Generalized Functions, Ex.3

1 Notation and Some Facts

2 QUESTION 1:

2.1 problem

Show that $C_c^\infty(\mathbb{R})$ is dance in $Dist(\mathbb{R})$, both in the strong and in the weak topology.

2.2 solution

As the strong topology is stronger, it will be sufficient to prove for the strong topology. As the solution we suggest is quite involved, we fix some notations:

- For $k > 0$ and a set $A \subseteq \mathbb{R}$, let $\rho_{A,k}(f) = \sum_{i=1}^{k}(sup_A\{|f^{(i)}(x)|\})$.
- Let ψ_n be a partition of unity of $\mathbb{R}$ with respect to $\{(-n-1/2, n+1/2) : n \in \mathbb{N}\}$. We can find such a partition with $\psi_{n+k}(x) = \psi_n(x-k)$, and we do so.
- For two sequences $\epsilon_i > 0$ and $\tau_i \in \mathbb{N}$ we let $U_{\{\epsilon_i\},\{\tau_i\}} = \{f \in C_c^\infty(\mathbb{R}) | \rho_{(-\infty,-n]\cup[n,\infty),k_n}(f) < a_n\}$. This is an open set in $C_c^\infty(\mathbb{R})$.
- Let $\chi_n := \sum_{k=-n}^{n}(\psi_k)$. This is a "bump function".

We solve the problem in two steps.

Step 1: In the strong topology, the space of compactly supported distributions is dance in the space of all distributions.

For given a distribution ξ. Let $T_k := sup_n\{||\chi_n^{(k)}||_\infty\}$ (this is $< \infty$ by construction). Consider the sequence $\xi_n := \chi_n \xi$. We claim that $\xi_n \xrightarrow{s} \xi$.

As ξ is a continuous functional on the space $C^\infty_{[-n,n]}(\mathbb{R})$, it is bounded with respect to one of the norms defining its topology. So, there's must be $k_n \in \mathbb{N}, C_n > 0$ such that

$$|\xi(f)| < C_n(\rho_{k_n,[-n,n]}(f)$$

. for every f with support in $[-n, n]$.

Let B be a bounded set, then for every $U = U_{\{\epsilon_i\},\{\tau_i\}}$, there is $\lambda > 0$ such that $\lambda U \supseteq B$.

So it will be sufficient to prove that $\xi_n - \xi$ converge to 0 uniformly on $U_{\{\lambda\epsilon_i\},\{\lambda\tau_i\}}$ for some choice of sequences ϵ_i, τ_i, and **every** λ.

But

$$\begin{aligned}
&|\xi_n(f)-\xi(f)| = |\xi((\chi_n-1)f)| < \\
&< \sum_{l=-N-1}^{-n+1}(|\xi(\psi_l f)|) + \sum_{l=n-1}^{N+1}(|\xi(\psi_l f)|) < \\
&< \sum_{l=-N-1}^{-n+1}(C_l\rho_{[-l-1,l+1],k_l}(\psi_l f)) + \sum_{l=n-1}^{N+1}(C_l\rho_{[-l-1,l+1],k_l}(\psi_l f)) = \\
&= \sum_{l=-N-1}^{-n+1}(C_l\rho_{[l-1,l+2],k_l}(\psi_l f)) + \sum_{l=n-1}^{N+1}(C_l\rho_{[l-1,l+2],k_l}(\psi_l f))
\end{aligned}$$

Now, by Liebnietz rule, $\rho_{A,k}(fg) < 2^k\rho_{A,k}(f)\rho_{A,k}(g)$. Substitute this, with $T_k = ||\psi_n^{(k)}||_\infty$ (which is clearly independent on n) we get

$$|\xi_n(f)-\xi(f)| < \sum_{l=-N-1}^{-n+1}(C_lT_{k_l}2^{k_l}\rho_{[l-1,l+2],k_l}(f)) + \sum_{l=n-1}^{N+1}(C_lT_{k_l}2^{k_l}\rho_{[l-1,l+2],k_l}(f)).$$

