Generalized Functions, Lesson 4

1 Linear Algebra

1.1 Symmetric and Anti-Symmetric Powers

Let V be a f.d. linear space. Consider the tensor product $V^* \otimes V^*$. It has a natural decomposition $V^* \otimes V^* \cong Sym^2(V^*) \oplus \bigwedge^2(V^*) = SymBil(V \times V, F) \otimes AntiSymBil(V, F)$, where $SymBil(V \times V, F)$ is the space of symmetric bilinear functions on V and $AntiSymBil(V, F)$ is the space of anti-symmetric bilinear forms. In general (for chararcteristic 0 only!) $V \otimes V$ is a direct sum of symmetric tensors $Sym^2(V) = span\{v \otimes u + u \otimes v\}$ and ati-symmetric tensors $\bigwedge^2(V) = span\{v \otimes u - u \otimes v\}$. One could define this spaces as quotient of $V \otimes V$ by appropriate relations, or as the universal domain for symmetric (resp. anti-symmetric) bilinear functions on V, i.e. $Hom(Sym^2(V), W) = SymBil(V \times V, W)$ and so on.

1.2 Some Facts on Tensor Product

Think of $\otimes$ as a multiplication operation on vector spaces, we can ask who is the identity and who are the invertibles under this operation. The identity is the trivial line F, as the multiplication by scalar map $F \otimes V \to V$ is a **canonical** isomorphism.

[Note: In this informal discussion we want to identify to spaces only if they are canonically isomorphic, whatever it means, since in the future, when a group action will be involved, there will be a big difference between canonical and non-canonical isomorphisms. So we write canonical morphisms as $\overset{\rightarrow}{can}$]

The answer for the invertibles question is also not surprising. these are the one dimentional linear spaces. On one hand, if $V \otimes W \cong_{can} F$ then $dim(V)dim(W) = dim(V \otimes W) = 1$ so V is one dimentional. On the other hand, we have a natural map $V \otimes V^* \to_{can} F$, namely the pairing $v \otimes \xi \to < \xi, v >$ between vectors and functionals. If $dim(V) = 1$ this canonical map is and isomorphism and then "$V^{-1} = V^*$".

Note that we also have a canonical map $F \to V \otimes V^*$ given by

$F \overset{1 \mapsto Id}{\to}_{can} End(V) \cong_{can} V \otimes V^*$. Under the identification $End(V) \cong_{can} V \otimes V^*$, the pairing map introduced above corresponds to the trace map $End(V) \to F$. This explains for example why the trace is "canonical", or invariant under conjugation by an invertible matrix.

2 Distributions with Values in a Vector Space

Definition 2.1 *$C_c^\infty(X, V)$ is the space of smooth functions with compact support from X to V, with the same convergence condition as in the usual $V = F$ case. Here the smoothness of a function is the usual coordinate-wise one.*

Exercise 2.1 *Prove that* $C_c^\infty(X,V) \cong C_c^\infty(X) \otimes_F V$ *as topological vector-spaces, the topology on* $C_c^\infty(X) \otimes_F V$ *is given by choosing a basis to identify* V *with* F^n *and then take the product topology on* $C_c^\infty(X) \otimes_F F^n \cong_{can} (C_c^\infty(X))^n$. *In particular, this topology is independent on a choice of a basis.*

2.1 Measures

By a measure we mean a measure defined on the algebra of Borel subsets of a locally compact top. space X. For us, the following definition is better:

Definition 2.2 *The space of signed measures on* X *is* $C_c(X)^*$, *i.e. a continuous functional on* $C_c^\infty(X)$. *A signed measure is a measure if it is non-negative on non-negative function.*

Let X be a f.d. linear space. As the space $C_c(X)$ is larger than $C_c^\infty(X)$, its dual is smaller. Specifically $C_c(X)^* \subseteq C_c^\infty(X)^*$, the inclusion is the dual of the obvious continuous map $C_c^\infty(X) \to C_c(X)$. Inside $C_c(X)^*$ there is a one dimensional space of Haar measures, which in this case is just the space of multiples of the Lebesgue measure.

Definition 2.3 *Let* X *be a linear space. The space of Haar measures on* V, *denoted* $h_V \subseteq C_c(V)^*$, *is the space of translation invariant measures.*

The fact that this space is one dimensional is non-trivial, but the intuition is as follows: A Borel measure on X is determined by its value on cubes with sides parallel to the axes planes of rational side length, as they form basis of the topology. It is not hard to see that if the measure is translation invariant, the measures of these cubes are determined by the measure of the unit cube.

