Generalized Functions: Homework 1

Exercise 1. *Prove that there exists a function $f \in C_c^\infty(\mathbb{R})$ which isn't the zero function.*

Solution: Consider the function:

$$g = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

We claim that $g \in C^\infty(\mathbb{R})$. Indeed, all of the derivatives of $e^{-\frac{1}{x}}$ are vanishing when $x \to 0$:

$$\lim_{x \to 0} \frac{d^n}{dx^n} e^{-\frac{1}{x}} = \lim_{x \to 0} q(x) \cdot e^{-\frac{1}{x}} = 0$$

where $q(x)$ is a rational function of x. The function $f = g(x) \cdot g(1-x)$ is smooth, as a multiplication of two smooth functions, has compact support (the interval $[0,1]$), and is non-zero ($f\left(\frac{1}{2}\right) = \frac{1}{e}$).

Exercise 2. *Find a sequence of functions $\{f_n\}_n \in \mathbb{Z}$ (when $f_n \in C_c^\infty(\mathbb{R})\ \forall n$) that weakly converges to Dirac Delta function.*

Solution: Consider the sequence of function:

$$\tilde{f_n} = g\left(\frac{1}{n} - x\right) \cdot g\left(\frac{1}{n} + x\right) \in C_c^\infty(\mathbb{R})$$

where $g(x) \in C_c^\infty(\mathbb{R})$ is the function defined in the first question. Notice that the support of f_n is the interval $\left[-\frac{1}{n}, \frac{1}{n}\right]$. Normalizing these functions so that the integral will be equal to 1 we get:

$$f_n = d(n)\tilde{f_n}, \quad d(n) = \left(\int_{-\infty}^{\infty} \tilde{f_n}(x)dx \right)^{-1}$$

For every test function $F(x) \in C_c^\infty(\mathbb{R})$, we can write $F(x) = F(0) + x \cdot G(x)$, where $G(x) \in C_c^\infty(\mathbb{R})$. Therefore we have:

$$\begin{aligned} \int_{-\infty}^{\infty} F(x) \cdot f_n(x)dx &= \int_{-1/n}^{1/n} F(x) \cdot f_n(x)dx \\ &= \int_{-1/n}^{1/n} F(0) \cdot f_n(x)dx + \int_{-1/n}^{1/n} xG(x) \cdot f_n(x)dx \\ &= F(0) + \int_{-1/n}^{1/n} xG(x) \cdot f_n(x)dx \end{aligned}$$

So $\int_{-\infty}^{\infty} F(x)\cdot f_n(x)dx - \int_{-\infty}^{\infty} F(x)\cdot\delta(x)dx = \int_{-1/n}^{1/n} xG(x)\cdot f_n(x)dx$. Notice that $G(x)$ is bounded by some $M\in\mathbb{R}$ and thus we get:

$$\begin{aligned}\left|\int\limits_{-1/n}^{1/n} xG(x)\cdot f_n(x)dx\right| &\le \int\limits_{-1/n}^{1/n} \left|xG(x)\cdot f_n(x)\right| dx \\ &\le \int\limits_{-1/n}^{1/n} \frac{1}{n}M\cdot f_n(x)dx \\ &= \frac{M}{n} \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

Therefore, the sequence of functions f_n weakly converges to Dirac Delta function.

Exercise 3. *Find a function $F\in L^1_{loc}$ for which $F'=\delta$.*

Solution: Consider Heaviside step function:

$$\Theta = \begin{cases} 1 & x\ge 0 \\ 0 & x<0\end{cases}$$

and let ξ_Θ be the functional defined by Θ. Then for every test function $\varphi\in C_c^\infty(\mathbb{R})$:

$$\begin{aligned}\xi'_\Theta(\varphi) &= -\xi_\Theta(\varphi') = -\int\limits_{-\infty}^{\infty}\varphi'(x)\cdot\Theta(x)dx \\ &= -\int\limits_{0}^{\infty}\varphi'(x)dx = -\left(\lim_{x\to\infty}\varphi(x)-\varphi(0)\right) \\ &= \varphi(0)=\delta(\varphi)\end{aligned}$$

Therefore, as distributions, $\Theta'=\delta$.