Take $\epsilon_{l+2} = (e^{-l}C_lT_{k_l}2^{k_l})^{-1}$, and $\tau_{l+2} = k_l$, we get that, if $f \in \lambda U_{\{\epsilon_i\},\{\tau_i\}}$ the last sum is bounded by

$$\begin{aligned}
&\sum_{l=-N-1}^{-n+1}(C_lT_{k_l}2^{k_l}\rho_{[l-1,l+2],k_l}(f)) + \sum_{l=n-1}^{N+1}(C_lT_{k_l}2^{k_l}\rho_{[l-1,l+2],k_l}(f)) < \\
&\sum_{l=-N-1}^{-n+1}(C_lT_{k_l}2^{k_l}(C_lT_{k_l}2^{k_l})^{-1}e^{-l}) + \sum_{l=n-1}^{N+1}(C_lT_{k_l}2^{k_l}(C_lT_{k_l}2^{k_l})^{-1}e^{-l}) < \\
&2\lambda(\sum_{l=n-1}^{N+1}e^{-l}) < 2\lambda\frac{e^{-n}}{1-e^{-1}} \to 0
\end{aligned}$$

Therefore, the convergence is uniform on B, and the strong convergense is proved.

Step 2: $C_c^\infty(\mathbb{R})$ is dance in the space of all compactly supported distributions.

Let $\xi \in Dist_c(\mathbb{R})$ be supported on $[-l,l]$. Let $\xi_n = \phi_n * \xi$, where $\phi_n(x) = n\phi(nx)$, $\phi(x)$ a positive function supported on $[-1,1]$ with total integral 1. These are smooth functions with compact support as convolution of smooth compactly supported function with compactly supported distribution is a smooth compactly supported function. We wish to show that $\xi_n \xrightarrow{s} \xi$. By definition $\xi_n(f) = \xi(\overline{g_n * \overline{f}})$. But as $supp(\xi) \subseteq [-l,l]$, $\xi(\overline{g_n * \overline{f}}) = \xi(\chi_{l+1}\overline{g_n * \overline{f}})$. As ξ is continuous, there is $k \in \mathbb{N}$ and $C > 0$ such that $|\xi(f)| < C\rho_{[-l-1,l+1],k}(f)$ for every f which is supported on $[-l-1,l+1]$. Therefore, for every $f \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
&|\xi_n(f)-\xi(f)| = \\
&|\xi(\chi_{l+1}(\overline{g_n * \overline{f}} - f))| \leq \\
&C\rho_{[-l-1,l+1],k}(\chi_{l+1}(\overline{g_n * \overline{f}} - f))
\end{aligned}$$

But, $||\overline{(g_n * \overline{f} - f)^{(k)}}||_\infty \leq 1/n||g_n||_1||f^{(k+1)}||_\infty = \frac{f^{(k+1)}}{n}$ as we integrate g_n against $f^{(k)}(x) - f^{(k)}(y)$ for $|y - x| < 1/n$. Substitute this, and using Liebnitz rule again, we get

$|\xi_n(f) - \xi(f)| < \frac{C2^k \rho_{[-l-1,l+1],k}(\chi_{l+1})\rho_{[-l-1,l+1],k+1}(f)}{n} \to 0$. and this convergence is uniform if we bound $\rho_{[-l-1,l+1],k+1}(f)$, which we can on any bounded set.

This completes the proof of step 3.

We proved that $C_c^\infty(\mathbb{R})$ is dance in $Dist_c(\mathbb{R})$ which is dance in $Dist(\mathbb{R})$. So $C_c^\infty(\mathbb{R})$ is dance in $Dist(\mathbb{R})$. ∎

3 Question 2

3.1 Problem

prove that $C^{-\infty}$ is a sheaf.