Exercise 2.2 *Let* $\xi \in C_c^\infty(X)^*$ *which is translation invariant. Prove that* ξ *is a Haar measure.*

We are interested in smooth measures, where:

Definition 2.4 *A measure* μ *on* X *is a* ***smooth measure*** *if* $\mu \in C^\infty(X)h_X$, *i.e.* $\mu = f(x)h$ *where* f *is smooth and* h *is a Haar measure.*

Note that by definition, the space of smooth measures on X is isomorphic to $C_c^\infty(V, h_V)$.

2.2 Generalized Functions Versus Distributions

We are now (hopefully :-)) in position to understand the difference between generalized functions and distributions.

Definition 2.5 *A* ***distribution*** *on* X *is continuous functional on the space of smooth functions with compact support:*

$$Dist(X) := C_c^\infty(X)^*$$

.

Definition 2.6 *A* ***Generalized Function*** *is a continuous functional on the space of smooth measures with compact support on* X, *i.e.*

$$C^{-\infty}(X) := C_c^\infty(X, h_X)^*.$$

As functions can be integrated against smooth measures, we have a pairing $C_c^\infty(X,h_X) \times C_c^\infty(X) \overset{<,>}{\to} F$. Though we have the following picture:

$$\begin{tikzcd}
C^{-\infty}(X) & Dist(X) \\
C_c^\infty(X) \arrow[u, hook] \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] & C_c^\infty(X,h_X) \arrow[u, hook] \arrow[ul, leftrightarrow]
\end{tikzcd}$$

We can also define generalized functions with value in a vector space, by

$$C^{-\infty}(X,V) := C^{-\infty}(X) \otimes V = C_c^\infty(X, h_X \otimes V^*)^*$$

. and then

$$C^{-\infty}(X,h_X) = C^{-\infty}(X) \otimes h_X = C_c^\infty(X)^* = Dist(X)$$

2.3 Generalized Functions With Support on a Subspace

Let $Y \subseteq X$ be a linear subspace. Recall that inside $C_c^\infty(X)^* = C^{-\infty}(X,h_X)$ there is a subspace $C_Y^{-\infty}(X,h_X) \subseteq C^{-\infty}(X,h_X)$ of distributions supported on Y.

This is locally the union of a filtration given as follows: Let $F_Y^i(X) = \{f \in C_c^\infty(X) : D_J f|_Y = 0, |J| \leq i\}$. Let $F_i(X)_Y = F_Y^i(X)^\perp$. We want to describe $F_i(X)_Y / F_{i-1}(X)_Y$ in canonical terms, i.e. in a way invariant under diffeomorphisms preserving Y.

Theorem 2.1

$$F_i(X)_Y / F_{i-1}(X)_Y \cong_{can} C_c^\infty(Y, Sym^i(Y^\perp))^*.$$

Note that $Y^\perp = (X/Y)^*$ and therefore $Sym^i(Y) = SymPoly(X/Y, ..., X/Y; \mathbb{R})$. the theorem is based on the following lemma:

Lemma 2.2 $F_i(X)_Y / F_{i-1}(X)_Y \cong (F_Y^{i-1}(X)/F_Y^i(X))^*$

Proof For $\phi \in F_i(X)_Y$, $\phi|_{F_Y^{i-1}(X)}$ vanish on $F_Y^i(X)$, and we send it to the iduced functional on $F_Y^{i-1}(X)/F_Y^i(X)$, $\overline{\phi|_{F_Y^{i-1}(X)}}$. This is an injective morphism, as if $\bar{\phi} = 0$ then $\phi|_{F_Y^{i-1}(X)} = 0$ so $\phi \in F_{i-1}(X)_Y$. The surjectivity of it follows from Haan-Banach theorem

Using this lemma, in order to prove the theorem it will be sufficient to prove that $F_Y^{i-1}(X)/F_Y^i(X) \cong C_c^\infty(Y, Sym^i(Y^\perp))$. What do we need to define foe each function? well, a smooth function which gives a symmetric multi-linear function on i variables from X/Y to $\mathbb{R}$. We do the natural thing- attach to f its i-th derivatives. In formula: Let

$$\Phi(f)(y)(v_1, ..., v_i) = \partial_{v_1} ... \partial_{v_i} f(y).$$

It is well defined as f vanish identically on Y, so this form kills all the tangential derivatives. It is one-to-one as if $\Phi(f) = 0$, then f vanish with all of its derivatives up to degree i so it is in $F^i(X)_Y$.

Exercise 2.3 *prove that Φ is onto, hence an isomorphism.*

This result is, in a sense, the first "Generalized-Functions Theoretic" non-trivial result we proved. It is useful if one want to prove a local statement on distributions/generalized-functions with support, one can do it grade-by-grade using this description.

