Category Theory Spring 2015 Exercise 3

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1. **[S]** Prove that a fully faithful functor is one-to-one on isomorphism classes.

2. Prove that the Schur functor corresponding to the trivial representation is the symmetric power and the one corresponding to the sign representation is the exterior power.

3. **[S]** Prove that an equivalence between additive categories is automatically an additive functor.

4. Determine which of the following functors are full, faithful, essentially surjective or additive.

   (a) The obvious fuctors

   $$\mathbf{AbGrp} \to \mathbf{Grp} \to \mathbf{Mon} \to \mathbf{SemiGrp} \to \mathbf{Set}$$

   (b) The obvious functors

   $$\begin{aligned} \text{metric spaces with isometries} \to & \quad \text{metric spaces with Lipschitz maps} \\ \to & \quad \text{metric spaces with continuous maps} \\ \to & \quad \mathbf{Top} \end{aligned}$$

   (c) **[S]** The duality $V \mapsto V^*$ considered as a functor $\mathbf{Vect} \to \mathbf{Vect}^{op}$ or $\mathbf{Vect}_{fd} \to \mathbf{Vect}_{fd}^{op}$

   (d) **[S]** $\mathrm{Sym}^n : \mathbf{Vect} \to \mathbf{Vect}$, $\Lambda^n : \mathbf{Vect} \to \mathbf{Vect}$, $\mathrm{Sym}^n : \mathbf{Vect}_{1d} \to \mathbf{Vect}_{1d}$, $\Lambda^n : \mathbf{Vect}_{1d} \to \mathbf{Vect}_{1d}$ where $\mathbf{Vect}_{1d}$ is the subcategory of 1-dimensional vector spaces. Note that the answer depends on the field.

   (e) $F : \mathbf{Set} \to \mathbf{Vect}_{\Bbbk}$ defined by $X \mapsto \Bbbk^X$, and same for finite sets.

   (f) $\mathcal{C}$- some additive category. $F : \mathcal{C} \to \mathcal{C}$ defined by $X \mapsto X \oplus A$ for some $A \in \mathrm{Ob}(\mathcal{C})$.

   (g) R-an algebra. $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ defined by $X \to X \otimes_R M$ for some $M \in \mathrm{Mod}_R$.

5. Prove that the following pairs of categories are equivalent (or contra-equivalent). You may use the condition in question 6 without proving it.

(a) **[S]** Coverings of X and $\pi_1(X)$-sets (for appropriate X).

(b) The fundamental groupoid of a path connected topological space X (the category with objects points of X and morphisms paths between them up to homotopy) and the category of universal covers of X (i.e. connected covers that are simply connected).

(c) The categories in (b) and $B(\pi_1(X))$ (i.e. the category with one object with endomorphisms $\pi_1(X)$).

(d) Any category to a category where each isomorphism class has one object (this is sometimes called a "skeleton" for the category).

(e) Algebraic closures of a field $\mathbb{k}$ and $B(Gal(\bar{\mathbb{k}}, \mathbb{k}))$ for some fixed algebraic closure $\bar{\mathbb{k}}$ of $\mathbb{k}$.

(f) **[S]** Extensions of a field $\mathbb{k}$ and $Gal(\bar{\mathbb{k}}, \mathbb{k})$ transitive sets.

(g) G-principle spaces and $B(G)$ (was done in class, just verify all the details)

(h) $\mathbf{Vect}_{fd}$ and $\mathbf{Vect}_{fd}^{op}$

6. * Prove that if a functor is fully faithful and essentially surjective then it has an inverse.