3.2 solution

For $V \supseteq U$ and $f \in C_c^\infty(U)$, let $ext_U^V(f)$ be the extention by 0 of f from U to V. We already saw in class that a distribution which vanish locally vanish also globally, using partition of unity. It remain to check the **patching condition**, namely that given a collection $\xi_i \in C^{-\infty}(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$, there is $\xi \in C^{-\infty}(\cap_i U_i)$ which agree with ξ_i on $C^{-\infty}(U_i)$. By compactness, we may restrict to finite covers. Let $U = U_1 \cap ... \cap U_n$, and consider the sequence

$$\bigoplus_{i<j} C_c^\infty(U_i \cap U_j) \xrightarrow{\beta} \bigoplus_i C_c^\infty(U_i) \xrightarrow{\alpha} C_c^\infty(U) \longrightarrow 0$$

where α is the sum of the natural inclusions, while $\beta(f_{i,j}) = ext_{U_i \cap U_i}^{U_j}(f_{i,j}) - ext_{U_i \cap U_i}^{U_i}(f_{i,j})$.

Let ξ_i be a compatible collection of distributions of the U_i-s. Consider the functional $\tilde{\xi} : \bigoplus_i C_c^\infty(U_i) \to \mathbb{R}$ given by the sum of the ξ_i-s. We wish to show that $\tilde{\xi}$ vanish on $Ker(\alpha)$, because then linearly it will descent to $Im(\alpha) = C_c^\infty(U)$, and the resulting functional will be continuous, as we can check on each $C_K^\infty(U)$ separately, and then $\alpha_K : \bigoplus_i C_K^\infty(U_i) \to C_K^\infty(U)$ is open by the open mapping theorem ($C_K^\infty(U)$ is Freshet).

Note that $\tilde{\xi} \circ \beta(\{f_{i,j}\}) = \sum_{i,j} \xi_i(f_{i,j}) - \xi_j(f_{i,j}) = 0$ by the compatibility of the collection ξ_i. Therefore, it will be sufficient to prove:

Lemma 3.1 *$Im(\beta) \supseteq Ker(\alpha)$*

We do this by induction on n. For $n = 1$, if $f_1 + f_2 = 0$ where $supp(f_i) \subseteq U_i, i = 1, 2$, then clearly $supp(f_i) \subseteq U_1 \cap U_2$ and we let $f_{1,2} = f_1$, and then $\beta(f_{1,2}) = (f_1, f_2)$.

Consider a collection of $n+1$ sets, $U_1, ..., U_{n+1}$. Let $(f_1, ..., f_{n+1})$ be in $Ker(\alpha)$. So $f1 + ... + f_{n+1} = 0$. For a point $x \in U_{n+1} - \bigcup_{i=1}^{n} supp(f_i)$, the only contribution to the sum $f_1(x) + ... + f_{n+1}(x)$ is of f_{n+1}, so f_{n+1} vanish outside of $\bigcup_{i=1}^n supp(f_i)$. This implies that $supp(f_{n+1}) \subseteq \bigcup_{i=1}^n (supp(f_i) \cap U_{n+1}) \subseteq \bigcup_{i=1}^n (U_i \cap U_{n+1})$. Let $\psi_1, ..., \psi_n, \psi$ be a partition of unity subordinate to the partition $U_{n+1} = (U_1 \cap U_{n+1} \cup ... \cup U_n \cap U_{n+1}) \cup (\bigcup_{i=1}^n supp(f_i))^C$. (this means that their sum is always 1 and each is supported in the corresponding open set. We saw that such things exists). Then, $f_{n+1} = \sum_{i=1}^n (\psi_i f_{n+1})$. taking $g_{i,n+1} = \psi_i f_{n+1}$ we get $(f_1, ..., f_{n+1}) - \sum_i \beta(g_{i,n+1}) = (h_1, ..., h_n, 0)$ for some $h_i \in C_c^\infty(U_i)$. By induction, $(h_1, ..., h_n, 0) = \beta(\gamma)$ for some γ. so $(f_1, ..., f_n) \in Im(\beta)$. ∎

4 Question 3:

4.1 Problem:

For $U \subseteq \mathbb{R}^n$ open, describe explicitly $\overline{C_c^\infty(\mathbb{R})}$

4.2 Solution:

Answer: $\overline{C_c^\infty(\mathbb{R})}$ is the space all functions which vanish outside U together with all of its derivatives.

