups
- $\mathbf{UCalg}_{\Bbbk}$ - category of unital commutative algebras over $\Bbbk$
- **Top** - category of topological spaces
- **Ban** - category of Banach spaces
- $\mathbf{Vect}_{\Bbbk}$ - category of vector spaces over $\Bbbk$
- $\mathbb{C}^{\times}$- mod - category of modules over $\mathbb{C}^{\times}$
- $\mathrm{Open}(T)$ - category of open sets of the topological space T.

1. **[S]** Which of the above categories are additive and which are also abelian?
2. **[S]** For those that are abelian, compute the Grothendieck group.
3. * Find a category $\mathcal{C}$ with 0 s.t. for any X, Y we have some isomorphism $X \times Y \cong X \sqcup Y$ but $\mathcal{C}$ is not semi-additive.
4. Find a semi-additive category which is not additive.
5. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. Denote $\mathrm{Sub}(X)$ the category of subobjects of X. Show that for any $U, V \in \mathrm{Sub}(X)$ we have $\# \mathrm{Hom}_{\mathrm{Sub}(X)}(U, V) \leq 1$. We will see that this means that $\mathrm{Sub}(X)$ is a poset.
6. **[S]** Show that the following definitions of $\mathrm{Ker}(f)$ are equivalent:

(a) The equalizer of f with the zero map.

(b) The fiber product of f with the zero map (from 0).

(c) A subobject that maps to 0 via f and is terminal with this property.

7. Show that in a semi-additive category, the addition structure defines a structure of commutative monoid on the Hom spaces, and that composition is a bi-additive map

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$$

8. **[S]** Show that in an abelian category, for any morphism f, $\mathrm{Im}(f) = \mathrm{Ker}(\mathrm{Coker}(f))$ (equality here means that the RHS satisfies the condition defining the LHS).

9. **[S]** Show that in an abelian category a morphism is an isomorphism iff it is a monomorphism and an epimorphism.