Exercise 4.

1. *Let U_1, U_2 be open subsets of $\mathbb{R}$. Show that if $\xi|_{U_1}\equiv\xi|_{U_2}\equiv 0$ then $\xi|_{U_1\cup U_2}\equiv 0$.*
2. *Show this also holds for any union of such open $\{U_i\}_{i\in I}$.*

Solution:

1. We define a smooth step function:

$$s(x) = \begin{cases} e^{-e^{-\tan\left(\frac{\pi x}{2}\right)}} & -1\le x\le 1 \\ 1 & 1<x \\ 0 & x<-1\end{cases}$$

and use it to write $f \in C_c^\infty(\mathbb{R})$, $supp(f) = U_1 \cup U_2$, as a sum $f_1 + f_2$ where $supp(f_i) = U_i$. Notice that $s(nx)$ is a step of width $\frac{1}{2n}$. Suppose $U_i = (a_i, b_i)$ are open intervals (in case U_i are finite union of open intervals the generalization is obvious; the case of infinite union is a result of section (b) of this question). If $U_1 \cap U_2 = \emptyset$ then $f_i = f|_{U_i}$. Otherwise, one of the edges of U_1 is in the interior of U_2, wlog $a_1 \in (a_2, b_2)$. Then:

$$f_1 \equiv f \cdot s\left(n\left(x - a_1\right)\right), \quad \frac{1}{2n} < \frac{1}{2}(b_2 - a_1)$$

and then take $f_2 = f - f_1$. Notice that $supp\left(f_2\right) \subset U_2$, since we chose n properly. Now, $\xi(f) = \xi\left(f_1\right) + \xi\left(f_2\right) = 0$, i.e. $\xi|_{U_1 \cup U_2} = 0$.

2. Notice that we are only interested in $\left\{U_i\right\}_{i \in I} \cap supp\left(f\right)$. In addition we can assume U_i are open intervals (by splitting U_i to its components). The union $\left\{U_i\right\}_{i \in I}$ covers $supp\left(f\right)$, which is compact, and therefore there is a finite sub-cover $\left\{U_{ij}\right\}_{j=1}^n$. So it is enough to present f as a sum $\sum_{j=1}^n f_j$ where $supp\left(f_j\right) = U_{ij}$. This we do as in the previous section.

Exercise 5. *Find all the generalized functions $\xi \in C_c^{-\infty}(\mathbb{R})$ for which $supp(\xi) = \{0\}$.*

Solution: For every $f \in C_c^\infty(\mathbb{R})$, $supp\left(f\right)$ is compact, and therefore, if $0 \notin supp\left(f\right)$ there exists a neighborhood of 0 in which $f = 0$. Therefore, for every n, $supp\left(\delta^{(n)}\right) = \{0\}$, and thus every generalized function of the form $\xi = \sum_{i=1}^n c_i \delta^{(i)}$ where $c_i \in \mathbb{R}$, has zero support. Now, suppose $\xi \in C^{-\infty}$, $supp\left(\xi\right) = \{0\}$, then by definition, $\xi|_{C_c^\infty(\mathbb{R}\setminus\{0\})} = 0$. Clearly this means that $\xi|_{\overline{C_c^\infty(\mathbb{R}\setminus\{0\})}} = 0$. For every $f \in C_c^\infty(\mathbb{R})$ such that $\forall n,\ f^{(n)}(0) = 0$, we can define a sequence $f_n(x) = w(nx) \cdot f(x)$, where $w_n(x) = 1 - s\left(2nx + 3\right) \cdot s\left(3 - 2nx\right)$ is a smooth symmetric "flat bump" function, that goes down from 1 to 0 in $\left[-\frac{2}{n}, -\frac{1}{n}\right]$ and up again in $\left[\frac{1}{n}, \frac{2}{n}\right]$. Notice that $f_n \in C_c^\infty(\mathbb{R}\setminus\{0\})$, and that $f_n \underset{n\to\infty}{\longrightarrow} f \in \overline{C_c^\infty(\mathbb{R}\setminus\{0\})}$, since at $x = 0$, $\forall k,\ f_n^{(k)}(x) = f^{(k)}(x) = 0$, and for $x \neq 0$, $f_n \equiv f$ in a neighborhood of x for n large enough. Define $\psi : C_c^\infty(\mathbb{R}) \to \mathbb{R}^\infty$ to be the vector of derivatives at zero:

$$\psi(f) = (f(0), f'(0), .., f^{(n)}(0), ...) \in \mathbb{R}^\infty$$

Clearly ψ is linear, and $\xi|_{\ker\psi} = 0$. Thus there exists a quotient map $\tilde{\xi} : L \to \mathbb{R}$ such that $\xi = \tilde{\xi} \circ \psi$.

$$\begin{array}{ccccc} C_c^\infty(\mathbb{R}) & \xrightarrow{\psi} & L & \subset \mathbb{R}^\infty \\ & \searrow^{\xi} & \downarrow^{\tilde{\xi}} & \\ & & \mathbb{R} & \end{array}$$

From this we can conclude that $\xi(f)$ must be a linear function, depending only on the derivatives of f at zero. Now notice that regarding the product topology on $\mathbb{R}^\infty$, the map ψ is continuous (wrt the topology we defined on $C_c^\infty(\mathbb{R})$, generated by the balls $b_{k,g,\varepsilon} = \left\{f : \left|f^{(k)}(x) - g^{(k)}(x)\right| < \varepsilon\right\}$):

$$\psi^{-1}(U) = \psi^{-1}(U_1 \times ... \times U_n \times \mathbb{R}^\infty) = \cap_{i=1}^n V_i$$

where V_i are unions of balls $b_{i,g,\varepsilon}$ and thus are open. So with respect to the product topology ψ is indeed continuous, and it is easy to see that if we would "allow" infinite product of opens to be open in $\mathbb{R}^\infty$, then we would get above infinite intersection of open sets which is in general not open. Now, for an open $V \in C_c^\infty(\mathbb{R})$, $V = \cup_{k,g,\varepsilon} b_{k,g,\varepsilon}$:

$$\psi(V) = \psi(\cup_{k,g,\varepsilon} b_{k,g,\varepsilon}) = \cup_{k,g,\varepsilon}\psi(b_{k,g,\varepsilon}) = \cup_{k,g,\varepsilon}\underbrace{\mathbb{R} \times .. \times \mathbb{R}}_{k-1} \times U_{k,g,\varepsilon} \times \mathbb{R}^\infty$$

where $U_{k,g,\epsilon} = \left\{f^{(k)}(0) : \left|f^{(k)}(x) - g^{(k)}(x)\right| < \varepsilon\right\} = \left\{f^{(k)}(0) : \left|f^{(k)}(0) - g^{(k)}(0)\right| < \varepsilon\right\}$. The last equality holds since for every element $f^{(k)}(0)$ in the RHS, we can define $\tilde{f}(x) = g(x) + \frac{x^k}{k!}\left(f^{(k)}(0) - g^{(k)}(0)\right)$ so:

$$\begin{aligned}\left|\tilde{f}^{(k)}(x) - g^{(k)}(x)\right| &= \left|g^{(k)}(x) + \left(f^{(k)}(0) - g^{(k)}(0)\right) - g^{(k)}(x)\right| \\ &= \left|f^{(k)}(0) - g^{(k)}(0)\right| < \varepsilon\end{aligned}$$