One direction is clear. In $C_c^\infty(\mathbb{R}^n)$ the closure of a set is the set of all limits of elements in it, as it is a direct limit of Freshet spaces. If $f_k \to f$ with $f_k \in C_c^\infty(U)$, and $J \in \mathbb{N}^n$, then $D_J(f_n)(x) \to Df(x) = 0$ if $x \notin U$.

We shall prove the converse.

Consider the norm $\rho_{l,A}(f) = \sum\limits_{J \in \mathbb{N}^n, |J| \le l} ||D_J(f)||_{\infty,A}$. It follows from Liebnietz rule that

$$\rho_{l,A}(fg) \le 2^l \rho_{l,A}(f)\rho_{l,A}(g) \tag{1}$$

Moreover, by the residue formula of Taylor approximation in follows that for every $x, y \in \mathbb{R}^n$,

$$|f(x) - f(y) - p_{l,x}(y)| \le \rho_{l+1,[x,y]}(f)|x-y|^{l+1} \tag{2}$$

where $p_{l,x}$ is the Taylor expansion of f around x. We will prove the converse using these two estimates.

Let f be a function which vanish along with all of its derivatives outside U. Let $K = [-1,1]^n$ be the unit cube.

Let $\{\psi_I(x)\}_{I \in \mathbb{Z}^n}$ be a collection of $C_c^\infty(\mathbb{R}^n)$ functions, such that:

- $supp(\psi_I) = \prod_{i=1}^n [-1.1 + 2I_i, 1.1 + 2I_i]$,
- $\psi_I(x) = \psi_0(x - 2I)$,
- $\sum_I (\psi_I(x)) \equiv 1$.

It is not hard to construct such a collection. For each $m > 0$, consider the partition of unity obtained by rescaling this partition by $1/m$, that is, let $\psi_{I,m}(x) := \psi_I(mx)$. Consider the function

$$f_m(x) = \sum_{supp(\psi_{I,m} \subseteq U)} (\psi_{I,m}(x) f(x))$$

We claim that $f_m \to f$. To see this, fix a compact $[-M,M]^n = K \supseteq supp(f)$, and choose it large enough so that for every $m \in \mathbb{N}$, $\sum\limits_{supp(\psi_{I,m}) \subseteq K} \psi_{I,m}(x) = 1, x \in supp(f)$.

then K contains all the supports of all the f_n-s. Let $l > 0$, we have to show that

$$\rho_{l,K}(f_m - f) = \rho_{l,K}\Big(\sum_{supp(\psi_{I,m}) \cap U^C \neq \emptyset} (\psi_{I,m} f)\Big) \to 0.$$

But

$$\begin{aligned}
&\rho_{l,K}(\sum_{supp(\psi_{I,m})\cap U^C\neq\emptyset}(\psi_{I,m}f)) \leq && \text{(By Triangle Inequality)}\\
&\sum_{supp(\psi_{I,m})\cap U^C\neq\emptyset}\rho_{l,K}(\psi_{I,m}f) \leq && \text{(By Liebnietz rule)}\\
&\sum_{supp(\psi_{I,m})\cap U^C\neq\emptyset}2^l\rho_{l,K}(\psi_{I,m})\rho_{l,supp(\psi_{I,m})}(f)
\end{aligned}$$

Now, $\rho_{l,K}(\psi_{I,m})$ is independent of I, as all the ψ-s are translates of each other. Moreover, as it is a rescale of ψ_I, we get $\rho_{l,K}(\psi_{I,m}) \leq m^l\rho_{l,K}(\psi_I) = m^lC_l$ where $C_l = \rho_{l,\mathbb{R}^n}(\psi_I)$ is independent of I and K. Moreover, using (2), we can estimate $\rho_{l,supp(\psi_{I,m})}(f)$ as follows. Firstly, $Diam(supp(\psi_{I,m})) \leq \frac{3}{m}$. Let $x \in supp(\psi_{I,m}) \cap U^C$ and $y \in supp(\psi_{I,m})$. Let J be a multi-index with $|J| \leq l$. Then

$$\begin{aligned}
&|D_Jf(x)| = |D_J(f(x)) - D_J(f(y)) - p_{N,x,D_Jf}(y)| \leq\\
&\rho_{N+1,supp(\psi_{I,m})}(D_Jf)|x-y|^{N+1} \leq\\
&\rho_{N+|J|+1,supp(\psi_{I,m})}(f)Diam(supp(\psi_{I,m}))^{N-|J|+1} \leq\\
&\rho_{N+|J|+1,supp(\psi_{I,m})}(f)Diam(\frac{3}{m})^{N-|J|+1} \leq
\end{aligned}$$