For the generalized-functions case, we get the same result by twisting with Haar measures. Indead, $F_i(X)_Y/F_{i-1}(X)_Y \cong C_c^{\infty}(Y, Sym^i(Y^{\perp}))^* = C^{-\infty}(Y, Sym^i(Y^{\perp}) \otimes h_Y)$. Take $\bar{F}_i(X)_Y = F_i(X)_Y \otimes h_X^* \subseteq C^{-\infty}(X)$. We get, by the compatibility of tensor and quotient,

$$\bar{F}_i(X)_Y/\bar{F}_{i-1}(X)_Y \cong C^{-\infty}(Y, Sym^i(Y^{\perp}) \otimes h_Y) \otimes h_X^* \cong C^{-\infty}(Y, Sym^i(Y^{\perp}) \otimes h_Y \otimes h_X^*)$$

But what is this (one dimensional) space $h_Y \otimes h_X^*$?

Exercise 2.4 *prove:*

- *If $Y \subseteq X$, then $h_Y \otimes h_{X/Y} \cong_{can} h_X$.*
- *$h_X^* = h_{X^*}$.*

It follows that $h_X^* \otimes h_Y = (h_X \otimes h_Y^*)^* = h_{X/Y}^* = h_{(Y^{\perp})^*}^* = h_{Y^{\perp}}$.

So,

Theorem 2.3

$$\bar{F}_i(X)_Y/\bar{F}_{i-1}(X)_Y \cong C^{-\infty}(Y, Sym^i(Y^{\perp}) \otimes h_{Y^{\perp}})$$

Remark 2.1 *Knowing how distributions supported on Y looks like can give information on X. This is because the exact sequence*

$$0 \longrightarrow C_Y^{-\infty}(X) \longrightarrow C^{-\infty}(X) \longrightarrow C^{-\infty}(X - Y)$$

which is not short exact, but it will be in the case of Shchwartz distributions.

3 More Linear Algebra

Let V be an n-dim linear space. Let $\Omega^n(V)$ be the space of anti-symmetric n-forms on V. It is a one-dimensional space, and $\Omega^n(V) \cong \bigwedge^n(V) \subseteq (V^*)^{\otimes_n}$. Let $\mathcal{V}$ be the space of bases of V. We can make the identification $\Omega^n(V) = \{f : \mathcal{B} \to \mathbb{R} : f(B_1) = det(M_{B_2}^{B_1})f(B_2)\}$. Then, we have two related spaces, the space of densities and the space fo orientations:

$$Dens(V) = \{f : \mathcal{B} \to \mathbb{R} : f(B_1) = |det(M_{B_2}^{B_1})|f(B_2)\}$$

$$Ori(V) = \{f : \mathcal{B} \to \mathbb{R} : f(B_1) = sign(det(M_{B_2}^{B_1}))f(B_2)\}$$

Exercise 3.1 *$\Omega^n(V) \cong Dens(V) \otimes Ori(V)$, via the tensor product fo the natural maps $\Omega^n(V) \to Dist(V)$ and $\Omega^n(V) \to Ori(V)$.*

Note that this space of orientation is a linear space and not two points as one expect from orientation. On the other hand, we have two distinguished points in $Ori(V)$, the two functions with absolute value 1. These are the usual orientations we used to think of. About densities, these are things we already know:

Exercise 3.2 *$Dens(V) \cong_{can} h_V$.*

We end up with the following definition, which will become important in the manifold case:

Definition 3.1 *A **top differential form** on V is an element of $C^{\infty}(V, \omega^n(V))$.*

to conclude, here are some linear algebra exercises added after the lecture:

Exercise 3.3 *prove that the following definitions of symmetric square are equivalent: a) a subspace of the tensor square which is generated by symmetric expressions (e.g. $v \otimes w + w \otimes v \in V \otimes V$)*
a') the subspace of the tensor square of all elements invariant under the flip automorphism $s : V \otimes V \to V \otimes V$. In order to define the flip automorphism, note that in general we have isomorphism $s : V \otimes W \to W \otimes V$. This isomorphism in the case $V = W$ is the flip automorphism.
b) the quotient of $V \otimes V$ by the relation generated by $s(v)$ v.
c) Universal property: a linear space $Sym^2(V)$ together with a symmetric bilinear map $V \times V to Sym^2(V)$ s.t. any symmetric bilinear map $V \times V to L$ factors in a unique way like this $V \times V to Sym^2(V) \to L$, where the second map is linear.

Exercise 3.4 *repeat the last Ex. for symmetric and anti-symmetric power.*

Exercise 3.5 *compute $dim(\bigwedge^k(V))$ and describe combinatorially $dim(Sym^k(V))$.*

That's it for now :-)