(since $f^{(k)}(0)$ is an element of the RHS). Therefore, $f^{(k)}(0) = \tilde{f}^{(k)}(0)$ is an element of the LHS. Now, the set $\left\{f^{(k)}(0) : \left|f^{(k)}(0) - g^{(k)}(0)\right| < \varepsilon\right\}$ is clearly open in $\mathbb{R}$, and thus $\psi(V)$ is open in $\mathbb{R}^\infty$, and ψ is an open map. Notice that ξ is continuous wrt to the mentioned topology on $C_c^\infty(\mathbb{R})$, and the usual topology on $\mathbb{R}$, and ψ is open (and continuous). Therefore $\tilde{\xi}$ is also continuous with respect to the product topology on $\mathbb{R}^\infty$ and the usual topology on $\mathbb{R}$, and would not be continuous wrt a finer topology on $\mathbb{R}^\infty$. Notice that since L is dense in $\mathbb{R}^\infty$, we can extend $\tilde{\xi}$ continuously to a function from $\mathbb{R}^\infty$ (from now on we regard $\tilde{\xi}$ as a map from $\mathbb{R}^\infty$ to $\mathbb{R}$). From this we conclude that ξ must by finite sum of $\delta^{(n)}$.

Exercise 6. *Show that for $\phi \in C_c^\infty(\mathbb{R})$ we get that $\xi * \phi$ is a smooth function.*

Solution: We consider the convolution $(\xi * \phi)(x) = \xi\left(\tilde{\phi_x}\right)$. Since $\tilde{\phi_x}(t) = \phi(x-t)$ is smooth and has compact support for every x, and ξ is a distribution, $\forall x\ \xi * \phi(x)$ is well defined. Since $\tilde{\phi_x} = \phi(x-t)$ is smooth with respect to x, and

ξ is a linear functional, $\xi * \phi(x)$ is continuous.

$$\begin{aligned}
\frac{d}{dx}(\xi * \phi)(x) &= \lim_{h\to 0} \frac{\xi\left(\tilde{\phi}_{x+h}(t)\right) - \xi\left(\tilde{\phi}_x(t)\right)}{h} \\
&\underset{\xi \text{ is linear}}{=} \lim_{h\to 0} \xi\left(\frac{\tilde{\phi}_{x+h}(t) - \tilde{\phi}_x(t)}{h}\right) \\
&= \lim_{h\to 0} \xi\left(\frac{\phi(x+h-t) - \phi(x-t)}{h}\right) \\
&\underset{\xi \text{ is continuous}}{=} \xi\left(\lim_{h\to 0}\frac{\phi(x+h-t) - \phi(x-t)}{h}\right) \\
&= \xi\left(\phi'(x-t)\right) = \xi * \phi'
\end{aligned}$$

This means that the function $\xi * \phi(x)$ has continuous derivatives of every order $(\xi * \phi^{(n)})$, i.e. $\xi * \phi(x)$ is smooth.

Exercise 7. *1. Show that $\delta * \eta = \eta$.*

2. Show that $\delta' * \eta = \eta'$.

3. Show the associativity: $\delta' * (\xi * \eta) = (\delta' * \xi) * \eta$.

4. Show that $(\xi * \eta)' = \xi' * \eta = \xi * \eta'$.

Solution:

1. By definition:

$$\begin{aligned}
(\delta * \eta)(\varphi) &= (\eta * \delta)(\varphi) = \eta\left(\overline{(\delta * \bar{\varphi})}\right) \\
&= \eta\left(\overline{\delta(\bar{\tilde{\varphi}}_x)}\right) = \eta\left(\overline{\overline{\varphi(x)}}\right) \\
&= \eta(\varphi)
\end{aligned}$$

   Notice that above, δ receives a function of t and η receives a function of x.

2.

$$\begin{aligned}
(\delta' * \eta)(\varphi) &= \eta\left(\overline{(\delta' * \bar{\varphi})}\right) \\
&= \eta\left(\overline{\delta'(\bar{\tilde{\varphi}}_x)}\right) = \eta\left(-\overline{\overline{\varphi'(x)}}\right) \\
&= \eta(-\varphi') = -\eta(\varphi') \\
&= \eta'(\varphi)
\end{aligned}$$

3.