We will specify N in a moment, if follows from these inequalities that
$\rho_{l,supp(\psi_{I,m})}(f) \leq \rho_{N+l+1,\mathbb{R}^n}(f)(\frac{3}{m})^{N-l+1}$.
Substitute these inequalities we get

$$\begin{aligned}
&\rho_{l,K}(f_m - f) \leq\\
&\sum_{supp(\psi_{I,m})\cap U^C\neq\emptyset}2^lC_lm^l\rho_{N+l+1,\mathbb{R}^n}(f)(\frac{3}{m})^{N-l+1} =\\
&\sum_{supp(\psi_{I,m})\cap U^C\neq\emptyset}2^lC_l3^{N-l+1}m^{2l-N-1}\rho_{N+l+1,\mathbb{R}^n}(f) \leq \quad \text{(As } |\{I : supp(\psi_{I,m}) \cap supp f \neq \emptyset\}| \leq (Mm)^n\text{)}\\
&M^nm^n2^lC_l3^{N-l+1}m^{2l-N-1}\rho_{N+l+1,\mathbb{R}^n}(f) =\\
&Cm^{2l+n-N-1}
\end{aligned}$$

Where C is a constant which doesn't depend on m (but it may depends on every other parameter, e.g. $l, N, f...$). Choose $N = 2l + n + 100$, the tail of the last inequality descent to 0. So $f_m \to f$ in the $\rho_{l,K}$ norm, and it follows that $f_m \to f$ in the topology of $C_c^\infty(\mathbb{R}^n)$. ∎

5 Question 4:

5.1 Problem

a. Find an example of a distribution $\xi \in C^{-\infty}_{\mathbb{R}^K}(\mathbb{R}^n)$, which is not in $\bigcup_{i=1}^\infty F_i$,
b. Show that locally $C^{-\infty}_{\mathbb{R}^K}(\mathbb{R}^n) = \bigcup_{i=1}^\infty F_i$.

5.2 Solution

a. Let $\xi = \sum_{i=1}^{\infty} \partial_{x_2}^i \delta_{(i,0,\ldots,0)}$. This distribution is clearly supported on $\mathbb{R}^k$ (even on $\mathbb{R}^1$), but for every $i \in \mathbb{N}$, this distribution don't vanish on $x_2^{i+1} \prod_{j=2}^n \phi(x_j)\phi(x_1 - i) \in F^i(\mathbb{R}^k, \mathbb{R}^n)$ where ϕ is as usual a bump function around 0, equal 1 in a neighborhood of 0 and supported in $[-1, 1]$.

b. Let $\xi \in C_{\mathbb{R}^K}^{-\infty}(\mathbb{R}^n)$, and let $p \in \mathbb{R}^k$. We wish to find $p \in U \subseteq \mathbb{R}^n$, $i \in \mathbb{N}$ and $\xi' \in F^i(\mathbb{R}^k, \mathbb{R}^n)$ such that $\xi'|_U = \xi|_U$. Let B be a closed ball centered at p. As ξ is a continuous functional on the Freshet space $C_B^\infty(\mathbb{R}^n)$, there is $N \in \mathbb{N}$ and $C > 0$ such that

$$\forall f \in C_B^\infty(\mathbb{R}^n), \quad |\xi(f)| \leq C \sum_{|J| \leq N} ||D_J(f)||_\infty.$$