$$\begin{aligned}
((\delta' * \xi) * \eta)(\varphi) &= (\delta' * \xi)\left(\overline{(\eta * \bar{\varphi})}\right) \\
&= \delta'\left(\overline{(\xi * (\eta * \bar{\varphi}))}\right) \\
&= \delta' * \left(\xi\left(\overline{(\eta * \bar{\varphi})}\right)\right) \\
&= (\delta' * (\xi * \eta))(\varphi)
\end{aligned}$$

4. Using the previous, $(\xi * \eta)' = \delta' * (\xi * \eta) = (\delta' * \xi) * \eta = \xi' * \eta$, and since $\xi * \eta = \eta * \xi$ we also get: $(\xi * \eta)' = \delta' * (\eta * \xi) = (\delta' * \eta) * \xi = \eta' * \xi = \xi * \eta'$.

Exercise 8. *Show that if φ is smooth, and supp(ξ) is compact, then $\xi * \varphi$ will still be smooth.*

Solution: The difference from question 6 is that φ is not compactly supported, so the integral (on unbounded domain) may diverge. However, since $supp(\xi)$ is compact, it is contained in an interval $[a, b]$. Thus $\forall g$ such that $g|_{[a,b]} = 0$ we have $\xi(g) = 0$. Multiplying by the step function from question 4(a) we can write $\varphi = \varphi_1 + \varphi_2$ such that $\varphi_1|_{[a,b]} = 0$ and $\varphi_2|_{\mathbb{R}\setminus[a-1,b+1]} = 0$. Then we have:

$$\begin{aligned}
(\xi * \varphi)(x) &= \xi(\varphi(x-t)) = \xi(\varphi_1(x-t) + \varphi_2(x-t)) \\
&= \xi(\varphi_2(x-t))
\end{aligned}$$

But since φ_2 is compactly supported, by question 6 $\xi(\varphi_2(x-t))$ is smooth.

Exercise 9. *Let A be a differential operator with fixed coefficients. Describe the Green function without using generalized functions.*

Solution: We defined the Green function in class as the solution $AG(t) = \delta_0$so when $t \neq 0$, $AG(t) = 0$. Notice that the Green function has to have a jump on it's nth derivative (where n is the order of A), since the integral in a small neighborhood of zero does not vanish. Specifically, Suppose $A = a(x)\partial^n + ...$, then:

$$\int_{-\varepsilon}^{\varepsilon} AGdx = \int_{-\varepsilon}^{\varepsilon} a(x)\partial^n G + ...dx = \int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1$$

$$\Rightarrow G^{(n)}(0_+) - G^{(n)}(0_-) = \frac{1}{a(0)} \quad (*)$$

Notice that G that satisfy $(*)$ indeed acts like a green function: Let $G(t)$ be a function such that $\forall t \neq 0,\ AG(t) = 0$, and such that $(*)$ holds, then:

$$\begin{aligned}
A\left(G * f\right)(x) &= \left(a(x)\frac{d^n}{dx^n} + ...\right)\left(\int\limits_{-\infty}^{\infty} G(x-t)f(t)dt\right) \\
&= \int\limits_{-\infty}^{\infty}\left(a(x)\frac{d^n}{dx^n} + ...\right)G(x-t)f(t)dt \\
&= \int\limits_{-\infty}^{\infty}\left(a(x)G^{(n)}(x-t) + ...\right)f(t)dt \\
&= \lim_{\varepsilon\to 0}\int\limits_{x-\varepsilon}^{x+\varepsilon}\left(a(x)G^{(n)}(x-t) + ...\right)f(t)dt \\
&= a(0)\left(G^{(n)}(0_+) - G^{(n)}(0_-)\right)f(x) \\
&= \frac{a(0)}{a(0)}f(x) = f(x)
\end{aligned}$$

Exercise 10. *Solve the equation* $\Delta f = \delta_0$.

Solution: Since δ_0 is radial symmetric (its values depend only on the radius and not on the direction) the solution would be radial symmetric too: $f = f(r)$. Now, writing δ_0 in spherical coordinates we must divide by the area element, so that the integral would stay 1:

$$\hat{\delta}(r) = \frac{\delta(r)}{4\pi r^2}$$

Writing the Laplace operator also in spherical coordinates we get the equation:

$$f''(r) + \frac{2}{r}f'(r) = \frac{\delta(r)}{4\pi r^2}$$

and we got an ODE.

$$\begin{aligned}
\frac{\delta(r)}{4\pi} &= r^2 f'' + 2rf' = \left(r^2 f'\right)' \\
r^2 f' &= \int \frac{\delta(r)}{4\pi}dr = \frac{1}{4\pi} \\
f &= \int \frac{1}{4\pi r^2}dr = -\frac{1}{4\pi r}
\end{aligned}$$

We prove that $g(r) = -\frac{1}{4\pi r}$ is indeed a solution of $\Delta f = \delta_0$. We actually need to prove that for any f, $\Delta\left(g * f\right)(x) = f(x)$. By properties of convolutions (easy change of variables) we get:

$$\Delta\left(g * f\right)(\mathbf{r}) = \nabla\cdot\nabla\left(g * f\right)(\mathbf{r}) = \left(\nabla g * \nabla f\right)(\mathbf{r})$$

So we wish to estimate the integral:

$$\int\limits_{\mathbb{R}^3} (\nabla g(r')) \cdot (\nabla f(\mathbf{r}-\mathbf{r}'))\, d\mathbf{r}' = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} (\nabla g(r')) \cdot (\nabla f(\mathbf{r}-\mathbf{r}'))\, d\mathbf{r}'.$$

Notice that for a vector function G and a scalar function f it holds that: $G\cdot\nabla f = \nabla(G\cdot f) - f\cdot\nabla G$. In our case $G=\nabla g$ so we get that $\nabla g\cdot\nabla f = \nabla(f\cdot\nabla g) - f\cdot\Delta g$. Using the Divergence theorem we get:

$$\begin{aligned}
\Delta(g*f)(\mathbf{r}) &= \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^3\setminus B(0,\varepsilon)} (\nabla g(r'))\cdot(\nabla f(\mathbf{r}-\mathbf{r}'))\,d\mathbf{r}' \\
&= \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^3\setminus B(0,\varepsilon)} \nabla\left(f(\mathbf{r}-\mathbf{r}')\cdot\nabla g(r')\right)d\mathbf{r}' - \int\limits_{\mathbb{R}^3\setminus B(0,\varepsilon)} f(\mathbf{r}-\mathbf{r}')\cdot\Delta g(r')d\mathbf{r}' \\
&= \lim_{\varepsilon\to 0}\int\limits_{\partial B(0,\varepsilon)} f(\mathbf{r}-\mathbf{r}')\cdot\nabla g(r')\cdot\hat{r'}dS - \int\limits_{\mathbb{R}^3\setminus B(0,\varepsilon)} f(\mathbf{r}-\mathbf{r}')\cdot\left(\frac{1}{r'^2}\frac{\partial}{\partial r'}r'^2\frac{\partial}{\partial r'}\right)g(r')d\mathbf{r}' \\
&= \lim_{\varepsilon\to 0}\int\limits_{\partial B(0,\varepsilon)} f(\mathbf{r}-\mathbf{r}')\cdot\frac{1}{4\pi r'^2}r'^2\sin\theta' d\theta' d\varphi' - 0 \\
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\partial B(0,\varepsilon)} f(\mathbf{r}-\mathbf{r}')\sin\theta' d\theta' d\varphi' \\
&= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\partial B(0,\varepsilon)} \Big(f(\mathbf{r}) + \mathbf{r}'\tilde{f}(\mathbf{r}-\mathbf{r}')\Big)\sin\theta' d\theta' d\varphi' \\
&= \frac{f(\mathbf{r})}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\partial B(0,\varepsilon)} \sin\theta' d\theta' d\varphi' + \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\partial B(0,\varepsilon)} \varepsilon\tilde{f}(\mathbf{r}-\mathbf{r}')\sin\theta' d\theta' d\varphi' \\
&= \frac{f(\mathbf{r})}{4\pi}4\pi + \frac{\varepsilon}{4\pi}\lim_{\varepsilon\to 0}\int\limits_{\partial B(0,\varepsilon)} \tilde{f}(\mathbf{r}-\mathbf{r}')\sin\theta' d\theta' d\varphi' = f(\mathbf{r}) + 0.
\end{aligned}$$

Exercise 11. *Find the order and the leading coefficients for every pole of* $\xi_\lambda \equiv x_+^\lambda$.