Consider again the bump function ϕ. Let $f \in F^{2N+2}(\mathbb{R}^k, \mathbb{R}^n) \cap C_B^\infty(\mathbb{R}^n)$, and let $f_m = \prod_{j=k+1}^n \phi(mx_j) f$. Then $\xi(f_m) = \xi(f)$ because ξ is supported on $\mathbb{R}^k$. On the other hand, it is not hard to see (using the same estimations as in question 3) that $||D_J(f_m)||\ m \overset{\to}{\to} \infty\ 0$ for every J with $|J| < 2N + 2$, (informally, this is because each derivation of ϕ contribute m to the norm while $|D_J(f)| = o(m^{-N-2})$). I follows that $\xi(f) = \xi(f_m) \to 0$ and as ξ is continuous, $\xi(f) = 0$. Now let $\xi'(x) = \phi((Diam(B)|x - p|)^2)\xi(x)$, we have by the consideration above $\xi' \in F_{2N+2}$ while $\xi|_{1/2B} = \xi'_{1/2B}$. This proves that ξ is locally F_{2N+2} near p. ∎

6 Question 5

6.1 problem:

a. Show that the filtration $F_i(\mathbb{R}^n)_{\mathbb{R}^k}$ of $C^{-\infty}(\mathbb{R}^n)$ is invariant under diffeomorphisms preserving $\mathbb{R}^k$.
b. show that the splitting $F_i = F_{i-1} \oplus \bigoplus_{J \subseteq \mathbb{N}^{n-k}, |J| = i} D_J C^{-\infty}(\mathbb{R}^k)$ is not invariant under diffeomorphisms.

6.2 solution:

a. Let $\phi : (\mathbb{R}^n, \mathbb{R}^k) \to (\mathbb{R}^n, \mathbb{R}^k)$ ne a diffeomorphism. It will clearly be sufficient to show that $\phi^*(F_{\mathbb{R}^k}^i(\mathbb{R}^n)) \subseteq \phi^*(F_{\mathbb{R}^k}^i(\mathbb{R}^n))$. Let D_J be a differential operator of degree $\leq i$, and assume $D_J f|_{\mathbb{R}}^k = 0$. We have to show that $D_J(f \circ \phi)|_{\mathbb{R}}^k = 0$. Write $D_J = \partial_{x_{i_1}} \ldots \partial_{x_{i_l}}$. Then

$$D_J(f \circ \phi)(x) = \partial_{x_{i_1}} \ldots \partial_{x_{l-1}} d\phi(x)(\partial_{x_{i_l}})(f)(\phi(x)) =$$
$$\partial_{x_{i_1}} \ldots \partial_{x_{l-1}} (\sum_k a_k(x) \partial_{x_k} f(\phi(x))) =$$
$$\sum_k \partial_{x_{i_1}} \ldots \partial_{x_{l-1}} (a_k(x) \partial_{x_k} f(\phi(x))) = \text{(By Liebnietz)}$$
$$\sum_k \sum_{I \subseteq i_1, \ldots, i_{l-1}} C_I D_I a_k(x) D_{I^C \cap \{i_1, \ldots, i_l\}}(f(\phi(x)))$$

Where C_I are some binomial coefficients.

Now, $\partial_{x_{i_l}} f$ is in F^{i-1}. By induction (the case i = 0 is trivial...) $\partial_{x_{i_l}} f \circ \phi$ is also in F^{i-1}. We conclude that all the summands in the sum above vanish, so $D_J(f \circ \phi) = 0$ and $f \circ \phi \in F^i$.

$\blacksquare$

b. A counter example: Let $n = 1, k = 0, \phi(x) = x + x^3$. Then

$$
\begin{aligned}
&< \delta''', (f \circ \phi) >= (f(x + x^3))'''|_{x=0} = \\
&[(1 + 3x^2)f'(x + x^3)]''|_{x=0} = [(1 + 3x^2)^2 f''(x + x^3) + 6xf'(x + x^3)]'|_{x=0} = \\
&f'''(0) + 12f''(0) + 6f'(0)
\end{aligned}
$$

and therefore $\psi_*(\delta''') = \delta''' + 12\delta'' + 6\delta'$. Thuse the space $span\delta'''$ is not invariant under diffeomorphisms preserving the origin.