Solution: Take $f\in C_c^\infty(\mathbb{R})$, then we can write it as a power series $f = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$. In addition, since f has compact support, we may have singularities only near $x=0$ (the integral $\int_1^\infty x^\lambda f(x)dx$ converges $\forall\lambda$) so:

$$\begin{aligned}\xi_\lambda(f) &= \int\limits_0^1 x^\lambda f(x)dx = \int\limits_0^1 x^\lambda \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}x^n dx \\ &= \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}\int\limits_0^1 x^{\lambda+n}dx \\ &= \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}\frac{x^{\lambda+n+1}}{\lambda+n+1}|_0^1 \\ &= \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}\frac{1}{\lambda+n+1}\end{aligned}$$

From this we can see that all of the poles are of order 1 and the residue of the pole $\lambda_n=-n$ is $\frac{f^{(n-1)}(0)}{(n-1)!}$.

Exercise 12. *Find an analytic continuation for* $P_\lambda = p_+(x_1,...x_n)^\lambda$ *in the case* $p(x,y,z)\equiv\sum_{i=1}^3 x_i^2 a$.

Solution: Consider the differential operator $\nabla^2=\sum_i \frac{\partial^2}{\partial x_i^2}$:

$$\begin{aligned}\nabla^2 p_+(x_1,...x_n)^\lambda &= \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}\left(\sum_{j=1}^3 x_j^2 a\right)^\lambda \\ &= \sum_{i=1}^3 \frac{\partial}{\partial x_i}\left(2\lambda x_i\left(\sum_{j=1}^3 x_j^2 a\right)^{\lambda-1}\right) \\ &= \sum_{i=1}^3 2\lambda\left(\sum_{j=1}^3 x_j^2 a\right)^{\lambda-1} + 4\lambda(\lambda-1)x_i^2\left(\sum_{j=1}^3 x_j^2 a\right)^{\lambda-2} \\ &= 6\lambda p^{\lambda-1}+4\lambda(\lambda-1)\left(p+a\right)p^{\lambda-2} \\ &= 6\lambda P_{\lambda-1}+4\lambda(\lambda-1)P_{\lambda-1}+4\lambda(\lambda-1)aP_{\lambda-2} \\ &\Rightarrow\ P_{\lambda-2}=\frac{\nabla^2 P_\lambda-\left(2\lambda+4\lambda^2\right)P_{\lambda-1}}{4\lambda(\lambda-1)a}\end{aligned}$$

Exercise 13. *Find an analytic continuation for* $p_+(x_1,..,x_n)^\lambda$ *in the case* $p(x,y,z)\equiv x^2+y^2-z^2$.

Solution: Consider the differential operator $L = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} - \frac{\partial^2}{\partial z_i}$:

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2} p_+(x_1, ...x_n)^\lambda &= \left(2\lambda p^{\lambda-1} + 4\lambda(\lambda-1)x^2 p^{\lambda-2}\right) \\
L p_+(x_1, ...x_n)^\lambda &= 2\lambda p^{\lambda-1} + 4\lambda(\lambda-1)\left(x^2 + y^2 - z^2\right) p^{\lambda-2} \\
&= 2\lambda p^{\lambda-1} + 4\lambda(\lambda-1) p^{\lambda-1} \\
&= \left(4\lambda^2 - 2\lambda\right) P_{\lambda-1} \\
&\Rightarrow \; P_{\lambda-1} = \frac{LP_\lambda}{(4\lambda-2)\lambda}
\end{aligned}$$